Manifestation of riddling in the presence of a small parameter mismatch between coupled systems

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Riddling bifurcation, i.e., the bifurcation in which one of the unstable periodic orbits embedded in a chaotic attractor becomes unstable transverse to the attractor, leads to the loss of chaos synchronization in coupled identical systems. We discuss here the manifestation of the riddling bifurcation for the case of a small parameter mismatch between coupled systems. We show that for slightly nonidentical coupled systems, the transverse growth of the synchronous attractor is mediated by transverse bifurcations of unstable periodic orbits embedded into the attractor. The desynchronization mechanism is shown to be similar to the case of chaos-hyperchaos transition.

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Consider two symmetrically coupled identical systems $dx/dt=f(x)$ and $dy/dt=f(y)$ and $x,y\in\mathcal{R}^n$, which evolve on an asymptotically stable bounded chaotic attractor A,

$$\frac{dx}{dt}=f(x)+C(y-x),\quad \frac{dy}{dt}=f(y)+C(x-y). \qquad (1)$$

Complete synchronization occurs when the coupled systems asymptotically exhibit identical behavior, i.e., $|x(t)-y(t)|\to 0$ as $t\to\infty$. The synchronous behavior takes place on the synchronization manifold $x=y$, which is invariant in the phase space of the coupled system (1) and has half the dimension of the full system. The synchronization loss in system (1) is initiated with the riddling bifurcation [1] when the first unstable periodic orbit (UPO) embedded into chaotic attractor A loses its transverse stability.

While the mechanisms of desynchronization are studied in more details for coupled identical systems, the following question still appears to be important: how desynchronization and, in particular, riddling bifurcation, manifests itself qualitatively and quantatively if the coupled systems are not completely identical. Sooner or later this question arises, having in mind practical applications of the synchronization theory or robustness of the results obtained for identical systems.

In this Brief Report we discuss the manifestation of the riddling bifurcation for the case of a small parameter mismatch between coupled systems. We give evidence that for slightly nonidentical coupled systems, the transverse growth of the synchronous attractor is mediated by transverse bifurcations of unstable periodic orbits embedded into the attractor. The desynchronization mechanism is shown to be similar to the bifurcation of chaos-hyperchaos transition [2]. We also note that the parameter mismatch leads to the increase of transverse instabilities after the riddling bifurcation.

Without loss of generality, a small difference between coupled systems can be incorporated in Eq. (1) as

$$\frac{dx}{dt}=f(x)+\alpha(x)+C(y-x),\quad \frac{dy}{dt}=f(y)+C(x-y), \qquad (2)$$

where $\alpha(x)$ describes parameter mismatch.[1] For sufficiently small α, system (2) is not, in general, topologically equivalent to system (1) in the neighborhood of the synchronous attractor [3], unless system (1) is structurally stable.[2] Nevertheless, one can prove (see Ref. [4]) that the attractor of system (2) is located in a small neighborhood of the hyperplane $x=y$. Moreover, for sufficiently small α, transverse stability of nondegenerate orbits embedded in A is preserved for perturbed system (2).

It is also meaningful to speak about transverse and longitudinal stability of saddle periodic orbits embedded in attractor A since a sufficiently small mismatch will cause only small perturbation of the local unstable and stable manifolds. Hence, we are still able to distinguish two directions: "transverse" and "longitudinal." As a result, the moment of riddling bifurcation of system (1) will correspond to the loss of transverse stability of some saddle orbit for system (2).

Therefore, the moment of riddling bifurcation will correspond to the loss of transverse stability of some orbit embedded in the attractor. Here, of course, the situation may arise when the above mentioned orbit leaves the attractor before its transverse destabilization, as it was described in Ref. [6]. In this situation, we may consider the remaining orbits that lose transverse stability with decrease of a coupling coefficient. In general, for nonidentical systems, we are dealing with a chaotic attractor which is no longer located in low-dimensional synchronization manifold but remains in the neighborhood of it. Moreover, periodic orbits embedded into this attractor are proved to loose transverse stability with the decrease of coupling [7]. Therefore, we have the same situation as for chaos-hyperchaos transition [2,5] where the growth of the attractor is mediated by doubly unstable orbits

[1]Linear diffusive coupling scheme (2) is one of the simplest configuration which arises, for example, when a diffusion process is involved and some of the components x_i and y_i denote concentrations. In this case, term $\alpha(x_i-y_i)$ gives a diffusive current (similar situation is considered in Ref. [11]).

[2]This is the case, for example, when the synchronous orbit in system (1) is normally hyperbolic torus or a periodic orbit with multipliers different from (1) in modulus.

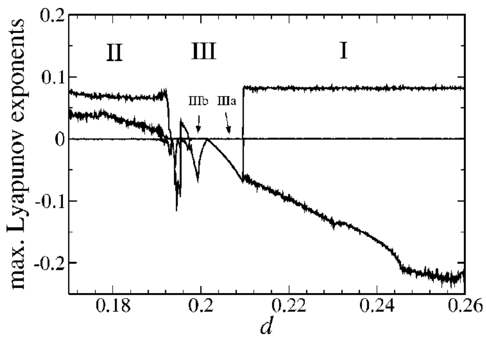

FIG. 1. Lyapunov exponents of system (3) vs. d; $\alpha=0.003$: (I) interval in which chaotic attractor A is located in the neighborhood of manifold $x=y$, (II) interval in which hyperchaotic attractor exists, and (III) interval where the chaotic attractor A loses stability and solution switches into stable limit cycle (IIIa) and torus (IIIb).

embedded in it. It was shown in Ref. [2] that this growth can be either smooth or abrupt depending on the type of "riddling" bifurcation.

In the following, as the numerical example, we consider two coupled Rössler systems:

$$\frac{dx}{dt}=f(x)+\overline{\alpha}+C(d)(y-x),\quad \frac{dy}{dt}=f(y)+C(d)(x-y), \tag{3}$$

where $C(d)=\mathrm{diag}\{d-0.6,1.0,-3.1d+0.7\}$, $\overline{\alpha}=(0,0,\alpha)$,

$$f(x)=[-x_2-x_3,x_1+0.42x_2,2+x_3(x_1-4)]^T.$$

The mismatch is introduced via parameter α.

It was shown in Ref. [9] that the corresponding system of identical coupled oscillators, i.e., $\alpha=0$, loses complete synchronization with the decrease of parameter d. In particular, the riddling bifurcation occurs at $d=0.241$ when the embedded period-1 cycle becomes transversely unstable via supercritical transverse period-doubling bifurcation. At $d\approx0.192$ the blowout bifurcation takes place when transverse Lyapunov exponent of the synchronous attractor becomes negative. Note also that using numerical simulation of coupled identical systems we were unable to detect bursts from the synchronization manifold for parameter values $d\in(0.22,0.24)$, i.e., where synchronous attractor has already lost its transverse stability but is still weakly stable.

In the case of systems with the mismatch, the above mentioned transverse period-doubling bifurcation persists and for $\alpha=0.003$ it takes place at $d=0.24$. Numerically computed Lyapunov exponents for system (3) are shown in Fig. 1. In interval I, the chaotic attractor A is located in the neighborhood of manifold $x=y$. We observe the growth of the second Lyapunov exponent, which is connected with the riddling bifurcation at $d=0.24$ and initiation of the chaos-hyperchaos transition. As it was shown in Ref. [2], this transition is mediated by the transverse destabilization of UPOs embedded in the chaotic attractor A. In interval II, system (3) has the stable hyperchaotic attractor with two positive Lyapunov exponents. At $d\approx0.21$ chaotic attractor A becomes unstable and disappears. The evolution of system (3) switches to the limit cycle (interval IIIa) and torus (interval IIIb).

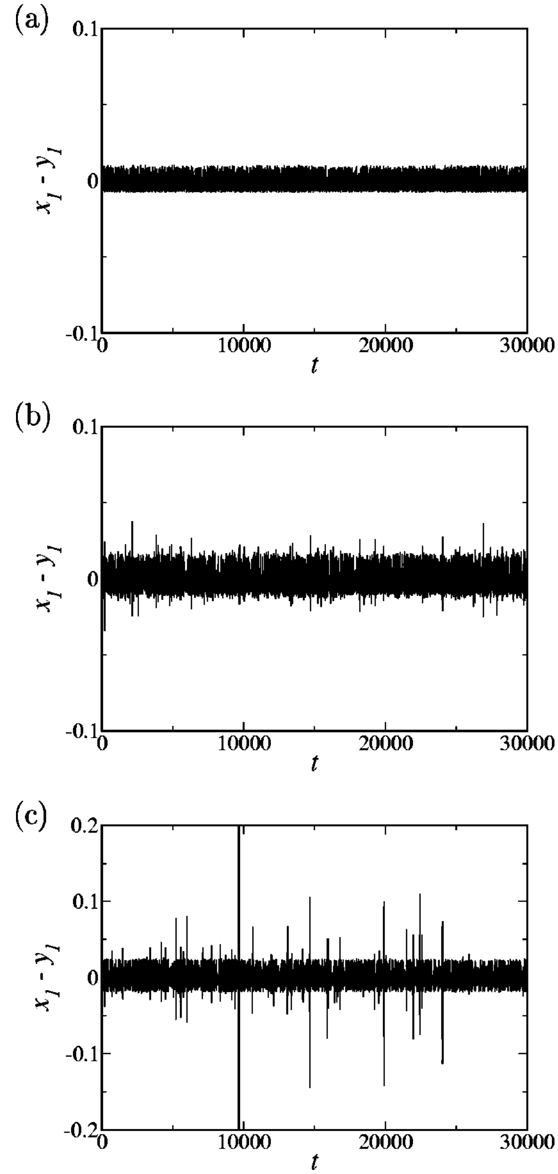

FIG. 2. Behavior of synchronization error x_1-y_1 for $\alpha=0.003$; (a) $d=0.25$, all UPO are transversely stable, (b) $d=0.23$, period-1 UPO is transversely unstable; and (c) $d=0.22$.

Figures 2(a–c) shows the behavior of synchronization error $x_1(t)-y_1(t)$ for different values of d. We can observe transverse bursts for the parameter values after the moment of riddling bifurcation [Figs. 2(b,c)]. More detailed information about the transverse size of the attractor can be seen in Fig. 3, where the maximum amplitude of bursts detected during time interval $T=200\,000$ versus coupling coefficient d is shown. It can be seen that the attractor already grows rapidly in transverse direction after the riddling bifurcation.

Figures 4(b,c) shows the Poincaré map[3] of system (3) for $d=0.22$ (after riddling) and Fig. 4(a) for $d=0.25$ (before

[3]The location of the Poincaré plane was chosen by setting the base point at some point after sufficiently long integration interval and the normal vector to be directed along the flow.

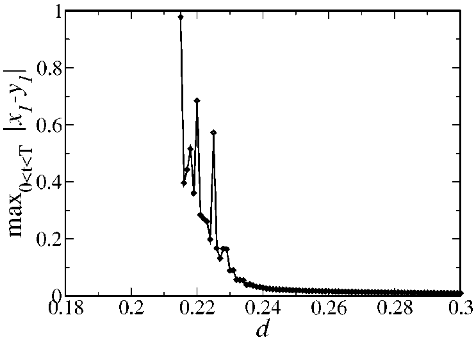

FIG. 3. Transverse growth of the attractor with decreasing of d; $\alpha = 0.003$.

riddling). Local directions of stable and unstable manifolds are indicated.

In the case of the ideal coupled systems the chaotic attractor A located at the invariant manifold $x = y$ can have locally or globally riddled basins of attraction. A is an attractor[4] with *locally riddled* basin if there is neighborhood U of A such that in any neighborhood V of any point in A, there is a set of points in $V \cap U$ of positive measure, which leaves U in a finite time. The trajectories which leave neighborhood U can either go to the other attractor (attractors) or after a finite number of iterations be diverted back to A. If there is neighborhood U of A such that in any neighborhood V of any point in U, there is a set of points of positive measure, that leaves U and goes to the other attractor (attractors), then the basin of A is *globally riddled*.

Let us consider the parameter values of two coupled systems which correspond to the moment after riddling bifurcation: $d_1 = 0.1$, $d_2 = 1.0$, and $d_3 = -1.62$. For these parameter values, period-1 UPO is transversally unstable. Moreover, as follows from the analysis in Ref. [9], for the case of identical systems ($\alpha = 0$), the basin of attraction of attractor A is globally riddled with the points diverging to infinity.

In the case of a small parameter mismatch ($\alpha = 10^{-4}, \ldots, 10^{-2}$), one can observe that for all initial conditions system trajectories eventually escape to infinity. The time intervals before escaping, T_{esc}, strongly depend on the initial conditions. In order to investigate the probability of the escape, 100 initial conditions are chosen in the vicinity of attractor A located in the neighborhood of the manifold $x = y$ and time intervals T_{esc} are averaged for each given value of the mismatch parameter α:$\langle T_{esc} \rangle = (1/100) \Sigma_{i=1}^{100} T_{esc}(x_i)$. The relation $\langle T_{esc} \rangle$ versus α is shown in Fig. 5. Similarly to the cases considered in Refs. [5,8], it is natural to expect the following scaling relation:

$$\langle T_{esc} \rangle (\alpha) \sim \alpha^{-\gamma}.$$

Least squares fitting gives coefficient $\gamma = -2.7$. Note that as α decreases T_{esc} increases significantly, suggesting that in

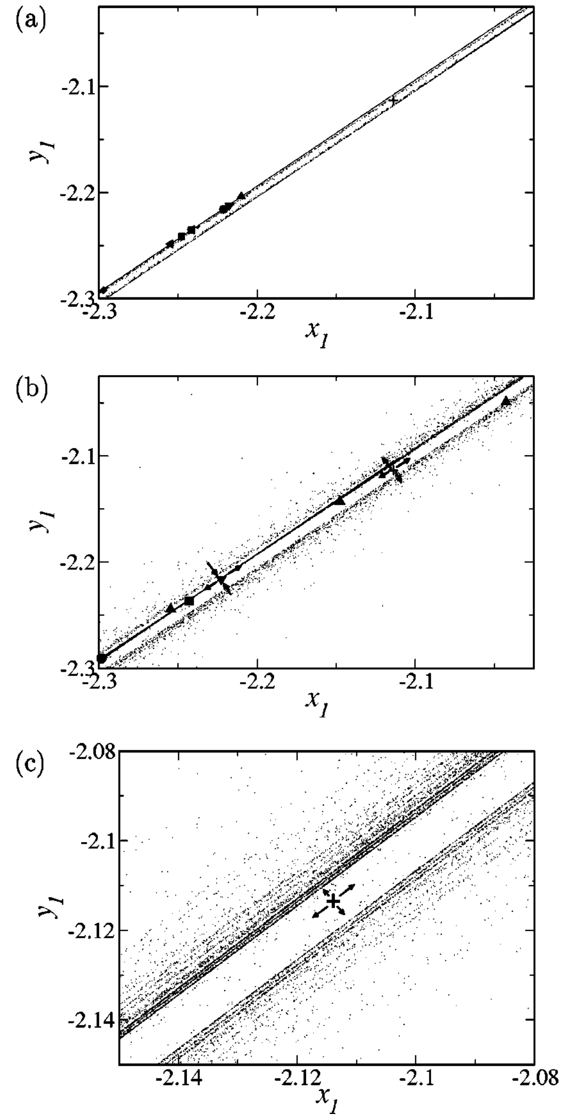

FIG. 4. Poincaré map of system (3): (a) $d = 0.25$, period-1 UPO (indicated with the cross) is transversely stable; (b) $d = 0.22$, period-1 UPO (indicated with the cross) is transversely unstable; and (c) enlargement of (b). Also, some low-periodic orbits are shown, which are transversely stable. The location of the Poincaré plane was chosen by setting the base point at some point after sufficiently long integration interval and the normal vector was directed along the flow.

the limit of identical coupled systems ($\alpha = 0$), there exists the riddled basin with the zero measure of points escaping to infinity.

In Ref. [8], scaling laws were obtained for the averaged switching time for on-off intermittent behavior in the case of maps with symmetry. It has been shown that algebraic relation

$$\langle T_{esc} \rangle (\alpha) \sim \alpha^{-\gamma}$$

holds, where α is a parameter of distance from the bifurcation point and γ is some constant. In Ref. [5] we have also

[4]Here we assume the attractor in the sense of Milnor [10].

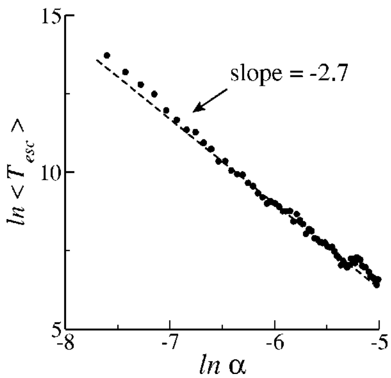

FIG. 5. Averaged time before escaping to infinity, $\langle T_{esc} \rangle$ vs. mismatch parameter α.

shown that the same law takes place in the case of symmetry, increasing bifurcation in two identical coupled Rossler systems.

In the present Brief Report, nonsymmetrical systems are studied and the scaling laws are obtained versus mismatch parameter. The scaling relation in Fig. 5 indicates that the mismatch parameter essentially influences the on-off intermittency phenomena in coupled systems: time switching decreases exponentially. It is interesting to note that a similar effect (with a similar scaling law) has been observed in the case when noise is added to the system with symmetry, as shown in Ref. [8]. Physically, the constant factor -2.7 means that the probability for the orbits to come to the escaping region (from which they may quickly go to infinity) is of order of $\alpha^{2.7}$.

In conclusion, we investigated the effect of riddling bifurcation on the chaotic attractor of the coupled systems with the parameter mismatch. After the onset of bifurcation, the system trajectory shows intermittency-like behavior with bursts away from manifold $x=y$. These bursts grow rapidly, resulting in the growth in size of the chaotic attractor. Contrary to the case of the coupled ideal systems, we have not observed globally riddled basins of the chaotic attractor located in the neighborhood of manifold $x=y$. The observed phenomena seem to be typical for a wide class of coupled, slightly nonidentical systems.

[1] Y.-Ch. Lai, C. Grebogi, J.A. Yorke, and S.C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996); V. Astakhov, A. Shabunin, T. Kapitaniak, and V.S. Anishchenko, *ibid.* **79**, 1014 (1997).

[2] T. Kapitaniak and W.-H. Steeb, Phys. Lett. A **152**, 33 (1991); T. Kapitaniak, *ibid.* **47**, R2975 (1993); K. Stefanski, Chaos, Solitons Fractals **9**, 83 (1998); M.A. Harrison and Y.-Ch. Lai, Phys. Rev. E **59**, R3799 (1999); T. Kapitaniak, Yu. Maistrenko, and S. Popovych, *ibid.* **62**, 1972 (2000); S. Yanchuk and T. Kapitaniak, Phys. Lett. A **290**, 139 (2001).

[3] G. Berkooz, Nonlinearity **7**, 313 (1994).

[4] G. Johnson, D. Mar, T. Carroll, and L. Pecola, Phys. Rev. Lett. **80**, 3956 (1998); S. Yanchuk, Yu. Maistrenko, B. Lading, and E. Mosekilde, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, 2629 (2000); K. Schneider and S. Yanchuk (unpublished).

[5] S. Yanchuk and T. Kapitaniak, Phys. Rev. E **64**, 056235 (2001).

[6] V. Astakhov, T. Kapitaniak, A. Shabunin, and V. Anishchenko, Phys. Lett. A **258**, 99 (1999).

[7] Y. Nagai, and Y.-Ch. Lai, Phys. Rev. E **55**, R1251 (1997); **56**, 4031 (1997),

[8] Y.-Ch. Lai, Phys. Rev. E **56**, 1407 (1997).

[9] S. Yanchuk, Yu. Maistrenko, and E. Mosekilde, Physica D **154**, 26 (2001).

[10] J. Milnor, Commun. Math. Phys. **99**, 177 (1985).

[11] B. Lading, E. Mosekilde, S. Yanchuk, and Yu. Maistrenko, Prog. Theor. Phys. Suppl. **139**, 114 (2000).