Global dissipativity of continuous-time recurrent neural networks with time delay

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This paper addresses the global dissipativity of a general class of continuous-time recurrent neural networks. First, the concepts of global dissipation and global exponential dissipation are defined and elaborated. Next, the sets of global dissipativity and global exponentially dissipativity are characterized using the parameters of recurrent neural network models. In particular, it is shown that the Hopfield network and cellular neural networks with or without time delays are dissipative systems.

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I. INTRODUCTION

Stability is one of the important properties for dynamic systems. From a systems-theoretic point of view, the global stability of recurrent neural networks is a very interesting issue for research because of the special nonlinear structure of recurrent neural networks. From a practical point of view, the global stability of neural networks is also very important because it is a prerequisite in many neural network applications such as optimization, control, and signal processing.

In recent years, the stability of continuous-time recurrent neural networks has received much attention in the literature, e.g., Refs. [1–29]. Among the numerous results, the stability of recurrent neural networks is characterized using symmetry of weight matrices [1], diagonal domination of matrices [13], positive definiteness of matrices [27], *M*-matrix characteristics [21], Lyapunov diagonal stability [5,17,25], and additive diagonal stability [28]. Despite the existence of many reported results in the literature, there are still needs for more in-depth and comprehensive investigations. For example, in almost all the existing results, the activation functions of the neural networks are limited to be sigmoid functions, piecewise linear monotone nondecreasing functions with bounded ranges.

The notion of dissipativity in dynamical systems was introduced in the early 1970s. This concept generalizes the idea of a Lyapunov function and has found applications in diverse areas such as stability theory, chaos and synchronization theory, system norm estimation, and robust control [30–32].

In this paper, we analyze the global dissipation and global exponential dissipation of several classes of continuous-time recurrent neural networks with general activation functions. The main contributions of this paper include the derivations of new global attractive sets and characterization of global dissipativity and global exponential dissipativity. These properties play an important role in studying the uniqueness of equilibria, global asymptotic stability, global exponential stability, instability, the existence of periodic solutions, and chaos control and synchronization.

The remaining paper is organized as follows. Section II describes some preliminaries. The main results are stated in Secs. III and IV. Illustrative results can be found in Sec. V. Finally, concluding remarks are made in Sec. VI.

II. PRELIMINARIES

Consider a general recurrent neural network model with multiple time delays

$$c_i \frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_{ij})) + u_i , \tag{1}$$

for $i=1,3,\ldots,n$; where $c_i>0$ and $d_i>0$ are positive parameters, x_i is the state variable of the ith neuron, u_i is an input (bias), a_{ij} and b_{ij} are connection weights from neuron i to neuron j, and $g_i(\cdot)$ is an activation function.

If $\tau_{ij}=\tau$ $(i,j=1,2,\ldots,n)$, then the recurrent neural network model with time delay can be described in a vector form:

$$C\frac{dx}{dt} = -Dx(t) + Ag(x(t)) + Bg(x(t-\tau)) + u, \tag{2}$$

where, $x=(x_1,x_2,\ldots,x_n)^T$ is the neuron state vector, $u=(u_1,u_2,\ldots,u_n)^T$ is the bias vector, $C=\mathrm{diag}(c_1,c_2,\ldots,c_n)$ and $D=\mathrm{diag}(d_1,d_2,\ldots,d_n)$, $A=(a_{ij})_{n\times n}, B=(b_{ij})_{n\times n}$ are connection weight matrices, and $g(\cdot)=(g_1(\cdot),g_2(\cdot),\ldots,g_n(\cdot))^T$ is a vector-valued activation function.

As a special case, the recurrent neural network model without any time delay can be viewed as $B\equiv 0$:

$$C\frac{dx}{dt} = -Dx(t) + Ag(x(t)) + u. \tag{3}$$

We assume that the activation function $g_j(\cdot)$ is continuous and monotonically nondecreasing with $g_i(0)=0$ (i.e., its Diniderivative $D^+ g_i \geqslant 0$, for the definition of Diniderivative, we refer to Ref. [34]) where $D^+ g_i(x) := \limsup_{h\to 0^+}(g_i(x+h)-g_i(x))/h$. Next, we define three classes of activation functions:

(1) The set of bounded activation functions defined as

$$\mathcal{B}\overset{def}{=}\{g(x)||g(x)|\leqslant k,\quad 0\leqslant D^+g(x)\leqslant \ell\},\tag{4}$$

where $k=(k_1,k_2,\ldots,k_n)^T$ and $\ell=(\ell_1,\ell_2,\ldots,\ell_n)^T$ with $0\leqslant k_i,\ell_i<+\infty$. The sigmoid activation functions used in the Hopfield networks [1] and the piecewise linear activation functions in cellular neural networks [33] are typical representatives of the bounded functions.

(2) The set of Lipschitz-continuous activation functions is defined as

$$\mathcal{L}\overset{def}{=}\left\{g(x)\left|0\leqslant\frac{g_i(x_i)-g_i(y_i)}{x_i-y_i}\leqslant\ell_i<\infty,\quad\forall\ x_i,y_i\in R\right.\right.$$
$$\left.i=1,2,\ldots,n\right\}.\tag{5}$$

A good example of the Lipschitz-continuous activation function is $g_i(x_i)=\max\{0,\xi_i x_i\}$, used in Ref. [18].

(3) The general set of monotone nondecreasing activation functions is denoted as

$$\mathcal{G}\overset{def}{=}\{g(x)|g(x)\in C[R,R],\quad D^+g_i(x_i)\geqslant 0,$$
$$i=1,2,\ldots,n\}.\tag{6}$$

Evidently, $\mathcal{B}\subset\mathcal{L}\subset\mathcal{G}$.

Definition 1. The neural network model (1) is said to be a dissipative system, if there exists a compact set $\mathcal{S}\subset R^n$, such that $\forall\ x_0\in R^n$, $\exists T>0$, when $t\geqslant t_0+T$, $x(t,t_0,x_0)\subseteq\mathcal{S}$, where $x(t,t_0,x_0)$ denotes the solution of Eq. (1) from initial state x_0 and initial time t_0. In this case, $\mathcal{S}$ is called a globally attractive set. A set $\mathcal{S}$ is called positive invariant, if $\forall\ x_0\in\mathcal{S}$ implies $x(t,t_0,x_0)\subseteq\mathcal{S}$ for $t\geqslant t_0$.

Definition 2. Let $\mathcal{S}$ is a globally attractive set of neural network model (1). The neural network model (1) is said to be globally exponentially dissipative system, if there exists a compact set $\mathcal{S}^*\supset\mathcal{S}$ in R^n such that $\forall\ x_0\in R^n\setminus\mathcal{S}^*$, there exists a constant $M(x_0)>0$ and $\alpha>0$ such that

$$\inf_{x\in R^n\setminus\mathcal{S}^*}\{\|x(t,t_0,x_0)-\tilde{x}\||\tilde{x}\in\mathcal{S}^*\}$$
$$\leqslant M(x_0)\exp\{-\alpha(t-t_0)\}.\tag{7}$$

The set $\mathcal{S}^*$ is called globally exponentially attractive set, where $x\in R^n\setminus\mathcal{S}^*$ means $x\in R^n$ but $x\notin\mathcal{S}^*$.

III. MAIN RESULTS

In this section, we present five theorems and two corollaries.

Theorem 1. Let $g(x)\in\mathcal{B}$. The neural network model (1) is a dissipative system and the set $\mathcal{S}=\mathcal{S}_1\cap\mathcal{S}_2$ is a positive invariant and globally attractive set, where

$$\mathcal{S}_1\overset{def}{=}\left\{x\left|\sum_{i=1}^n d_i\left[|x_i|-\frac{1}{2d_i}\left(\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right)\right]^2\right.\right.$$
$$\left.\leqslant\sum_{i=1}^n\frac{1}{4d_i}\left[\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right]^2\right\},\tag{8}$$

$$\mathcal{S}_2\overset{def}{=}\left\{x||x_i|\leqslant\frac{1}{d_i}\left[\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right]\right.$$
$$\left.\overset{def}{=}M_i,\quad i=1,2,\ldots,n\right\}.\tag{9}$$

Proof. First, we employ a radically unbounded and positive definite Lyapunov function as $V(x)=\sum_{i=1}^n c_i x_i^2/2$. Computing dV/dt along the positive half trajectory of (1), we have

$$\left.\frac{dV}{dt}\right|_{(1)}=\sum_{i=1}^n c_i x_i\frac{dx_i}{dt}$$
$$\leqslant\sum_{i=1}^n\left[\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j|x_i|-d_i x_i^2+|u_i||x_i|\right]$$
$$=\sum_{i=1}^n\left\{-\left[d_i x_i^2-\left(\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right)|x_i|\right]\right.$$
$$-\frac{1}{d_i}\left[\frac{1}{2}\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right]^2$$
$$\left.+\frac{1}{d_i}\left[\frac{1}{2}\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right]^2\right\}$$
$$=\sum_{i=1}^n-d_i\left[|x_i|-\frac{1}{2d_i}\left(\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right)\right]^2$$
$$+\sum_{i=1}^n\frac{1}{4d_i}\left[\sum_{j=1}^n(|a_{ij}|+|b_{ij}|)k_j+|u_i|\right]^2<0,\tag{10}$$

when $x\in R^n\setminus\mathcal{S}_1$; i.e., $x\notin\mathcal{S}_1$. Equation (10) implies that $\forall\ x_0\in\mathcal{S}_1$ holds $x(t,t_0,x_0)\subseteq\mathcal{S}_1,t\geqslant t_0$. For $x_0\notin\mathcal{S}_1$, there exists $T>0$ such that

$$x(t,t_0,x_0)\subseteq\mathcal{S}_1,\quad\forall\ t\geqslant T+t_0,$$

i.e., the neural network model (1) is a dissipative system and $\mathcal{S}_1$ is a positive invariant and attractive set.

Second, we define a radically unbounded and positive definite Lyapunov function

$$V_i=c_i|x_i|,\quad i=1,2,\ldots,n.\tag{11}$$

Calculating the right-upper Diniderivative D^+V_i, one obtains

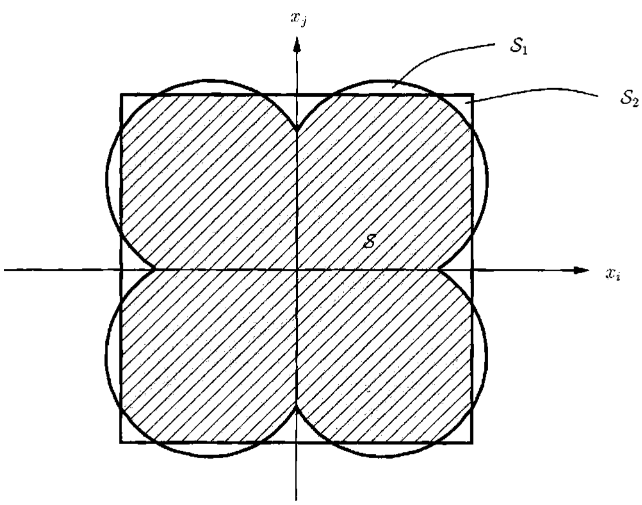

FIG. 1. Profile of the positive variant and globally attractive set $\mathcal{S}$.

$$D^+V_i|_{(1)}\leqslant -d_i|x_i|+\sum_{j=1}^{n}(|a_{ij}|+|b_{ij}|)k_j+|u_i|<0,$$

$$i=1,2,\ldots,n, \tag{12}$$

when $x\in R^n\setminus\mathcal{S}_2$. So, $\mathcal{S}_2$ is also a positive invariant and globally attractive set.

Combining the above proof, we know that $\mathcal{S}=\mathcal{S}_1\cap\mathcal{S}_2$ is a positive invariant and globally attractive set. Theorem 1 is proved.

Figure 1 illustrates a profile of $\mathcal{S}$.

Corollary 1. Let $g(x)\in\mathcal{B}$. The neural network (3) is a dissipative system and the $\hat{\mathcal{S}}_1\cap\hat{\mathcal{S}}_2$ is a positive invariant and globally attractive set, where

$$\hat{\mathcal{S}}_1\overset{def}{=}\left\{x\Big|\sum_{i=1}^{n}d_i\left[|x_i|-\frac{1}{2d_i}\left(\sum_{j=1}^{n}|a_{ij}|k_j+|u_i|\right)\right]^2\right.$$
$$\left.\leqslant\sum_{i=1}^{n}\frac{1}{4d_i}\left(\sum_{j=1}^{n}|a_{ij}|k_j+|u_i|\right)^2\right\}, \tag{13}$$

$$\hat{\mathcal{S}}_2\overset{def}{=}\left\{x||x_i|\leqslant\frac{1}{d_i}\left(\sum_{j=1}^{n}|a_{ij}|k_j+|u_i|\right)\overset{def}{=}\hat{M}_i,\ \ i=1,\ldots,n\right\}. \tag{14}$$

Corollary 1 is an improvement and extension of Theorem 3 in Ref. [2] where the global attractive set is a sphere defined by $\mathcal{S}_s=\{x\in R^n|\,||x||_2\leqslant||RL||_2\}$. However, the global attractive set given in Corollary 1 is $\hat{\mathcal{S}}_1\cap\hat{\mathcal{S}}_2$. It can be seen that $\hat{\mathcal{S}}_2\subset\mathcal{S}_s$.

Now, we construct a new set as

$$\mathcal{S}^*\overset{def}{=}\left\{x||x_i|\leqslant\frac{d_iM_i}{d_i-\varepsilon c_i}\overset{def}{=}M_i^*\right\}, \tag{15}$$

where $0<\varepsilon<\max_i d_i/c_i$ and M_i is defined in $\mathcal{S}_2$ of Eq. (9) (see Fig. 2).

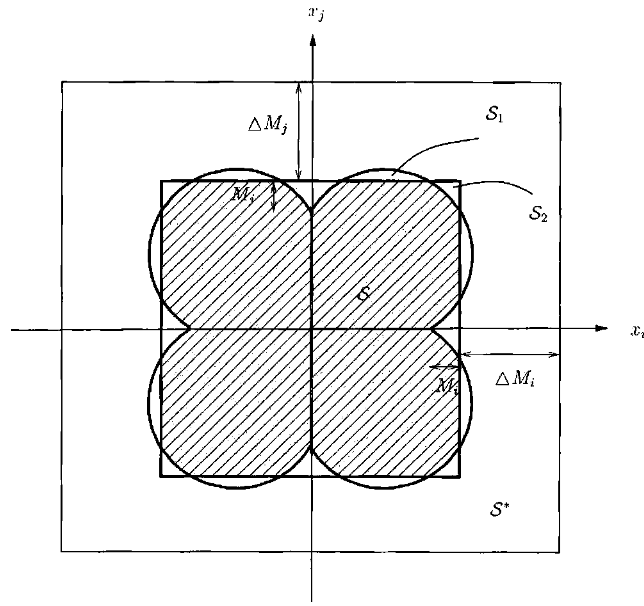

FIG. 2. Profile of the positive variant and globally exponential attractive set $\mathcal{S}^*$.

Theorem 2. Let $g(x)\in\mathcal{B}$. the neural network model (1) is globally exponentially dissipative and set $\mathcal{S}^*$ is a positive invariant and globally exponentially attractive set.

Proof. Obviously, $\mathcal{S}^*$ is a positive invariant set because $\mathcal{S}^*\supset\mathcal{S}_2$. Now choosing $0<\varepsilon<\max_i(d_i/c_i)$, by Eq. (11) we have

$$D^+\exp\{\varepsilon t\}V_i|_{(1)}\leqslant\exp\{\varepsilon t\}\left(\varepsilon c_i|x_i|-d_i|x_i|+\sum_{j=1}^{n}|a_{ij}|k_j\right.$$
$$\left.+\sum_{j=1}^{n}|b_{ij}|k_j+|u_i|\right)$$
$$=\exp\{\varepsilon t\}[-(d_i-\varepsilon c_i)|x_i|+d_iM_i]<0,$$
$$i=1,2,\ldots,n, \tag{16}$$

when $|x_i|>M_i^*$; i.e., $x\in R^n\ \mathcal{S}^*$. Integrating two sides of Eq. (16) from 0 to an arbitrary $t>0$, we have

$$\exp\{\varepsilon t\}V_i(x(t),t)\leqslant V_i(x_i(0),0).$$

Therefore, we have

$$c_i|x_i(t)|\leqslant\exp\{-\varepsilon t\}c_i|x_i(0)|$$

or

$$|x_i(t)|\leqslant\exp\{-\varepsilon t\}|x_i(0)|, \tag{17}$$

when $x_0\in R^n\setminus\mathcal{S}^*$. Equation (17) means that set $\mathcal{S}^*$ is globally exponentially attractive, i.e., Eq. (1) is globally exponentially dissipative. The proof is complete.

Corollary 2. Let $g(x)\in\mathcal{B}$. The neural network (3) without any time delay is globally exponentially dissipative, and $\hat{\mathcal{S}}^*$ is a positive invariant and globally exponentially attractive set where

$$\hat{\mathcal{S}}^* \overset{def}{=} \left\{ x \left| |x_i| \leqslant \frac{d_i M_i^*}{d_i - \lambda c_i}, \quad 0 < \frac{\lambda c_i}{d_i} < 1; \quad i = 1, \ldots, n \right. \right\}.$$

In the following, we suppose $\tau_{ij} = \tau_j$.

Theorem 3. Let $g(x) \in \mathcal{L}, g(0) = 0$ and $|g(x)| \to +\infty$ as $|x_i| \to +\infty$. If the following matrix Q is negative semidefinite, then the neural network model (1) is a dissipative system and set $\mathcal{S}_3 = \{x | \ |g_i(x_i)| \leqslant \ell_i |u_i| / d_i i = 1,2, \ldots, n\}$ is a positive invariant and globally attractive set, where

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix};$$

$Q_{11} = ((A + A^T)/2 + I_{n\times n})$, $Q_{12} = B/2$, and $Q_{22} = -I_{n\times n}$, where $I_{n\times n}$ is an $n \times n$ identity matrix.

Proof. Let us employ the radically unbounded and positive definite Lyapunov function as

$$V(x(t),t) = \sum_{i=1}^{n} c_i \int_0^{x_i} g_i(x_i) dx_i + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i(x_i(\xi))^2 d\xi. \tag{18}$$

Computing the derivative of $V(x(t),t)$ along the positive half trajectory of Eq. (1), we have

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(1)} &= \sum_{i=1}^{n} c_i g_i(x_i(t)) \frac{dx_i}{dt} + \sum_{i=1}^{n} [g_i(x_i(t))^2 - g_i(x_i(t-\tau_i))^2] \\
&\leqslant \sum_{i=1}^{n} \left[-\frac{d_i}{\ell_i} g_i(x_i(t))^2 + \sum_{j=1}^{n} a_{ij} g_i(x_i(t)) g_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_i(x_i(t)) g_j(x_j(t-\tau_j)) + g_i(x_i(t)) u_i \right] \\
&\quad + \sum_{i=1}^{n} g_i(x_i(t))^2 - \sum_{i=1}^{n} g_i(x_i(t-\tau_i))^2 \\
&= \begin{pmatrix} g(x(t)) \\ g(x(t-\tau)) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} g(x(t)) \\ g(x(t-\tau)) \end{pmatrix} - \sum_{i=1}^{n} \frac{d_i}{\ell_i} |g_i(x_i(t))| \left[|g_i(x_i(t))| - |u_i| \frac{\ell_i}{d_i} \right] \\
&\leqslant -\sum_{i=1}^{n} \frac{d_i}{\ell_i} |g_i(x_i(t))| \left[|g_i(x_i(t))| - |u_i| \frac{\ell_i}{d_i} \right] \\
&< 0 \quad \text{when} \quad g_i(x_i) \in R^n \setminus \mathcal{S}_3.
\end{aligned}$$

So set $\mathcal{S}_3$ is a positive invariant and globally attractive set.

Theorem 4. Let $g(x) \in \mathcal{G}, g(0) = 0$, and $D^+ g_i(x_i) \leqslant +\infty$, $i = 1,2, \ldots, n$. If there exist $\varepsilon > 0$ and $\lambda > 0$ such that the given matrix

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}$$

is negative semidefinite, where $Q_{11} = ((A + A^T)/2 + (\lambda + \varepsilon) I_{n\times n})$, $Q_{12} = B/2$, and $Q_{22} = -\lambda I_{n\times n}$, then the neural network model (1) is a dissipative system and the set

$$\mathcal{S}_4 \overset{def}{=} \left\{ x \left| \sum_{i=1}^{n} \left(g_i(x_i) - \frac{u_i}{2\varepsilon} \right)^2 \leqslant \sum_{i=1}^{n} \frac{u_i^2}{4\varepsilon^2}, \quad i = 1,2, \ldots, n \right. \right\}$$

is a positive invariant and globally attractive set.

Proof. Let us employ the Lyapunov function

$$V(x(t),t) = \sum_{i=1}^{n} c_i \int_0^{x_i} g_i(x_i) dx_i + \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \lambda g_i(x_i(\xi))^2 d\xi. \tag{19}$$

Then we have

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(1)} &\leqslant \begin{pmatrix} g(x(t)) \\ g(x(t-\tau)) \end{pmatrix}^T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} \begin{pmatrix} g(x(t)) \\ g(x(t-\tau)) \end{pmatrix} \\
&\quad - \varepsilon \sum_{i=1}^{n} g_i(x_i(t))^2 + \sum_{i=1}^{n} u_i g_i(x_i(t)) \\
&\leqslant -\varepsilon \sum_{i=1}^{n} g_i(x_i(t))^2 + \sum_{i=1}^{n} u_i g_i(x_i(t)) \\
&\leqslant -\varepsilon \sum_{i=1}^{n} \left[\left(g_i(x_i) - \frac{u_i}{2\varepsilon} \right)^2 - \frac{u_i^2}{4\varepsilon^2} \right] \\
&< 0 \quad \text{when} \quad g_i(x_i) \in R^n \setminus \mathcal{S}_4.
\end{aligned}$$

Therefore, the proof is complete.

Theorem 5. Let $g(x) \in \mathcal{G}$. If there exists a positive diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ with $p_i > 0$ such that

$$Q \overset{def}{=} P(A - L^{-1}D) + (A - L^{-1}D)^T P \tag{20}$$

is negative definite, then the neural network (3) without time delay is a dissipative system and the set

$$\mathcal{S}_5 \overset{def}{=} \left\{ x \Big| \sum_{i=1}^{n} \left(g_i(x_i) + \frac{p_i u_i}{2\lambda_Q} \right)^2 \leqslant \sum_{i=1}^{n} \frac{(p_i u_i)^2}{4\lambda_Q^2}, \quad i=1,2,\ldots,n \right\}$$

is a positive invariant and globally attractive set, where $L = \mathrm{diag}(l_1, l_2, \ldots, l_n)$ abd $\lambda_Q = \max_{1 \leqslant i \leqslant n} \lambda_i(Q)$ is the maximum eigenvalue of Q. If $PA + A^T P$ is negative semidefinite, then Eq. (3) is still a dissipative system with a positive invariant and globally attractive set

$$\mathcal{S}_6 \overset{def}{=} \left\{ x \,\Big|\, |x_i| \leqslant \frac{|u_i|}{d_i}, \quad i=1,2,\ldots,n \right\}.$$

Proof. Let a positive definite and radially unbounded Lyapunov function be

$$V(x) = \sum_{i=1}^{n} c_i p_i \int_0^{x_i} g_i(x_i) dx_i.$$

Calculating the time derivative of $V(x)$, we have

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(1)} &= \sum_{i=1}^{n} \left[-p_i d_i x_i g_i(x_i) + \sum_{j=1}^{n} p_i a_{ij} g_i(x_i) g_j(x_j) \right] + \sum_{i=1}^{n} p_i g_i(x_i) u_i \\
&\leqslant g(x)^T Q g(x) + \sum_{i=1}^{n} p_i g_i(x_i) u_i \\
&\leqslant \lambda_Q \sum_{i=1}^{n} g_i(x_i)^2 + \sum_{i=1}^{n} p_i g_i(x_i) u_i \\
&= \lambda_Q \sum_{i=1}^{n} \left[\left(g_i(x_i) + \frac{p_i u_i}{2\lambda_Q} \right)^2 - \frac{(p_i u_i)^2}{4\lambda_Q^2} \right] \\
&< 0,
\end{aligned}$$

when $g_i(x_i) \in R^n \setminus \mathcal{S}_5$. So, $\mathcal{S}_5$ is a positive invariant and globally attractive set.

When $PA + A^T P$ is negative semidefinite,

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(1)} &\leqslant \sum_{i=1}^{n} [-p_i d_i x_i g_i(x_i) + p_i g_i(x_i) u_i] \\
&\leqslant \sum_{i=1}^{n} p_i |g_i(x_i)| [-d_i |x_i| + u_i] \\
&< 0, \quad \text{for} \quad x \in R^n \setminus \mathcal{S}_6.
\end{aligned}$$

So $\mathcal{S}_6$ is a positive invariant and globally attractive set. Theorem 5 is proven.

IV. TWO PROPOSITIONS

Proposition 1. The continuous-time Hopfield network with or without any time delay is a dissipative system.

In the Hopfield network, its sigmoid activation function satisfies $|g_i(x_i)| \leqslant 1$ and $0 \leqslant D^+ g_i(x_i) \leqslant 1$. According to Theorem 1, a globally attractive set is $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ as defined in Eqs. (13) and (14).

Proposition 2. The cellular neural network with or without any time delay is a dissipative system.

In a cellular network, the activation function $g_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|)$ satisfies $|g_i(x_i)| \leqslant 1$ and $0 \leqslant D^+ g_i(x_i) \leqslant 1$. According to Theorem 1, a global attractive set is $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$ as defined in Eqs. (13) and (14).

In Theorem 1 of Ref. [33], an estimation is given as

$$\forall |x_i(0)| \leqslant 1, \quad |x_i(t)| \leqslant 1 + \max_{1 \leqslant i \leqslant n} \frac{1}{d_i} \left(\sum_{j=1}^{n} |a_{ij}| + |u_i| \right).$$

There is no conclusion for $|x_i(0)| > 1$.

According to Corollary 1, we have

$$\hat{\mathcal{S}}_2 \overset{def}{=} \left\{ x \,\Big|\, |x_i| \leqslant \frac{1}{d_i} \left(\sum_{j=1}^{n} |a_{ij}| k_j + |u_i| \right), \ i=1,\ldots,n \right\}.$$

Because $\hat{\mathcal{S}}_2$ is positive invariant, (1) when $(\Sigma_{j=1}^n |a_{ij}| k_j + |u_i|)/d_i \geqslant 1$, $\forall |x_i(0)| \leqslant 1$, $x_i(t) \in \hat{\mathcal{S}}_2$ (i.e., $|x_i(t)| \leqslant \max_{1 \leqslant i \leqslant n} (\Sigma_{j=1}^n |a_{ij}| + |u_i|)/d_i$); (2) when $(\Sigma_{j=1}^n |a_{ij}| k_j + |u_i|)/d_i < 1$, $\forall |x_i(0)| \leqslant 1, |x_i(t)| \leqslant (\Sigma_{j=1}^n |a_{ij}| + |u_i|)/d_i \leqslant 1$; (3) $\forall\, x(0) \in R^n$, $\exists T > 0$ such that when $t \geqslant T$, $|x_i(t)| \leqslant (\Sigma_{j=1}^n |a_{ij}| + |u_i|)/d_i$.

Therefore, the above result improves and extends that in Ref. [33].

V. THREE EXAMPLES

Example 1. Consider a two-neuron neural network:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} g_1(x_1) \\ g_2(x_2) \end{pmatrix}.$$

Here, $c_i = 1$, $d_i = 2$, $a_{ii} = 3$, $a_{ij} = 1$, $b_{ij} = 0$, and $u_i = 0 (i,j = 1,2)$. Suppose that $g(x) \in \mathcal{B}$, $g(0) = 0$, $|g_i(x_i)| \leqslant 10$, and $\sup_{x_i \in R} D^+ g_i(x_i) = D^+ g_i(0) = 4$. Obviously, $(0,0)^T$ is the equilibrium point of the neural network. Linearizing g_i at the origin, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Since the coefficient matrix of the linearized system is positive definite with position eigenvalues, it is not stable at the equilibrium. From the Lyapunov theorem and approximation theory, we can see that the original neural network is also instable at the equilibrium.

According to Corollary 2, however, the above neural network is exponentially dissipative. Let $\epsilon=1$, then $M_1=M_2=(3+1)10/2=20$. Therefore, a globally exponential attractive set is

$$\mathcal{S}^* \overset{def}{=} \left\{ x \middle| |x_i| \leq \frac{2\times 20}{2-1} = 40, \ \ i=1,2 \right\}.$$

Example 2. Consider a two-neuron neural network with time delay:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 & -18 \\ 18 & -2 \end{pmatrix} \begin{pmatrix} g_1(x_1) \\ g_2(x_2) \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} g_1(x_1(t-\tau)) \\ g_2(x_2(t-\tau)) \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

Suppose that $g(x) \in \mathcal{L}$ and $\ell_i \overset{def}{=} \sup_{x_i \in R} D^+ g_i(x_i) = 100$ $(i=1,2)$.

From Theorem 3,

$$Q_{11}=Q_{22}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_{12}=\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Hence, Q is negative semidefinite. According to Theorem 3, the above delayed neural network is dissipative with a globally attractive set

$$\mathcal{S} \overset{def}{=} \{x | g_i(x_i) \leq \ell_i |u_i| / d_i = 500/1 = 500, \quad i=1,2\}.$$

Example 3. Consider another two-neuron neural network with time delay:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.02 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -4.1 & a \\ -a & -4.1 \end{pmatrix} \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1(t-\tau)^3 \\ x_2(t-\tau)^3 \end{pmatrix} + \begin{pmatrix} 20 \\ 20 \end{pmatrix},$$

where a is any constant. Obviously, $g(x) \in \mathcal{G}$.

Let $\lambda=2$ and $\epsilon=0.1$. From Theorem 4,

$$Q_{11}=Q_{22}=\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad Q_{12}=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence, Q is negative semidefinite. According to Theorem 4, the above delayed neural network is dissipative with a globally attractive set

$$\mathcal{S} \overset{def}{=} \left\{ x \middle| \sum_{i=1}^{2} (g_i(x_i)-100)^2 \leq 20\,000, \quad i=1,2 \right\}.$$

It is worth noting that the equilibrium point of the above neural network may not be existent or unique. So, the discussion of the global stability may not be meaningful. However, the study of the global dissipativity is feasible.

VI. CONCLUDING REMARKS

In this paper, we discuss the global disspativity of a class of continuous-time recurrent neural networks. Several theorems and corollaries are presented to characterize global dissipation and global exponential dissipation together with their sets of attraction. The theorems and corollaries herein imply that the equilibrium of a neural network lies in the positive invariant and globally attractive set only, the globally asymptotic stability is equivalent to the asymptotic stability in the attractive set, any properties of activation function over the set can be utilized, and the condition using the LaSalle invariant principle is given.

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