Generation of large-scale vorticity in a homogeneous turbulence with a mean velocity shear

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An effect of a mean velocity shear on a turbulence and on the effective force which is determined by the gradient of the Reynolds stresses is studied. Generation of a mean vorticity in a homogeneous incompressible nonhelical turbulent flow with an imposed mean velocity shear due to an excitation of a large-scale instability is found. The instability is caused by a combined effect of the large-scale shear motions (''skew-induced'' deflection of equilibrium mean vorticity) and ''Reynolds stress-induced'' generation of perturbations of mean vorticity. Spatial characteristics of the instability, such as the minimum size of the growing perturbations and the size of perturbations with the maximum growth rate, are determined. This instability and the dynamics of the mean vorticity are associated with Prandtl's turbulent secondary flows.

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I. INTRODUCTION

Vorticity generation in turbulent and laminar flows was studied experimentally, theoretically, and numerically in a number of publications (see, e.g., Refs. [1–16]). For instance, a mechanism of the vorticity production in laminar compressible fluid flows consists in the misalignment of pressure and density gradients [12,13]. It was shown in Ref. [13] that the vorticity generation represents a generic property of any slow nonadiabatic laminar gas flow. In incompressible nonhelical flows, this effect does not occur. The role of small-scale vorticity production in incompressible turbulent flows was discussed in Ref. [10].

On the other hand, generation and dynamics of the mean vorticity are associated with turbulent secondary flows (see, e.g., Refs. [1,3–8]). These flows, e.g., arise at the lateral boundaries of three-dimensional thin shear layers whereby longitudinal (streamwise) mean vorticity is generated by a lateral deflection or ''skewing'' of an existing shear layer [3]. The skew-induced streamwise mean vorticity generation corresponds to Prandtl's first kind of secondary flows. In turbulent flows, e.g., in straight noncircular ducts, streamwise mean vorticity can be generated by the Reynolds stresses. The latter is Prandtl's second kind of turbulent secondary flow, and it ''has no counterpart in laminar flow and cannot be described by any turbulence model with an isotropic eddy viscosity'' [3].

Note that the effect of the generation of large-scale vorticity in the helical turbulence due to hydrodynamical α effect was considered in Refs. [14–16], where α is determined by the hydrodynamical helicity of the turbulent flow.

In the present study, we demonstrated that in a homogeneous incompressible nonhelical turbulent flow with an imposed mean velocity shear (when the α effect does not exist), a large-scale instability can be excited which results in a mean-vorticity production. This instability is caused by a *combined* effect of the large-scale shear motions (''skew-induced'' deflection of equilibrium mean vorticity) and ''Reynolds stress-induced'' generation of perturbations of mean vorticity. The skew-induced deflection of the equilibrium mean vorticity $\bar{\mathbf{W}}^{(s)}$ is determined by $(\bar{\mathbf{W}}^{(s)}\cdot\boldsymbol{\nabla})\tilde{\mathbf{U}}$ term in the equation for the mean vorticity, where $\tilde{\mathbf{U}}$ are perturbations of the mean velocity (see below). The Reynolds stress-induced generation of a mean vorticity is determined by $\boldsymbol{\nabla}\times\boldsymbol{\mathcal{F}}$, where $\boldsymbol{\mathcal{F}}$ is an effective force caused by a gradient of the Reynolds stresses.

This paper is organized as follows. In Sec. II, the governing equations are formulated. In Sec. III, the general form of the Reynolds stresses in a homogeneous nonhelical turbulence with an imposed mean-velocity shear is found using simple symmetry reasoning, and the mechanism for the large-scale instability caused by a combined effect of the large-scale shear motions and Reynolds stress-induced generation of perturbations of the mean vorticity is discussed. In Sec. IV, the equation for the second moment of velocity fluctuations in a homogeneous turbulence with an imposed mean velocity shear is derived. This allows us to study an effect of a mean velocity shear on a nonhelical turbulence and to calculate the effective force determined by the gradient of the Reynolds stresses. Using the derived mean-field equation for vorticity, in Sec. IV we studied the large-scale instability which causes the mean vorticity production.

II. THE GOVERNING EQUATIONS

Our goal is to study an effect of the mean velocity shear on a nonhelical turbulence and on the dynamics of a mean vorticity. The equation for the evolution of vorticity $\mathbf{W} \equiv \boldsymbol{\nabla}\times\mathbf{v}$ reads

$$\frac{\partial \mathbf{W}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{W}-\nu\boldsymbol{\nabla}\times\mathbf{W}), \tag{1}$$

where $\mathbf{v}$ is the fluid velocity with $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ and ν is the kinematic viscosity. This equation follows from the Navier-

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Stokes equation. In this study, we use a mean-field approach whereby the velocity and vorticity are separated into the mean and fluctuating parts: $\mathbf{v}=\overline{\mathbf{U}}+\mathbf{u}$ and $\mathbf{W}=\overline{\mathbf{W}}+\mathbf{w}$, the fluctuating parts have zero mean values, and $\overline{\mathbf{U}}=\langle\mathbf{v}\rangle$, $\overline{\mathbf{W}}=\langle\mathbf{W}\rangle$. Averaging Eq. (1) over an ensemble of fluctuations, we obtain an equation for the mean vorticity $\overline{\mathbf{W}}$:

$$\frac{\partial\overline{\mathbf{W}}}{\partial t}=\boldsymbol{\nabla}\times(\overline{\mathbf{U}}\times\overline{\mathbf{W}}+\langle\mathbf{u}\times\mathbf{w}\rangle-\nu\boldsymbol{\nabla}\times\overline{\mathbf{W}}). \tag{2}$$

Note that the effect of turbulence on the mean vorticity is determined by the Reynolds stresses $\langle u_i u_j\rangle$, because

$$\langle\mathbf{u}\times\mathbf{w}\rangle_i=-\boldsymbol{\nabla}_j\langle u_i u_j\rangle+\tfrac{1}{2}\boldsymbol{\nabla}_i\langle\mathbf{u}^2\rangle, \tag{3}$$

and the curl of the last term in Eq. (3) vanishes.

We consider a turbulent flow with an imposed mean velocity shear $\boldsymbol{\nabla}_i\overline{\mathbf{U}}^{(s)}$, where $\overline{\mathbf{U}}^{(s)}$ is a steady-state solution of the Navier-Stokes equation for the mean-velocity field. In order to study a stability of this equilibrium, we consider perturbations $\tilde{\mathbf{U}}$ of the mean velocity, i.e., the total mean velocity is $\overline{\mathbf{U}}=\overline{\mathbf{U}}^{(s)}+\tilde{\mathbf{U}}$. Similarly, the total mean vorticity is $\overline{\mathbf{W}}=\overline{\mathbf{W}}^{(s)}+\tilde{\mathbf{W}}$, where $\overline{\mathbf{W}}^{(s)}=\boldsymbol{\nabla}\times\overline{\mathbf{U}}^{(s)}$ and $\tilde{\mathbf{W}}=\boldsymbol{\nabla}\times\tilde{\mathbf{U}}$. Thus, the linearized equation for the small perturbations of the mean vorticity, $\tilde{\mathbf{W}}=\overline{\mathbf{W}}-\overline{\mathbf{W}}^{(s)}$, is given by

$$\frac{\partial\tilde{\mathbf{W}}}{\partial t}=\boldsymbol{\nabla}\times(\overline{\mathbf{U}}^{(s)}\times\tilde{\mathbf{W}}+\tilde{\mathbf{U}}\times\overline{\mathbf{W}}^{(s)}+\boldsymbol{\mathcal{F}}-\nu\boldsymbol{\nabla}\times\tilde{\mathbf{W}}), \tag{4}$$

where $\mathcal{F}_i=-\boldsymbol{\nabla}_j[f_{ij}(\overline{\mathbf{U}})-f_{ij}(\overline{\mathbf{U}}^{(s)})]$ is the effective force and $f_{ij}=\langle u_i u_j\rangle$. Equation (4) is derived by subtracting Eq. (2) written for $\overline{\mathbf{W}}^{(s)}$ from the corresponding equation for the mean vorticity $\overline{\mathbf{W}}$. In order to obtain a closed system of equations, in Sec. IV we derived an equation for the effective force $\boldsymbol{\mathcal{F}}$. Equation (4) determines the dynamics of perturbations of the mean vorticity. In the following sections, we will show that under certain conditions the large-scale instability can be excited which causes the mean vorticity production.

III. THE QUALITATIVE DESCRIPTION

In this section, we discuss the mechanism of the large-scale instability. The mean velocity shear can affect a turbulence. The reason is that additional strongly anisotropic velocity fluctuations can be generated by tangling of the mean velocity gradients with the isotropic and homogeneous background turbulence (see Fig. 1). The source of energy of this "tangling turbulence" is the energy of the background turbulence [17].

The tangling turbulence is a universal phenomenon, e.g., it was introduced by Wheelon [18] and Batchelor *et al.* [19] for a passive scalar and by Golitsyn [20] and Moffatt [21] for a passive vector (magnetic field). Anisotropic fluctuations of a passive scalar (e.g., the number density of particles or temperature) are produced by tangling of gradients of the mean passive scalar field with a random velocity field. Similarly, anisotropic magnetic fluctuations are generated by tangling

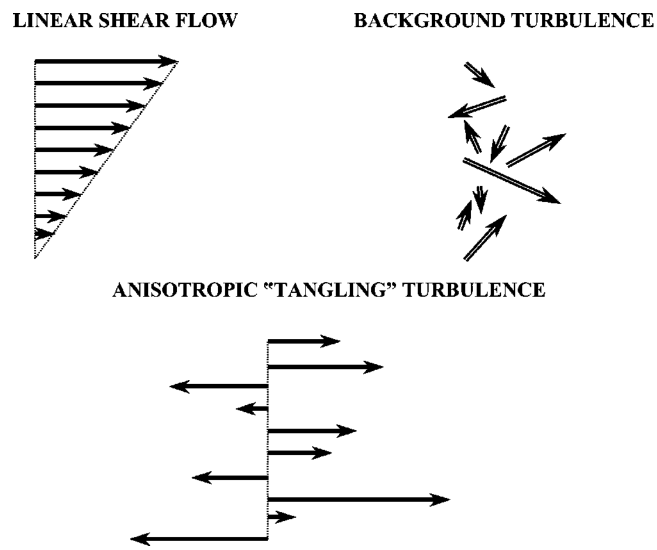

FIG. 1. Mechanism for the tangling turbulence: generation of anisotropic velocity fluctuations by tangling of the mean velocity gradients with the background turbulence.

of the mean magnetic field with the velocity fluctuations. The Reynolds stresses in a turbulent flow with a weak mean velocity shear form another example of a tangling turbulence. Indeed, they are strongly anisotropic in the presence of shear and have a steeper spectrum ($\propto k^{-7/3}$) than a Kolmogorov background turbulence (see, e.g., Refs. [17,22–25]). The anisotropic velocity fluctuations of tangling turbulence were studied first by Lumley [22]. Different properties of anisotropic turbulence, such as anisotropic scaling behavior and anomalous scalings, were studied recently in [26–28].

We assumed that the generated tangling turbulence does not affect the background turbulence. This implies that we considered one-way coupling due to a weak large-scale velocity shear.

The general form of the Reynolds stresses in a turbulent flow with a mean-velocity shear can be obtained from simple symmetry reasoning. Indeed, the Reynolds stresses $\langle u_i u_j\rangle$ form a symmetric true tensor. In a turbulent flow with an imposed mean-velocity shear, the Reynolds stresses depend on the true tensor $\boldsymbol{\nabla}_j\overline{U}_i$, which can be written as a sum of the symmetric and antisymmetric parts, i.e., $\boldsymbol{\nabla}_j\overline{U}_i=(\partial\hat{U})_{ij}-(1/2)\varepsilon_{ijk}\overline{W}_k$, where $(\partial\hat{U})_{ij}=(\boldsymbol{\nabla}_i\overline{U}_j+\boldsymbol{\nabla}_j\overline{U}_i)/2$ is the true tensor and $\overline{\mathbf{W}}=\boldsymbol{\nabla}\times\overline{\mathbf{U}}$ is the mean vorticity (pseudo-vector). We take into account the effect which is linear in perturbations $(\partial\tilde{U})_{ij}$ and $\tilde{\mathbf{W}}$, where $(\partial\tilde{U})_{ij}=(\boldsymbol{\nabla}_i\tilde{U}_j+\boldsymbol{\nabla}_j\tilde{U}_i)/2$. Thus, the general form of the Reynolds stresses can be found using the following true tensors: $(\partial\tilde{U})_{ij}$, M_{ij}, N_{ij}, H_{ij}, and G_{ij}, where

$$M_{ij}=(\partial\overline{U}^{(s)})_{im}(\partial\tilde{U})_{mj}+(\partial\overline{U}^{(s)})_{jm}(\partial\tilde{U})_{mi}, \tag{5}$$

$$N_{ij}=\tilde{W}_n[\varepsilon_{nim}(\partial\overline{U}^{(s)})_{mj}+\varepsilon_{njm}(\partial\overline{U}^{(s)})_{mi}], \tag{6}$$

$$H_{ij}=\overline{W}_n^{(s)}[\varepsilon_{nim}(\partial\tilde{U})_{mj}+\varepsilon_{njm}(\partial\tilde{U})_{mi}], \tag{7}$$

$$G_{ij}=\overline{W}_i^{(s)}\tilde{W}_j+\overline{W}_j^{(s)}\tilde{W}_i, \tag{8}$$

$(\partial\overline{U}^{(s)})_{ij}=(\nabla_i\overline{U}_j^{(s)}+\nabla_j\overline{U}_i^{(s)})/2$ and ε_{ijk} is the fully antisymmetric Levi-Civita tensor (pseudotensor). Therefore, the Reynolds stresses have the following general form:

$$f_{ij}(\tilde{\mathbf{U}})=-2\nu_T(\partial\tilde{U})_{ij}-l_0^2(B_1M_{ij}+B_2N_{ij}+B_3H_{ij}+B_4G_{ij}), \tag{9}$$

where B_k are the unknown coefficients, l_0 is the maximum scale of turbulent motions, $\nu_T=l_0u_0/a_*$ is the turbulent viscosity with the factor $a_*\approx 3$–6, and u_0 is the characteristic turbulent velocity in the maximum scale of turbulent motions l_0. The parameter l_0^2 in Eq. (9) was introduced using dimensional arguments. The first term on the right hand side of Eq. (9) describes the standard isotropic turbulent viscosity, whereas the other terms are determined by fluctuations caused by the imposed velocity shear $\nabla_i\overline{U}_j^{(s)}$.

Let us study the evolution of the mean vorticity using Eqs. (4) and (9), where $\mathcal{F}_i=-\nabla_j f_{ij}(\tilde{\mathbf{U}})$ is the effective force. We consider a homogeneous divergence-free turbulence with a mean-velocity shear, e.g., $\overline{\mathbf{U}}^{(s)}=(0,Sx,0)$ and $\overline{\mathbf{W}}^{(s)}=(0,0,S)$. For simplicity, we use perturbations of the mean vorticity in the form $\tilde{\mathbf{W}}=(\tilde{W}_x(z),\tilde{W}_y(z),0)$. Then Eq. (4) can be written as

$$\frac{\partial\tilde{W}_x}{\partial t}=S\tilde{W}_y+\nu_T\tilde{W}_x'', \tag{10}$$

$$\frac{\partial\tilde{W}_y}{\partial t}=-\beta Sl_0^2\tilde{W}_x''+\nu_T\tilde{W}_y'', \tag{11}$$

where $\tilde{W}_x''=\partial^2\tilde{W}_x/\partial z^2$, $\beta=[B_1+2(B_2+B_3)-4B_4]/4$. In Eq. (10), we took into account that $l_0^2\tilde{W}_y''\ll\tilde{W}_y$, i.e., the characteristic scale L_W of the mean vorticity variations is much larger than the maximum scale of turbulent motions l_0. This assumption corresponds to the mean-field approach. For derivation of Eqs. (10) and (11), we used the identities presented in Appendix A.

We seek for a solution of Eqs. (10) and (11) in the form $\propto\exp(\gamma t+iKz)$. Thus when $\beta>0$, perturbations of the mean vorticity can grow in time and the growth rate of the instability is given by

$$\gamma=\sqrt{\beta}Sl_0K-\nu_TK^2. \tag{12}$$

The maximum growth rate of perturbations of the mean vorticity, $\gamma_{\max}=\beta(Sl_0)^2/4\nu_T$, is attained at $K=K_m=\sqrt{\beta}Sl_0/2\nu_T$. The sufficient condition $\gamma>0$ for the excitation of the instability reads $L_W/l_0>2\pi/(a_*\sqrt{\beta}\tau_0S)$, where $L_W\equiv 2\pi/K$ and we consider a weak velocity shear $\tau_0S\ll 1$.

Now let us discuss the mechanism of this instability using a terminology from Ref. [3]. The first term $S\tilde{W}_y=(\overline{\mathbf{W}}^{(s)}\cdot\nabla)\tilde{U}_x$ in Eq. (10) describes a skew-induced generation of perturbations of the mean vorticity $\tilde{W}_x$ by quasi-inviscid deflection of the equilibrium mean vorticity $\overline{\mathbf{W}}^{(s)}$. In particular, the mean vorticity $\tilde{W}_x\mathbf{e}_x$ is generated from $\tilde{W}_y\mathbf{e}_y$ by equilibrium shear motions with the mean vorticity $\overline{\mathbf{W}}^{(s)}=S\mathbf{e}_z$, i.e., $\tilde{W}_x\mathbf{e}_x\propto(\overline{\mathbf{W}}^{(s)}\cdot\nabla)\tilde{U}_x\mathbf{e}_x\propto\tilde{W}_y\mathbf{e}_y\times\overline{\mathbf{W}}^{(s)}$ (see Fig. 2). Here $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit vectors along x, y, and z axis, respectively. On the other hand, the first term, $-\beta Sl_0^2\tilde{W}_x''$ in Eq. (11) determines a Reynolds stress-induced generation of perturbations of the mean vorticity $\tilde{W}_y$ by turbulent Reynolds stresses (see Fig. 3). In particular, this term is determined by $(\nabla\times\boldsymbol{\mathcal{F}})_y$, where $\boldsymbol{\mathcal{F}}$ is a gradient of the Reynolds stresses. This implies that the mean vorticity $\tilde{W}_y\mathbf{e}_y$ is gener-ated by an effective anisotropic viscous term $\propto -l_0^2\Delta(\tilde{W}_x\mathbf{e}_x\cdot\nabla)\overline{U}^{(s)}(x)\mathbf{e}_y\propto -l_0^2S\tilde{W}_x''\mathbf{e}_y$. This mechanism of the generation of perturbations of the mean vorticity $\tilde{W}_y\mathbf{e}_y$ can be interpreted as a stretching of the perturbations of the mean vorticity $\tilde{W}_x\mathbf{e}_x$ by the equilibrium shear motions $\overline{\mathbf{U}}^{(s)}=Sx\mathbf{e}_y$ during the turnover time of turbulent eddies (see Fig. 3). The growth rate of this instability is caused by a combined effect of the sheared motions (skew-induced generation) and the Reynolds stress-induced generation of perturbations of the mean vorticity.

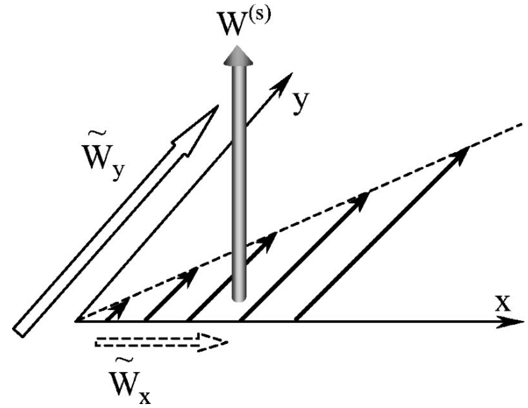

FIG. 2. Mechanism for "skew-induced" generation of perturbations of the mean vorticity $\tilde{W}_x$ by quasi-inviscid deflection of the equilibrium mean vorticity $\overline{W}^{(s)}\mathbf{e}_z$. In particular, the mean vorticity $\tilde{W}_x\mathbf{e}_x$ is generated by an interaction of perturbations of the mean vorticity $\tilde{W}_y\mathbf{e}_y$ and the equilibrium mean vorticity $\overline{\mathbf{W}}^{(s)}=S\mathbf{e}_z$, i.e., $\tilde{W}_x\mathbf{e}_x\propto(\overline{\mathbf{W}}^{(s)}\cdot\nabla)\tilde{U}_x\mathbf{e}_x\propto\tilde{W}_y\mathbf{e}_y\times\overline{\mathbf{W}}^{(s)}$.

This large-scale instability is similar to a mean-field magnetic dynamo instability caused by the $\mathbf{\Omega}\times\mathbf{J}$ effect (see, e.g., Refs. [29,30]) or the shear-current effect [31], where $\mathbf{J}$ is the electric current and Ω is the angular velocity. Indeed, the first term in Eq. (10) is similar to the differential rotation (or large-scale shear motions) which causes a generation of a toroidal mean magnetic field by a stretching of the poloidal mean magnetic field with the differential rotation (or by large-scale shear motions). On the other hand, the first term in Eq. (11) is similar to the $\mathbf{\Omega}\times\mathbf{J}$ effect (see, e.g., Refs. [29,30]), or to the shear-current effect [31]. These effects result in the generation of a poloidal mean magnetic field from the toroidal mean magnetic field. The $\mathbf{\Omega}\times\mathbf{J}$ effect is related to an interaction of the mean rotation and electric current in a homogeneous turbulent flow, while the shear-current effect occurs due to an interaction of the mean vorticity and electric current in a homogeneous aniso-

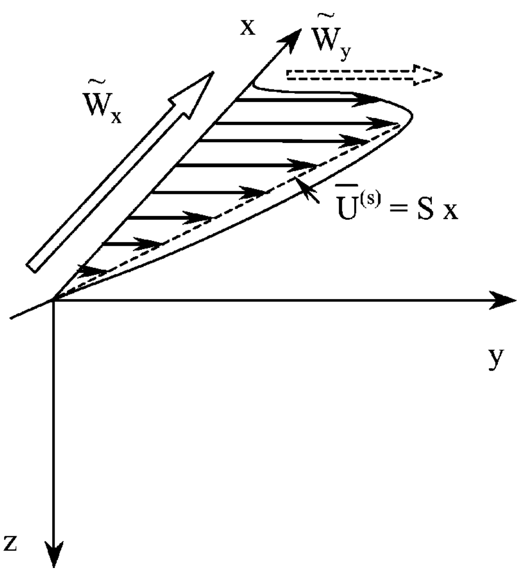

FIG. 3. Mechanism for a ‘‘Reynolds stress-induced’’ generation of perturbations of the mean vorticity $\tilde{W}_y \mathbf{e}_y$ by turbulent Reynolds stresses. In particular, the mean vorticity $\tilde{W}_y \mathbf{e}_y$ is generated by an effective anisotropic viscous term $\propto -l_0^2 \Delta(\tilde{W}_x \mathbf{e}_x \cdot \boldsymbol{\nabla}) \overline{U}^{(s)}(x) \mathbf{e}_y \propto -l_0^2 S \tilde{W}_x'' \mathbf{e}_y$. This mechanism of the generation of perturbations of the mean vorticity $\tilde{W}_y \mathbf{e}_y$ can be interpreted as a stretching of the perturbations of the mean vorticity $\tilde{W}_x \mathbf{e}_x$ by the equilibrium shear motions $\overline{\mathbf{U}}^{(s)} = Sx\mathbf{e}_y$ during the turnover time of turbulent eddies.

tropic turbulent flow of a conducting fluid. The magnetic dynamo instability is a combined effect of the nonuniform mean flow (differential rotation or large-scale shear motions) and turbulence effects (anisotropic turbulent motions which cause the $\boldsymbol{\Omega} \times \mathbf{J}$ effect and the shear-current effect).

On the other hand, the kinematic magnetic dynamo instability is different from the instability of the mean vorticity although they are governed by similar equations. The mean vorticity $\overline{\mathbf{W}} = \boldsymbol{\nabla} \times \overline{\mathbf{U}}$ is directly determined by the velocity field $\overline{\mathbf{U}}$, while the magnetic field depends on the velocity field through the induction equation.

IV. EFFECT OF A MEAN-VELOCITY SHEAR ON A TURBULENCE AND LARGE-SCALE INSTABILITY

In this section, we study quantitatively an effect of a mean-velocity shear on a nonhelical turbulence. This allows us to derive an equation for the effective force $\boldsymbol{\mathcal{F}}$ and to study the dynamics of the mean vorticity.

A. Method of derivations

To study an effect of a mean-velocity shear on a turbulence, we used an equation for fluctuations $\mathbf{u}(t,\mathbf{r})$ that is obtained by subtracting equation for the mean field from the corresponding equation for the total field:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\overline{\mathbf{U}} \cdot \boldsymbol{\nabla})\mathbf{u} - (\mathbf{u} \cdot \boldsymbol{\nabla})\overline{\mathbf{U}} - \frac{\boldsymbol{\nabla} p}{\rho} + \mathbf{F}^{(\mathrm{st})} + \mathbf{U}_N, \qquad (13)$$

where p are the pressure fluctuations, ρ is the fluid density, $\mathbf{F}^{(\mathrm{st})}$ is an external stirring force with a zero-mean value, and $U_N = \langle (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \rangle - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \nu \Delta \mathbf{u}$. We consider a turbulent flow with large Reynolds numbers ($\mathrm{Re} = l_0 u_0 \nu \gg 1$). We assumed that there is a separation of scales, i.e., the maximum scale of turbulent motions l_0 is much smaller than the characteristic scale of the inhomogeneities of the mean fields. Using Eq. (13), we derived an equation for the second moment of the turbulent velocity field $f_{ij}(\mathbf{k},\mathbf{R}) \equiv \int \langle u_i(\mathbf{k} + \mathbf{K}/2) u_j(-\mathbf{k} + \mathbf{K}/2) \rangle \exp(i\mathbf{K} \cdot \mathbf{R}) d\mathbf{K}$:

$$\frac{\partial f_{ij}(\mathbf{k},\mathbf{R})}{\partial t} = I_{ijmn}(\overline{\mathbf{U}}) f_{mn} + F_{ij} + f_{ij}^{(N)} \qquad (14)$$

(see Appendix B), where

$$I_{ijmn}(\overline{\mathbf{U}}) = \Bigg(2k_{iq}\delta_{mp}\delta_{jn} + 2k_{jq}\delta_{im}\delta_{pn} - \delta_{im}\delta_{jq}\delta_{np} - \delta_{iq}\delta_{jn}\delta_{mp} + \delta_{im}\delta_{jn} k_q \frac{\partial}{\partial k_p} \Bigg) \boldsymbol{\nabla}_p \overline{U}_q, \qquad (15)$$

and $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small scales (see Appendix B), $k_{ij} = k_i k_j / k^2$, $f_{ij}^{(N)}(\mathbf{k},\mathbf{R})$ is the third moment appearing due to the nonlinear term, $\boldsymbol{\nabla} = \partial / \partial \mathbf{R}$,

$$F_{ij}(\mathbf{k},\mathbf{R}) = \langle \tilde{F}_i(\mathbf{k},\mathbf{R}) u_j(-\mathbf{k},\mathbf{R}) \rangle + \langle u_i(\mathbf{k},\mathbf{R}) \tilde{F}_j(-\mathbf{k},\mathbf{R}) \rangle$$

and

$$\tilde{\mathbf{F}}(\mathbf{k},\mathbf{R},t) = -\mathbf{k} \times [\mathbf{k} \times \mathbf{F}^{(\mathrm{st})}(\mathbf{k},\mathbf{R})]/k^2.$$

Equation (14) is written in a frame moving with a local velocity $\overline{\mathbf{U}}$ of the mean flow. In Eqs. (14) and (15), we neglected small terms which are of the order of $O(|\nabla^3 \overline{\mathbf{U}}|)$. Note that Eqs. (14) and (15) do not contain terms proportional to $O(|\nabla^2 \overline{\mathbf{U}}|)$.

The total mean-velocity is $\overline{\mathbf{U}} = \overline{\mathbf{U}}^{(s)} + \tilde{\mathbf{U}}$, where we considered a turbulent flow with an imposed mean-velocity shear $\boldsymbol{\nabla}_i \overline{\mathbf{U}}^{(s)}$. Now let us introduce a background turbulence with zero gradients of the mean fluid velocity $\boldsymbol{\nabla}_i \overline{U}_j = 0$. The background turbulence is determined by the equation $\partial f_{ij}^{(0)} / \partial t = F_{ij} + f_{ij}^{(N0)}$, where the superscript (0) corresponds to the background turbulence, and we assumed that the tensor $F_{ij}(\mathbf{k},\mathbf{R})$, which is determined by a stirring force, is independent of the mean-velocity. Equation for the deviations $f_{ij} - f_{ij}^{(0)}$ from the background turbulence is given by

$$\frac{\partial (\hat{f} - \hat{f}^{(0)})}{\partial t} = [\hat{I}(\overline{\mathbf{U}}^{(s)}) + \hat{I}(\tilde{\mathbf{U}})] \hat{f} + \hat{f}^{(N)} - \hat{f}^{(N0)}, \qquad (16)$$

where we used the following notations: $\hat{f} \equiv f_{ij}(\mathbf{k},\mathbf{R})$, $\hat{f}^{(N)} \equiv f_{ij}^{(N)}(\mathbf{k},\mathbf{R})$, $\hat{f}^{(N0)} \equiv f_{ij}^{(N0)}(\mathbf{k},\mathbf{R})$, $\hat{f}^{(0)} \equiv f_{ij}^{(0)}(\mathbf{k},\mathbf{R})$, and $\hat{I}(\overline{\mathbf{U}}) \hat{f} \equiv I_{ijmn}(\overline{\mathbf{U}}) f_{mn}(\mathbf{k},\mathbf{R})$.

Equation (16) for the deviations of the second moments in $\mathbf{k}$ space contains the deviations of the third moments and a problem of closing the equations for the higher moments arises. Various approximate methods have been proposed for the solution of problems of this type (see, e.g., Refs. [32–34]). The simplest procedure is the τ approximation which was widely used for the study of different problems of turbulent transport (see, e.g., Refs. [32,35–37]). It allows us to express the deviations of the third moments $\hat{f}^{(N)}-\hat{f}^{(N0)}$ in $\mathbf{k}$ space in terms of those for the second moments $\hat{f}-\hat{f}^{(0)}$ by assuming that

$$\hat{f}^{(N)}-\hat{f}^{(N0)}=-\frac{\hat{f}-\hat{f}^{(0)}}{\tau(k)}, \tag{17}$$

where $\tau(k)$ is the correlation time of the turbulent velocity field. Here we assumed that the time $\tau(k)$ is independent of the gradients of the mean fluid velocity because in the framework of the mean-field approach we may only consider a weak shear: $\tau_0|\boldsymbol{\nabla}\overline{U}|\ll 1$, where $\tau_0=l_0/u_0$.

The τ approximation is, in general, similar to the eddy damped Quasinormal Markowian (EDQNM) approximation. However, there is a principlal difference between these two approaches (see Refs. [32,34]). The EDQNM closures do not relax to the equilibrium, and do not describe properly the motions in the equilibrium state. In the EDQNM theory, there is no dynamically determined relaxation time, and no slightly perturbed steady state can be approached [32]. In the τ approximation, the relaxation time for small departures from the equilibrium is determined by the random motions in the equilibrium state, but not by the departure from the equilibrium [32]. Analysis performed in Ref. [32] showed that the τ approximation describes the relaxation to the equilibrium state (the background turbulence) more accurately than the EDQNM approach.

Note that we applied the τ approximation (17) only to study the deviations from the background turbulence which are caused by the spatial derivatives of the mean-velocity. The background turbulence is assumed to be known. In this study, we used the model of an isotropic, homogeneous, and nonhelical background turbulence:

$$f_{ij}^{(0)}(\mathbf{k},\mathbf{R})=\frac{u_0^2}{8\pi k^2}P_{ij}(\mathbf{k})\mathcal{E}(k), \tag{18}$$

where $P_{ij}(\mathbf{k})=\delta_{ij}-k_{ij}$, δ_{ij} is the Kronecker tensor, $\tau(k)=2\tau_0\overline{\tau}(k)$, $\mathcal{E}(k)=-d\overline{\tau}(k)/dk$, $\overline{\tau}(k)=(k/k_0)^{1-q}$, $1<q<3$ is the exponent of the kinetic energy spectrum (e.g., $q=5/3$ for Kolmogorov spectrum), and $k_0=1/l_0$. We assumed that the generated tangling turbulence does not affect the background turbulence. This implies that we considered a one-way coupling due to a weak large-scale velocity shear.

B. Equation for the second moment of velocity fluctuations

We assume that the characteristic time of variation of the second moment $f_{ij}(\mathbf{k},\mathbf{R})$ is substantially larger than the correlation time $\tau(k)$ for all turbulence scales. Thus in a steady state, Eq. (16) reads

$$\hat{L}(\hat{f}-\hat{f}^{(0)})=\tau(k)[\hat{I}(\overline{\mathbf{U}}^{(s)})+\hat{I}(\tilde{\mathbf{U}})]\hat{f}^{(0)}, \tag{19}$$

where $\hat{L}\equiv L_{ijmn}=\delta_{im}\delta_{jn}-\tau(k)[I_{ijmn}(\overline{\mathbf{U}}^{(s)})+I_{ijmn}(\tilde{\mathbf{U}})]$, and we used Eq. (17). The solution of Eq. (19) yields the expression for the second moment $\hat{f}\equiv f_{ij}(\mathbf{k},\mathbf{R})$:

$$\hat{f}\approx\hat{f}(\overline{\mathbf{U}}^{(s)})+\tau(k)[\hat{I}(\tilde{\mathbf{U}})+\hat{I}(\overline{\mathbf{U}}^{(s)})\tau(k)\hat{I}(\tilde{\mathbf{U}}) + \hat{I}(\tilde{\mathbf{U}})\tau(k)\hat{I}(\overline{\mathbf{U}}^{(s)})]\hat{f}^{(0)}, \tag{20}$$

where

$$\hat{f}(\overline{\mathbf{U}}^{(s)})=\hat{f}^{(0)}+\tau(k)[\hat{I}(\overline{\mathbf{U}}^{(s)})+\hat{I}(\overline{\mathbf{U}}^{(s)})\tau(k)\hat{I}(\overline{\mathbf{U}}^{(s)})]\hat{f}^{(0)}.$$

In Eq. (20), we neglected terms which are of the order of $O(|\boldsymbol{\nabla}\tilde{\mathbf{U}}|^2)$ and $O(|\boldsymbol{\nabla}\overline{\mathbf{U}}^{(s)}|^2|\boldsymbol{\nabla}\tilde{\mathbf{U}}|)$. The first term in the equation for $\hat{f}(\overline{\mathbf{U}}^{(s)})$ is independent of the mean-velocity shear and it describes the background turbulence, while the second and the third terms in this equation determine an effect of the mean-velocity shear on turbulence.

C. Effective force

Equation (20) allows us to determine the effective force: $\mathcal{F}_i=-\boldsymbol{\nabla}_j\int\tilde{f}_{ij}(\mathbf{k},\mathbf{R})d\mathbf{k}$, where $\tilde{f}=\hat{f}-\hat{f}(\overline{\mathbf{U}}^{(s)})$, and we used the notation $\tilde{f}\equiv\tilde{f}_{ij}(\mathbf{k},\mathbf{R})$. The integration in $\mathbf{k}$ space yields the second moment $\tilde{f}_{ij}(\mathbf{R})=\int\tilde{f}_{ij}(\mathbf{k},\mathbf{R})d\mathbf{k}$:

$$\tilde{f}_{ij}(\mathbf{R})=-2\nu_T(\partial\tilde{U})_{ij}-l_0^2 \times[4C_1M_{ij}+C_2(N_{ij}+H_{ij})+C_3G_{ij}], \tag{21}$$

where $C_1=8(q^2-13q+40)/315$, $C_2=2(6-7q)/45$, $C_3=-2(q+2)/45$, and the tensors M_{ij}, N_{ij}, H_{ij}, and G_{ij} are determined by Eqs. (5)–(8). In Eq. (21), we omitted terms $\propto\delta_{ij}$ because they do not contribute to $\boldsymbol{\nabla}\times\boldsymbol{\mathcal{F}}$ [see Eq. (4) for perturbations of the mean vorticity $\tilde{\mathbf{W}}$]. To derive Eq. (21), we used the identities presented in Appendix A. Equations (9) and (21) yield $B_1=4C_1$, $B_2=B_3=C_2$, and $B_4=C_3$.

Note that the mean velocity gradient $\boldsymbol{\nabla}_i\overline{\mathbf{U}}^{(s)}$ causes generation of anisotropic velocity fluctuations (tangling turbulence). Inhomogeneities of perturbations of the mean velocity $\tilde{\mathbf{U}}$ produce additional velocity fluctuations, so that the Reynolds stresses $\tilde{f}_{ij}(\mathbf{R})$ are the result of a combined effect of two types of velocity fluctuations produced by the tangling of mean gradients $\boldsymbol{\nabla}_i\overline{\mathbf{U}}^{(s)}$ and $\boldsymbol{\nabla}_i\tilde{\mathbf{U}}$ by a small-scale Kolmogorov turbulence. Equation (21) allows us to determine the effective force $\mathcal{F}_i=-\boldsymbol{\nabla}_j\tilde{f}_{ij}(\mathbf{R})$.

D. The large-scale instability in a homogeneous turbulence with a mean-velocity shear

Let us study the evolution of the mean vorticity using Eq. (21) for the Reynolds stresses. Consider a homogeneous nonhelical turbulence with a mean-velocity shear, e.g., $\overline{\mathbf{U}}^{(s)}=(0,Sx,0)$ and $\overline{\mathbf{W}}^{(s)}=(0,0,S)$. For simplicity, we consider perturbations of the mean vorticity in the form $\tilde{\mathbf{W}} = (\tilde{W}_x(z), \tilde{W}_y(z), 0)$. Then Eq. (4) reduces to Eqs. (10) and (11), where $\beta = C_1 + C_2 - C_3 = 4(2q^2 - 47q + 108)/315$. We seek for a solution of Eqs. (10) and (11) in the form $\propto \exp(\gamma t + iKz)$. Thus, the growth rate of perturbations of the mean vorticity is given by $\gamma = \sqrt{\beta} S l_0 K - \nu_T K^2$. The maximum growth rate of perturbations of the mean vorticity, $\gamma_{\rm max} = \beta (S l_0)^2 / 4\nu_T \approx 0.1 a_* S^2 \tau_0$, is attained at $K = K_m = \sqrt{\beta} S l_0 / 2\nu_T$. Here we used that for a Kolmogorov spectrum ($q = 5/3$) of the background turbulence, the factor $\beta \approx 0.45$. The sufficient condition $\gamma > 0$ for the excitation of the instability reads $L_W / l_0 > 2\pi / (a_* \sqrt{\beta} \tau_0 S)$. Since $\tau_0 S \ll 1$ (we considered a weak velocity shear), the scale $L_W \gg l_0$ and, therefore, there is a separation of scales as required in our model.

Now we consider perturbations of the mean vorticity which depend on x and z, i.e., $\tilde{\mathbf{W}} = (\tilde{W}_x(x,z), \tilde{W}_y(x,z), 0)$. Then Eq. (4) reduces to

$$\Delta \frac{\partial \tilde{W}_x}{\partial t} = S \frac{\partial^2 \tilde{W}_y}{\partial z^2} + \nu_T \Delta^2 \tilde{W}_x , \qquad (22)$$

$$\frac{\partial^2}{\partial z^2} \frac{\partial \tilde{W}_y}{\partial t} = \Delta \left[\beta S l_0^2 \left(\beta_* \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right) \tilde{W}_x + \nu_T \frac{\partial^2}{\partial z^2} \tilde{W}_y \right], \qquad (23)$$

where $\beta_* = 2(C_1 - C_2)/\beta$. For $9 = 5/3$, the parameter $\beta_* \approx 3.5$. Therefore, the growth rate of perturbations of the mean vorticity is given by

$$\gamma = S l_0 K \sqrt{\beta [1 - (1 + \beta_*) \sin^2 \theta]} - \nu_T K^2 . \qquad (24)$$

The maximum growth rate of perturbations of the mean vorticity, $\gamma_{\rm max} = \beta (S l_0)^2 [1 - (1 + \beta_*) \sin^2 \theta] / 4\nu_T$, is attained at $K = K_m = S l_0 \sqrt{\beta [1 - (1 + \beta_*) \sin^2 \theta]} / 2\nu_T$. The large-scale instability may occur when $|\theta| < 28°$. In the case of $\theta = 0$, the large-scale instability is more effective, i.e., the mean vorticity grows with a maximum possible rate. The mechanism of this instability is discussed in Sec. III and it is associated with a combined effect of the skew-induced deflection of equilibrium mean vorticity due to the sheared motions and the Reynolds stress-induced generation of perturbations of mean vorticity.

V. CONCLUSIONS AND APPLICATIONS

We discussed an effect of a mean-velocity shear on a homogeneous nonhelical turbulence and on the effective force which is determined by the gradient of the Reynolds stresses. We demonstrated that in a homogeneous incompressible nonhelical turbulent flow with an imposed mean velocity shear, a large-scale instability can be excited which results in a mean vorticity production. This instability is caused by a *combined* effect of the large-scale shear motions (skew-induced deflection of equilibrium mean vorticity) and Reynolds stress-induced generation of perturbations of mean vorticity. We determined the spatial characteristics of the instability, such as the minimum size of the growing perturbations and the size of perturbations with the maximum growth rate.

The results obtained in this study are different from those discussed in Refs. [14–16], whereby the generation of large-scale vorticity in the helical turbulence occurs due to hydrodynamic α effect. The latter effect is associated with the term $\alpha \tilde{\mathbf{W}}$ in the equation for the mean vorticity, where α is determined by the hydrodynamic helicity of the turbulent flow. We considered a nonhelical background homogeneous turbulence which implies that Eq. (4) for the mean vorticity $\tilde{\mathbf{W}}$ does not have the term $\alpha \tilde{\mathbf{W}}$. We studied a linear stage of the large-scale instability which is saturated by nonlinear effects, but not a finite time growth of large-scale vorticity as described in Ref. [16].

The analyzed effect of the mean vorticity production may be of relevance in different turbulent industrial, environmental, and astrophysical flows (see, e.g., Refs. [3–8,38–41]). Thus, e.g., the suggested mechanism can be used in the analysis of the flows associated with Prandtl's turbulent secondary flows (see, e.g., Refs. [3–8]). These flows, e.g., arise in straight noncircular ducts, at the lateral boundaries of three-dimensional thin shear layers, etc.

However, the results obtained in this study cannot be valid in the most general cases. We made the following assumptions about the turbulence. We considered a homogeneous, isotropic, incompressible, and nonhelical background turbulence (i.e., the turbulence without the large-scale shear). The weak mean velocity shear affects the background turbulence, i.e., it causes generation of the additional strongly anisotropic velocity fluctuations (tangling turbulence) by tangling of the mean-velocity gradients with the background turbulence. We assumed that the generated tangling turbulence does not affect the background turbulence. This implies that we considered a one-way coupling due to a weak large-scale velocity shear. In Secs. III and IV D, we considered a linear velocity shear to derive the specific results for the large-scale instability. Thus, we studied simple physical mechanisms to describe an initial stage of the mean vorticity generation. The simple model considered in our paper can only mimic the flows associated with the turbulent secondary flows. Clearly, the comprehensive theoretical and numerical studies are required for a quantitative description of the secondary flows.

The obtained results may be also important in astrophysics, e.g., in extragalactic clusters and in interstellar clouds. The extragalactic clusters are nonrotating objects with a homogeneous turbulence in the center of an extragalactic cluster. Sheared motions between interacting clusters can cause an excitation of the large-scale instability which results in the mean vorticity production and formation of large-scale vortices. Dust particles can be trapped by these vortices to enhance agglomeration of material and formation of particle inhomogeneities [39–41]. The sheared motions can also occur between interacting interstellar clouds, whereby the dynamics of the mean vorticity is important.

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APPENDIX A: IDENTITIES USED FOR DERIVATION OF EQS. (10), (11), AND (21)

To derive Eqs. (10) and (11), we used the following identities:

$$\boldsymbol{\nabla}\times\mathbf{M}=(SK^2/4)(\tilde{W}_y,\tilde{W}_x,0),$$

$$\boldsymbol{\nabla}\times(\tilde{\mathbf{U}}\times\overline{\mathbf{W}}^{(s)})=S(\tilde{W}_y,-\tilde{W}_x,0),$$

$$\boldsymbol{\nabla}\times(\overline{\mathbf{U}}^{(s)}\times\tilde{\mathbf{W}})=S(0,\tilde{W}_x,0),$$

$$\boldsymbol{\nabla}\times\mathbf{J}=(SK^2/2)(\tilde{W}_y,\tilde{W}_x,0),$$

$$\boldsymbol{\nabla}\times[(\overline{\mathbf{W}}^{(s)}\cdot\boldsymbol{\nabla})\tilde{\mathbf{W}}]=SK^2(\tilde{W}_y,-\tilde{W}_x,0),$$

where

$$M_i=\boldsymbol{\nabla}_j M_{ij},$$

$$J_i=\boldsymbol{\nabla}_j(\tilde{W}_n[\varepsilon_{nim}(\partial\overline{U}^{(s)})_{mj}+\varepsilon_{njm}(\partial\overline{U}^{(s)})_{mi}]),$$

and we also took into account that

$$\boldsymbol{\nabla}_j G_{ij}=(\overline{\mathbf{W}}^{(s)}\cdot\boldsymbol{\nabla})\tilde{W}_i,$$

$$\boldsymbol{\nabla}_j\{\overline{W}_n^{(s)}[\varepsilon_{nim}(\partial\tilde{U})_{mj}+\varepsilon_{njm}(\partial\tilde{U})_{mi}]\}$$
$$=[\boldsymbol{\nabla}_i(\overline{\mathbf{W}}^{(s)}\cdot\tilde{\mathbf{W}})-(\overline{\mathbf{W}}^{(s)}\cdot\boldsymbol{\nabla})\tilde{W}_i]/2.$$

To derive Eq. (21), we used the following identities for the integration over the angles in $\mathbf{k}$ space:

$$\int k_{ijmn}\,d\hat{\Omega}=\frac{4\pi}{15}\Delta_{ijmn},$$

$$T_{ijmnpq}\equiv\int k_{ijmnpq}d\hat{\Omega}$$
$$=\frac{4\pi}{105}(\Delta_{mnpq}\delta_{ij}+\Delta_{jmnq}\delta_{ip}+\Delta_{imnq}\delta_{jp}+\Delta_{jmnp}\delta_{iq}$$
$$+\Delta_{imnp}\delta_{jq}+\Delta_{ijmn}\delta_{pq}-\Delta_{ijpq}\delta_{mn}),$$

$$T_{ijmnpq}(\nabla_m\overline{U}_n^{(s)})(\nabla_p\tilde{U}_q)=\frac{4\pi}{105}(4M_{ij}+\delta_{ij}M_{pp}),$$

$k_{ijmn}=k_ik_jk_mk_n/k^4$, $d\hat{\Omega}=\sin\theta\,d\theta\,d\varphi$, $k_{ijmnpq}=k_{ijmn}k_{pq}$, and $\Delta_{ijmn}=\delta_{ij}\delta_{mn}+\delta_{im}\delta_{nj}+\delta_{in}\delta_{mj}$.

APPENDIX B: DERIVATION OF EQ. (14)

In order to derive Eq. (14), we use a two-scale approach, i.e., a correlation function is written as follows:

$$\langle u_i(\mathbf{x})u_j(\mathbf{y})\rangle$$
$$=\int\langle u_i(\mathbf{k}_1)u_j(\mathbf{k}_2)\rangle\exp[i(\mathbf{k}_1\cdot\mathbf{x}+\mathbf{k}_2\cdot\mathbf{y})]d\mathbf{k}_1\,d\mathbf{k}_2$$
$$=\int f_{ij}(\mathbf{k},\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r})d\mathbf{k},$$

$$f_{ij}(\mathbf{k},\mathbf{R})=\int\langle u_i(\mathbf{k}+\mathbf{K}/2)u_j(-\mathbf{k}+\mathbf{K}/2)\rangle$$
$$\times\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}$$

(see, e.g., Refs. [42,43]), where $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small scales, i.e., $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$, $\mathbf{r}=\mathbf{x}-\mathbf{y}$, $\mathbf{K}=\mathbf{k}_1+\mathbf{k}_2$, $\mathbf{k}=(\mathbf{k}_1-\mathbf{k}_2)/2$. This implies that we assumed that there exists a separation of scales, i.e., the maximum scale of turbulent motions l_0 is much smaller than the characteristic scale of the inhomogeneities of the mean fields.

Now we calculate

$$\frac{\partial f_{ij}(\mathbf{k}_1,\mathbf{k}_2)}{\partial t}\equiv\left\langle P_{in}(\mathbf{k}_1)\frac{\partial u_n(\mathbf{k}_1)}{\partial t}u_j(\mathbf{k}_2)\right\rangle$$
$$+\left\langle u_i(\mathbf{k}_1)P_{jn}(\mathbf{k}_2)\frac{\partial u_n(\mathbf{k}_2)}{\partial t}\right\rangle, \tag{B1}$$

where we multiplied equation of motion (13) rewritten in $\mathbf{k}$ space by $P_{ij}(\mathbf{k})=\delta_{ij}-k_{ij}$ in order to eliminate the pressure term from the equation of motion, δ_{ij} is the Kronecker tensor, and $k_{ij}=k_ik_j/k^2$. Thus, the equation for $f_{ij}(\mathbf{k},\mathbf{R})$ is given by Eq. (14).

For the derivation of Eq. (14), we used the following identity:

$$ik_i\int f_{ij}(\mathbf{k}-\tfrac{1}{2}\mathbf{Q},\mathbf{K}-\mathbf{Q})\overline{U}_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}\,d\mathbf{Q}$$
$$=-\frac{1}{2}\overline{U}_p\boldsymbol{\nabla}_i f_{ij}+\frac{1}{2}f_{ij}\boldsymbol{\nabla}_i\overline{U}_p-\frac{i}{4}(\boldsymbol{\nabla}_s\overline{U}_p)\left(\boldsymbol{\nabla}_i\frac{\partial f_{ij}}{\partial k_s}\right)$$
$$+\frac{i}{4}\left(\frac{\partial f_{ij}}{\partial k_s}\right)(\boldsymbol{\nabla}_s\boldsymbol{\nabla}_i\overline{U}_p). \tag{B2}$$

To derive Eq. (B2), we multiply the equation $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, written in $\mathbf{k}$ space for $u_i(\mathbf{k}_1-\mathbf{Q})$, by $u_j(\mathbf{k}_2)\overline{U}_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})$, and integrate over $\mathbf{K}$ and $\mathbf{Q}$, and average over the ensemble of velocity fluctuations. Here $\mathbf{k}_1=\mathbf{k}+\mathbf{K}/2$ and $\mathbf{k}_2=-\mathbf{k}+\mathbf{K}/2$. This yields

$$\int i\left(k_i+\frac{1}{2}K_i-Q_i\right)\left\langle u_i\left(\mathbf{k}+\frac{1}{2}\mathbf{K}-\mathbf{Q}\right)u_j(-\mathbf{k}+\tfrac{1}{2}\mathbf{K})\right\rangle$$
$$\times\overline{U}_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}\,d\mathbf{Q}=0. \tag{B3}$$

Then we introduce new variables: $\tilde{\mathbf{k}}_1=\mathbf{k}+\mathbf{K}/2-\mathbf{Q}$, $\tilde{\mathbf{k}}_2=-\mathbf{k}+\mathbf{K}/2$, and $\tilde{\mathbf{k}}=(\tilde{\mathbf{k}}_1-\tilde{\mathbf{k}}_2)/2=\mathbf{k}-\mathbf{Q}/2$, $\tilde{\mathbf{K}}=\tilde{\mathbf{k}}_1+\tilde{\mathbf{k}}_2=\mathbf{K}-\mathbf{Q}$. This allows us to rewrite Eq. (B3) in the form

$$\int i\left(k_i+\frac{1}{2}K_i-Q_i\right)f_{ij}\left(\mathbf{k}-\frac{1}{2}\mathbf{Q},\mathbf{K}-\mathbf{Q}\right)\times\overline{U}_p(\mathbf{Q})\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}\,d\mathbf{Q}=0. \tag{B4}$$

Since $|\mathbf{Q}|\ll|\mathbf{k}|$, we can use the Taylor expansion

$$f_{ij}(\mathbf{k}-\mathbf{Q}/2,\mathbf{K}-\mathbf{Q})\simeq f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})-\frac{1}{2}\frac{\partial f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})}{\partial k_s}Q_s+O(\mathbf{Q}^2). \tag{B5}$$

We also use the following identities:

$$[f_{ij}(\mathbf{k},\mathbf{R})\overline{U}_p(\mathbf{R})]_{\mathbf{K}}=\int f_{ij}(\mathbf{k},\mathbf{K}-\mathbf{Q})\overline{U}_p(\mathbf{Q})d\mathbf{Q},$$

$$\nabla_p[f_{ij}(\mathbf{k},\mathbf{R})\overline{U}_p(\mathbf{R})]=\int iK_p[f_{ij}(\mathbf{k},\mathbf{R})\overline{U}_p(\mathbf{R})]_{\mathbf{K}}\exp(i\mathbf{K}\cdot\mathbf{R})d\mathbf{K}. \tag{B6}$$

Therefore, Eqs. (B4)–(B6) yield Eq. (B2).

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