Exact steady-state solution of the Boltzmann equation: A driven one-dimensional inelastic Maxwell gas

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The exact nonequilibrium steady-state solution of the nonlinear Boltzmann equation for a driven inelastic Maxwell model was obtained by Ben-Naim and Krapivsky [Phys. Rev. E **61**, R5 (2000)] in the form of an infinite product for the Fourier transform of distribution function $f(c)$. In this paper we have inverted the Fourier transform to express $f(c)$ in the form of an infinite series of exponentially decaying terms. The dominant high-energy tail is exponential, $f(c)\simeq A_0 \exp(-a|c|)$, where $a\equiv 2/\sqrt{1-\alpha^2}$ and amplitude A_0 is given in terms of a converging sum. This is explicitly shown in the totally inelastic limit ($\alpha\to 0$) and in the quasielastic limit ($\alpha\to 1$). In the latter case, the distribution is dominated by a Maxwellian for a very wide range of velocities, but a crossover from a Maxwellian to an exponential high-energy tail exists for velocities $|c-c_0|\sim 1/\sqrt{q}$ around a crossover velocity $c_0\simeq \ln q^{-1}/\sqrt{q}$, where $q\equiv(1-\alpha)/2\ll 1$. In this crossover region the distribution function is extremely small, $\ln f(c_0)\simeq q^{-1}\ln q$.

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I. INTRODUCTION

In kinetic theory there is a long standing interest in overpopulated high-energy tails of velocity distribution functions [1] because of chemical reactions and other activated processes that occur only at energies far above thermal. This interest has been considerably increased in the past ten years because of research in granular fluids with dissipative or inelastic interactions. The velocity distributions in fluidized systems have been studied theoretically [2–8] and measured in Monte Carlo [8–10] and molecular dynamics simulations [11], and in numerous laboratory experiments [12].

Very recently, a revival in this field occurred when Baldassarri *et al.* [13,14] discovered an exact scaling solution, with an algebraic high-energy tail, of the nonlinear Boltzmann equation for an inelastic one-dimensional freely cooling gas (without energy input) with a collision frequency independent of the energy of the colliding particles. This model, called inelastic Maxwell model (IMM), was introduced by Ben-Naim and Krapivsky [15]. It is in fact an inelastic modification of Ulam's stochastic model to illustrate the velocity relaxation of elastic one-dimensional point particles towards a Maxwellian [16]. A three-dimensional version of it has been constructed by Bobylev *et al.* [17,18]. For a recent review on inelastic Maxwell models, see Refs. [19,20].

Baldassarri *et al.* have demonstrated the importance of this type of solutions in Ref. [13] with the help of Monte Carlo simulations of the nonlinear Boltzmann equation for one-dimensional and two-dimensional IMM's. It appeared that solution $F(v,t)$ for large classes of initial distributions $F(v,0)$ (e.g., uniform or Gaussian) and for all values of the inelasticity could be collapsed for large times on a scaling form $v_0^{-d}(t)f(v/v_0(t))$, where $v_0(t)=\langle v^2\rangle^{1/2}$ is the rms velocity. In one dimension, the scaling form was given by $f(c)=(2/\pi)(1+c^2)^{-2}$, which has a heavily overpopulated algebraic tail $\sim c^{-4}$ when compared to a Maxwellian. In two dimensions the solutions also approached a scaling form with an algebraic tail, $f(c)\sim c^{-d-a}$ with an exponent $a(q)$ that depends on degree of inelasticity $q=\frac{1}{2}(1-\alpha)$, where α is the coefficient of restitution. Soon after, Ben-Naim and Krapivsky [21] and Ernst and Brito [22] obtained asymptotic solutions with algebraic tails for the velocity distribution in d-dimensional freely cooling IMM's from self-consistently determined solutions of the Boltzmann equation. Using methods previously developed for the inelastic hard sphere case, the asymptotic solutions were also extended to nonequilibrium steady states (NESS) in d-dimensional systems driven by Gaussian white noise and other thermostats [5,23]. There the tails exhibited overpopulations of exponential type, $\sim\exp(-a|c|)$, for all d-dimensional IMM's [23]. For inelastic hard spheres, which is the prototypical model for granular gases, the velocity distribution function shows an overpopulated exponential tail in free cooling [5,8,9] and a stretched exponential tail $\sim\exp(-a|c|^{3/2})$ when driven by white noise [5,8,10].

For the case of d-dimensional free IMM's, the approach of $F(v,t)$ to a scaling form with an algebraic tail has also been rigorously proven, for initial distributions in the $\mathcal{L}_1$ function space, satisfying the physical requirements of finite mass and energy, i.e., $\int d\mathbf{v}\{1,v^2\}F(v,0)<\infty$ [24].

What about exact and/or more explicit results for the distribution function in the one-dimensional IMM, driven by Gaussian white noise? The exact solution of the nonlinear Boltzmann equation for this case is given in the form of an infinite product for the Fourier transform of the distribution

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function [15]. Nienhuis and van der Hart [25] made an extensive numerical analysis of this solution and demonstrated exponential decay, in agreement with the predictions of Ref. [23]. More numerical evidence for exponential high-energy tails in the one-dimensional driven IMM was given recently by Marconi and Puglisi [26], and by Antal *et al.* [27]. In a recent paper [20], Ben-Naim and Krapivsky have also used the Fourier transform method to show that the high-energy tail is exponential for any inelasticity, but with an amplitude that diverges in the quasielastic limit. On the other hand, the problem of determining for what range of velocities the exponential tail actually applies remains open. This is one of the points addressed in this paper.

The plan of the paper is as follows. In the remainder of this section we present the nonlinear Boltzmann equation for velocity distribution function $F(v)$ or $f(c)$, driven by Gaussian white noise, and we discuss qualitatively the physical properties of the model in different limiting cases. In Sec. II the exact solution $\phi(k)=\int dc\, e^{-ikc}f(c)$ of the Fourier transformed Boltzmann equation in the NESS is presented in the form of an infinite product and its large- and small-k properties are analyzed. In Sec. III we determine inverse Fourier transform $f(c)$ in the form of an infinite series of exponentially decaying terms. In the limit of totally inelastic collisions $(\alpha\to 0)$, substantial simplifications occur. The rather singular quasielastic limit $(\alpha\to 1)$ is studied in Sec. IV, where the crossover from Maxwellian to exponential decay is also analyzed. We end with some comments in Sec. V and some technical details are moved to Appendixes A and B.

Before concluding this introduction we present the Boltzmann equation for the one-dimensional IMM [15] driven by Gaussian white noise and discuss some of its important properties. The time evolution of a spatially homogeneous isotropic velocity distribution function $F(v,t)=F(|v|,t)$ is described by the nonlinear Boltzmann equation

$$\frac{\partial F(v)}{\partial t}-D\frac{\partial^2 F(v)}{\partial v^2}=\int dv_1\left[\frac{1}{\alpha}F(v'')F(v_1'')-F(v)F(v_1)\right]$$
$$=-F(v)+\frac{1}{p}\int du F(u)F\left(\frac{v-qu}{p}\right)$$
$$\equiv I(v|F). \tag{1.1}$$

All velocity integrations extend over interval $(-\infty,+\infty)$. The diffusion term represents the (heating) effect of the Gaussian white noise with noise strength D. The nonlinear collision term represents the inelastic collisions, where $v''=v-\frac{1}{2}(1+\alpha^{-1})(v-v_1)$ and $v_1''=v_1+\frac{1}{2}(1+\alpha^{-1})(v-v_1)$ denote restituting velocities. Here, $\alpha=2p-1=1-2q$ with $0<\alpha<1$ is the coefficient of restitution. The mass is normalized as $\int dv F(v)=1$ and the mean square velocity or temperature as $\langle v^2\rangle(t)=\int dv\, v^2F(v)\equiv v_0^2(t)$. Rate equation

$$\partial_t\langle v^2\rangle=2D-2pq\langle v^2\rangle, \tag{1.2}$$

obtained from Eq. (1.1), describes the approach to the NESS with width $\langle v^2\rangle=D/pq$, where heating rate D caused by the random forces is balanced by loss rate $pq\langle v^2\rangle=\frac{1}{4}(1-\alpha^2)\langle v^2\rangle$ caused by the inelastic collisions.

To understand the physical processes involved, we first discuss in a qualitative way the relevant limiting cases. Without the heating term $(D=0)$, Eq. (1.1) reduces to the freely cooling IMM, whose exact solution has been discussed in Refs. [13,14]. If one takes, in addition, the elastic limit $(\alpha\to 1$ or $q\to 0)$, the collision laws reduce in the *one-dimensional* case to $v''=v_1$, $v_1''=v$, i.e., an exchange of particle labels, the collision term vanishes identically, every $F(v,t)=F(v)$ is a solution, there is no randomization or relaxation of the velocity distribution through collisions, and the model becomes trivial at the Boltzmann level of description whereas the distribution function in the presence of *infinitesimal* dissipation $(\alpha\to 1)$ approaches a Maxwellian.

If we turn on the noise $(D\neq 0)$ at vanishing dissipation $(q=0)$, the exact solution of Eq. (1.1) in Fourier representation is $\hat{F}(k,t)=\exp(-Dk^2t)\hat{F}(k,0)$ and granular temperature $v_0^2(t)=v_0^2(0)+2Dt$ increases linearly with time. With stochastic heating *and* dissipation (even in infinitesimal amounts) the system reaches a NESS and it is the goal of this paper to determine the NESS distribution function.

To expose the universality of this NESS it is convenient to measure velocities $c=v/v_0(\infty)$ in units of its typical size $v_0(\infty)$, i.e., the rms velocity or width of the velocity distribution $v_0(\infty)$,

$$F(v,\infty)=v_0^{-1}(\infty)f[v/v_0(\infty)], \tag{1.3}$$

which obeys normalizations $\int dc\{1,c^2\}f(c)=\{1,1\}$. Different normalizations have been used as well [28].

The rescaled velocity distribution in the NESS is then the solution of scaling equation

$$I(c|f)=-\frac{D}{v_0^2(\infty)}f''(c)=-pqf''(c), \tag{1.4}$$

where primes denote c derivatives. The first equality may suggest that $f(c)$ may depend on noise strength D and possibly on the initial distribution via $v_0(\infty)$. By eliminating $v_0(\infty)$ with the help of Eq. (1.2) in the NESS we have shown that the scaling form of distribution function $f(c)$ is a *universal* function that does not depend on strength D of this thermostat, nor on any property of the initial distribution. It only depends on the type of thermostat used.

II. FOURIER TRANSFORM OF IMM BOLTZMANN EQUATION

The nonlinear Boltzmann equation for characteristic function $\phi(k)=\int dc\,e^{-ikc}f(c)$ is obtained by Fourier transformation of Eq. (1.4) with result

$$(1+pqk^2)\phi(k)=\phi(pk)\phi(qk). \tag{2.1}$$

The simple structure of the equation for Fourier transform $\phi(k)$ follows because the nonlinear collision operator for (in)elastic Maxwell models is a convolution in the velocity variables [1]. Equation (2.1) is a nonlinear finite difference equation that can be solved by iteration. A simple way to construct the exact solution is to introduce $\psi(k)\equiv\ln\phi(k)$, which satisfies

$$\psi(k)=\psi(pk)+\psi(qk)-\ln(1+pqk^2). \tag{2.2}$$

The normalization of mass and energy implies that $\phi(k)\approx1-\frac{1}{2}k^2$ and $\psi(k)\approx-\frac{1}{2}k^2$ at small k. The solution to Eq. (2.2) can be found iteratively starting from $\psi_0(k)=-\ln(1+pqk^2)$ and inserting $\psi_n(k)$ on the right-hand side of Eq. (2.2) to get $\psi_{n+1}(k)$ on the left-hand side. By taking limit $\psi(k)=\lim_{n\to\infty}\psi_n(k)$, one finally obtains

$$\psi(k)=-\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\nu_{m\ell}\ln[1+p^{2\ell}q^{2(m-\ell)}pqk^2],$$

$$\phi(k)=\prod_{m=0}^{\infty}\prod_{\ell=0}^{m}[1+p^{2\ell}q^{2(m-\ell)}pqk^2]^{-\nu_{m\ell}}, \tag{2.3}$$

where $\nu_{m\ell}=\binom{m}{\ell}$. These solutions satisfy the required boundary conditions at $k=0$. We further note that $\overline{\psi}(k)=\psi(k)-\lambda|k|$ with λ being an arbitrary complex number is also a solution of Eq. (2.2) but in general does not satisfy the boundary conditions at small k. This property is a reflection of the Galilean invariance of the original Boltzmann equation.

Equations (2.3) provide an exact representation in Fourier space of the solution of the Boltzmann equation (1.1). Series (2.3) converges rapidly, even for large k. By expanding the logarithm in powers of k^2 and summing a geometric series, we obtain

$$\psi(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{(k^2pq)^n}{1-p^{2n}-q^{2n}}. \tag{2.4}$$

It converges for $k^2\leqslant1/pq$ and $\psi(k)$ has a branch point singularity at $k^2=-1/pq$, as is apparent from Eq. (2.2). Equation (2.4) allows one to get cumulants C_{2n} defined by

$$\psi(k)=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}C_{2n}k^{2n}, \tag{2.5}$$

with result

$$C_{2n}=\frac{(2n)!}{n}\frac{(pq)^n}{1-p^{2n}-q^{2n}}. \tag{2.6}$$

In particular, $C_2=\langle c^2\rangle=1$. Since $1-p^{2n}-q^{2n}>0$, it follows that *all* cumulants are positive, indicating already an overpopulation of the high-energy tails. So far, a summary of the results obtained in Ref. [15]. We note that the Stirling approximation shows that the cumulants at fixed α or q and $n>e/(2\sqrt{pq})$ are rapidly diverging with increasing n, as $C_{2n}\sim2\sqrt{\pi/n}(2n\sqrt{pq}/e)^{2n}$.

The exact solution $\phi(k)$ in Eq. (2.3) has an infinite sequence of poles of multiplicity $\nu_{m\ell}$ in the complex k plane, all of which contribute to the amplitude of the asymptotic high-energy tail of $f(c)$. This makes a numerical inversion of $\phi(k)$ to obtain $f(c)$ a bit tricky. To determine $f(c)$, several authors [25,27] have performed numerical inversions of $\phi(k)$, starting from the infinite product (2.3) or from the more convenient series expansion (2.4). However, the latter one is only convergent for $pqk^2<1$. To facilitate such numerical procedures, we have derived an expansion in powers of k^{-2}, convergent in the complementary region $pqk^2>1$ of the complex k plane. This rather technical part is deferred to Appendix A. The results can be found in Eqs. (A2), (A4), and (A7).

III. HIGH-ENERGY TAIL

On account of Eq. (2.3), characteristic function $\phi(k)$ can be written as

$$\phi(k)=\prod_{m=0}^{\infty}\prod_{\ell=0}^{m}(1+k^2/k_{m\ell}^2)^{-\nu_{m\ell}}, \tag{3.1}$$

where $k_{m\ell}\equiv ap^{-\ell}q^{-(m-\ell)}$ with $a\equiv1/\sqrt{pq}$. Thus, $\phi(k)$ has poles at $k=\pm ik_{m\ell}$ with multiplicity $\nu_{m\ell}$. Velocity distribution

$$f(c)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{ikc}\phi(k) \tag{3.2}$$

can then be obtained by contour integration. As $f(c)$ is an even function, we only need to evaluate the integral in Eq. (3.2) for $c>0$. Replacement $c\to|c|$ then gives the result for all c. By closing the contour through an infinite upper half circle and applying the residue theorem, we obtain

$$f(c)=\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}e^{-k_{m\ell}|c|}\sum_{n=0}^{\nu_{m\ell}-1}|c|^nA_{m\ell n}, \tag{3.3}$$

where

$$A_{m\ell n}=\frac{i^{n+1}k_{m\ell}^{2\nu_{m\ell}}}{n!(\nu_{m\ell}-1-n)!}\lim_{k\to ik_{m\ell}}\left(\frac{\partial}{\partial k}\right)^{\nu_{m\ell}-1-n}\times(k+ik_{m\ell})^{-\nu_{m\ell}}\tilde{\phi}_{m\ell}(k), \tag{3.4}$$

with

$$\tilde{\phi}_{m\ell}(k)\equiv\prod_{m'=0}^{\infty}\prod_{\ell'=0}^{m'}(1+k^2/k_{m'\ell'}^2)^{-\nu_{m'\ell'}(1-\delta_{mm'}\delta_{\ell\ell'})}. \tag{3.5}$$

Note that the factor labeled $(m',\ell')=(m,\ell)$ is absent. The dominant terms in Eq. (3.3) for large $|c|$ correspond to the smallest values of $k_{m\ell}$. The two smallest ones are $k_{00}=a$ and $k_{11}=a/p$. Consequently, the leading and subleading terms are

$$f(c)\approx A_0e^{-a|c|}+A_1e^{-a|c|/p}+\cdots, \tag{3.6}$$

where

$$A_n\equiv A_{nn0}=(a/2p^n)\tilde{\phi}_{nn}(ia/p^n). \tag{3.7}$$

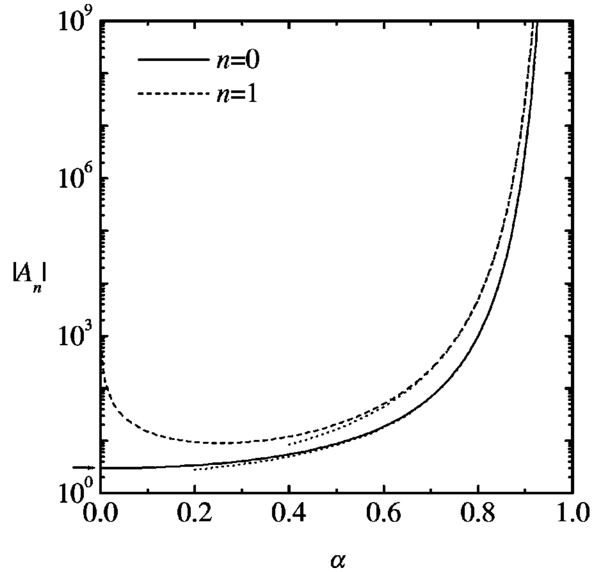

FIG. 1. Logarithmic plot of amplitudes $A_0 \equiv A_{000}$ (solid line) and $-A_1 \equiv -A_{110}$ (dashed line) as functions of coefficient of restitution. The arrow indicates the value $A_0 \simeq 2.958\,389$ at $\alpha=0$. The dotted lines represent asymptotic form (4.11) for small q.

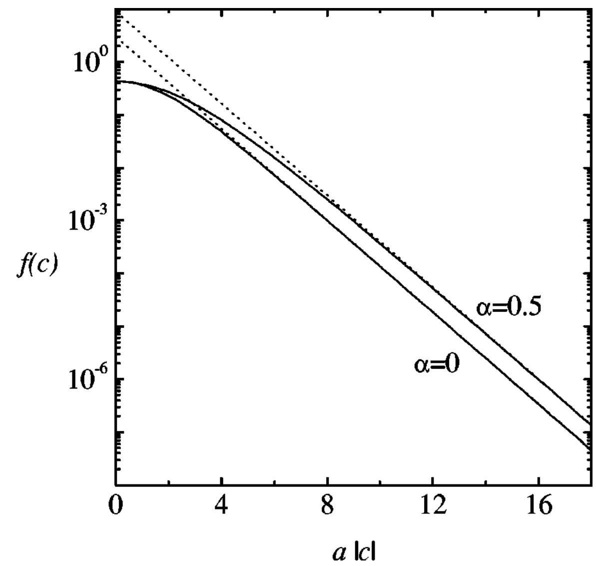

FIG. 2. Logarithmic plot of $f(c)$ vs $a|c|$ for $\alpha=0$ and $\alpha=0.5$. The dotted lines are asymptotic forms $f(c) \approx A_0 e^{-a|c|}$ at $\alpha=0$ and $\alpha=0.5$, with A_0 obtained from Eq. (3.8).

We calculate the first two explicitly, i.e.,

$$A_0 = \frac{a}{2} \exp\left[\sum_{m=1}^{\infty} \frac{p^{2m}+q^{2m}}{m\ (1-p^{2m}-q^{2m})} \right], \tag{3.8}$$

$$A_1 = \frac{-ap^3}{2(1-p^2)(p-q)} \exp\left[\sum_{m=1}^{\infty} \frac{p^{-2m}(p^{2m}+q^{2m})^2}{m(1-p^{2m}-q^{2m})} \right]. \tag{3.9}$$

In the last equalities we have followed steps similar to those used to obtain Eq. (2.4) from Eq. (2.3). Results (3.3)–(3.9) exhibit the full analytical structure of the dominant and subdominant high-energy tails of the velocity distribution in the NESS, as already demonstrated numerically for the one-dimensional case in Refs. [25–27] and derived in Ref. [23] for d-dimensional IMM's on the basis of self-consistent solutions. Moreover, we have obtained here explicit expressions for amplitudes A_0 and A_1 in the form of sums that are rapidly converging when q is not too small. Coefficients $A_0 \equiv A_{000}$ and $A_1 \equiv A_{110}$ are shown in Fig. 1 as functions of α, where $A_{110} \propto 1/(p-q) = 1/\alpha$ diverges according to Eq. (3.9). The next term to those explicitly given in Eq. (3.6) corresponds either to $k_{22} = a/p^2$ if $p^2 > q$ (i.e., if $\alpha > \sqrt{5}-2 \simeq 0.236$) or to $k_{10} = a/q$ if $p^2 < q$. Note that amplitude A_{100} of $\exp(-k_{10}|c|)$ can be obtained from A_1 in Eq. (3.9) by interchanging $p \leftrightarrow q$. Figure 2 compares asymptotic form $f(c) \approx A_0 e^{-a|c|}$ with function $f(c)$ obtained by numerically inverting $\phi(k)$ for $\alpha=0$ and $\alpha=0.5$. We observe that the asymptotic behavior is reached for $a|c| \gtrsim 4$ if $\alpha=0$ and for $a|c| \gtrsim 8$ if $\alpha=0.5$. As $a = 1/\sqrt{pq}$, this corresponds to velocities far above the rms velocity.

There are two interesting limiting cases: the *quasielastic* limit $(\alpha \to 1, q \to 0)$ and the *totally inelastic* limit $(\alpha \to 0, p \to \frac{1}{2}^+, q \to \frac{1}{2}^-)$. We start with the latter. In the totally inelastic limit $(\alpha \to 0)$, subdominant terms $A_{110} e^{-a|c|/p}$ and $A_{100} e^{-a|c|/q}$ become equally important, i.e., the single poles in Eq. (3.1) at $k_{11} = a/p$ and $k_{10} = a/q$ coalesce and Eq. (3.6) no longer describes the subdominant large-c behavior correctly. Moreover, $A_{110} \simeq -A_{100} \propto 1/\alpha$, as can be seen in Fig. 1 for A_{110}. In fact, poles $k_{m\ell} \to k_m \equiv 2^m a$ coalesce for all ℓ, some of coefficients $A_{m\ell n}$ diverge, e.g., $A_{mm0} \propto (1/\alpha)^{2^m-1}$, and the expansion makes no sense anymore. So, we analyze case $\alpha=0$ separately. In this case the characteristic function is according to Eq. (2.3):

$$\phi(k) = \prod_{m=0}^{\infty} (1+k^2/k_m^2)^{-\nu_m}, \tag{3.10}$$

where $\nu_m \equiv 2^m$ and $k_m \equiv 2^m a$ with $a = 1/\sqrt{pq} = 2$. Then, the distribution function is

$$f(c) = \sum_{m=0}^{\infty} e^{-k_m|c|} \sum_{n=0}^{\nu_m-1} |c|^n A_{mn}, \tag{3.11}$$

where the residues or amplitudes are given by

$$A_{mn} = \frac{i^{n+1} k_m^{2\nu_m}}{n!(\nu_m-1-n)!} \lim_{k \to ik_m} \left(\frac{\partial}{\partial k} \right)^{\nu_m-1-n} (k+ik_m)^{-\nu_m} \tilde{\phi}_m(k) \tag{3.12}$$

and $\tilde{\phi}_n(k)$ is defined as

$$\tilde{\phi}_n(k) \equiv \prod_{m=0}^{\infty} (1+k^2/k_m^2)^{-\nu_m(1-\delta_{nm})}. \tag{3.13}$$

For large $|c|$, the distribution function becomes

$$f(c) \approx A_{00} e^{-2|c|} + (A_{10} + A_{11}|c|) e^{-4|c|} + \cdots. \tag{3.14}$$

To calculate the amplitudes of the dominant terms, we derive from Eq. (3.13),

$$\ln \tilde{\phi}_0(k) = -\sum_{m=1}^{\infty} 2^m \ln(1+2^{-2m}k^2/a^2)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{(k/a)^{2n}}{2^{2n-1}-1}$$

$$\ln \tilde{\phi}_1(k) = -\ln(1+k^2/a^2) + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{(k/2a)^{2n}}{2^{2n-1}-1}. \tag{3.15}$$

Definitions (3.12)–(3.13) with $k_m = 2^m a (a=2)$ then yield

$$A_{00} = \frac{1}{2} a \tilde{\phi}_0(ia) = e^{S_0} \simeq 2.958\,389,$$

$$A_{11} = a^2 \tilde{\phi}_1(2ia) = -\frac{4}{3} A_{00}^2 \simeq -11.669\,422,$$

$$A_{10} = \frac{1}{2} a \tilde{\phi}_1(2ia) - ia^2 \tilde{\phi}_1'(2ia) = \frac{4}{3}\left(S_1 - \frac{11}{12}\right) A_{00}^2$$

$$\simeq 3.138\,267, \tag{3.16}$$

where we have used the rapidly converging sums

$$S_0 = \sum_{n=1}^{\infty} \frac{1}{n} (2^{2n-1}-1)^{-1} \simeq 1.084\,645,$$

$$S_1 = \sum_{n=1}^{\infty} (2^{2n-1}-1)^{-1} \simeq 1.185\,597. \tag{3.17}$$

In fact, results (3.14) could have been derived directly from Eqs. (3.3)–(3.9) after lengthy calculations, by expanding A_{110} and A_{100} in powers of α, with the result

$$A_{1s0} = (-1)^{s+1} A_{11}/(8\alpha) + \frac{1}{2} A_{10} + O(\alpha) \tag{3.18}$$

with $s = 0,1$. Insertion of these results in Eq. (3.3) yields Eq. (3.14). Limit $\alpha \to 1$ is discussed in the following Section.

IV. QUASIELASTIC LIMIT

As already mentioned in the introduction, the velocity distribution approaches a NESS even in the presence of an *infinitesimal* dissipation ($\alpha \to 1$, $q \to 0$), balanced by a ditto amount of stochastic heating. This limit is referred to as the quasielastic limit. For the rescaled functions $f(c)$ and $\phi(k)$ it simply refers to limit $q \to 0$.

Once we have *first* taken the large-$|c|$ limit at *fixed* $\alpha < 1$, as has been done in the preceding section, we can *next* take the quasielastic limit $\alpha \to 1$. When the limits are taken in that order, the asymptotic behavior is still of form $e^{-a|c|}$, where the decay constants are $k_{mm} = a/p^m \to a$ and the amplitudes may diverge. On the other hand, if the limits are taken in the reverse order, first $\alpha \to 1$ at fixed $|c|$ and next $|c| \to \infty$, the behavior is in general totally different.

First consider the second case and observe that $\psi(k)$ in Eq. (2.4) has, at small q, form $\psi(k) = -\frac{1}{2}k^2 + \sum_{n=2}^{\infty} a_{2n}(q) k^{2n}$ with rapidly decreasing coefficients $a_{2n} \simeq (-1)^n q^{n-1}(1-\frac{1}{2}q)/(2n^2)$ for $n \geqslant 2$. Consequently, $\phi(k) = e^{\psi(k)}$ can be expanded as

$$\phi(k) = e^{-(1/2)k^2}\left[1 + \sum_{n=2}^{\infty} \mu_{2n}(q) k^{2n}\right], \tag{4.1}$$

where the relation between a_{2n} and μ_{2n} is the same as between cumulants and moments after setting $a_2 = \mu_2 = 0$. Coefficients μ_{2n} are, to dominant order in q^2, given by

$$\mu_4 = a_4 = \frac{1}{8} q\left(1 - \frac{1}{2}q - \frac{3}{4}q^2\right) + O(q^4),$$

$$\mu_6 = a_6 = -\frac{1}{18} q^2\left(1 - \frac{1}{2}q\right) + O(q^4),$$

$$\mu_8 = \frac{1}{2} a_4^2 + a_8 = \frac{1}{128} q^2(1+3q) + O(q^4),$$

$$\mu_{10} \simeq a_4 a_6 = -\frac{1}{144} q^3 + O(q^4),$$

$$\mu_{12} \simeq \frac{1}{6} a_4^3 = \frac{1}{3072} q^3 + O(q^4), \tag{4.2}$$

and in general, $\mu_{4n-2} \sim \mu_{4n} \sim O(q^n)$ for $n \geqslant 2$. The series above can be Fourier inverted termwise, using the following relation:

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikc} e^{-(1/2)k^2} k^{2n}$$

$$= (-1)^n \left(\frac{d}{dc}\right)^{2n} \exp\left(-\frac{1}{2}c^2\right) \Big/ \sqrt{2\pi}$$

$$= (-1)^n He_{2n}(c) f_0(c)$$

$$= 2^n n! L_n^{(-1/2)}\left(\frac{1}{2}c^2\right) f_0(c), \tag{4.3}$$

where $f_0(c) = \exp(-\frac{1}{2}c^2)/\sqrt{2\pi}$ is the Maxwellian. In the last two equalities Rodrigues' formula for the Hermite polynomials has been used, as well as their relation to the generalized Laguerre or Sonine polynomials [see Ref. [29], Eqs. (22.11.88), (22.5.18) and (22.5.40)]. The resulting Sonine polynomial expansion of the velocity distribution in the NESS then reads

$$f(c) = f_0(c)\left[1 + \sum_{n=2}^{\infty} (-1)^n \mu_{2n}(q) He_{2n}(c)\right]. \tag{4.4}$$

Similar expansions of the NESS-distribution function in low order Hermite or Sonine polynomials have also been derived for inelastic hard spheres in d dimensions [5] and for a three-dimensional IMM in Ref. [18].

Next, we consider the case where *first* $|c|\to\infty$ at finite $\alpha<1$ and *next* $\alpha\to 1$, or $q\to 0$. The large-c behavior at fixed α has already been discussed in Eq. (3.3)–(3.9), and we observe that the terms in Eq. (3.3) at large c, associated with all poles of form $k_{n\ell}=a/p^{\ell}q^{n-\ell}(\ell<n)$ decay rapidly as $q\to 0$ and only poles with $k_{nn}=a/p^n$ need to be considered:

$$f(c)=\sum_{n=0}^{\infty} A_n e^{-k_{nn}|c|}. \tag{4.5}$$

We will analyze the behavior of associated amplitudes A_n by combining Eq. (3.7) with Eq. (3.5), i.e.,

$$\begin{aligned}\ln A_n&=\ln(a/2p^n)-\sum_{m=0}^{\infty}\sum_{\ell=0}^{m}\binom{m}{\ell}(1-\delta_{mn}\delta_{\ell n})\\&\quad\times\ln[1-p^{-2(n-\ell)}q^{2(m-\ell)}]\\&\equiv B_n^{(1)}+B_n^{(2)}+B_n^{(3)}+\ln(a/2p^n),\end{aligned} \tag{4.6}$$

where

$$B_n^{(1)}=-\sum_{m=0}^{n-1}\sum_{\ell=0}^{m}\binom{m}{\ell}\ln[1-p^{-2(n-\ell)}q^{2(m-\ell)}],$$

$$B_n^{(2)}=-\sum_{\ell=0}^{n-1}\binom{n}{\ell}\ln[1-(q/p)^{2(n-\ell)}],$$

$$B_n^{(3)}=-\sum_{m=n+1}^{\infty}\sum_{\ell=0}^{m}\binom{m}{\ell}\ln[1-p^{-2(n-\ell)}q^{2(m-\ell)}]. \tag{4.7}$$

Now we take limit $q\to 0$ at *finite* n and retain terms to order q. The dominant small-q contribution to $B_n^{(1)}$ comes from $\ell=m$, i.e.,

$$\begin{aligned}B_n^{(1)}&=-\sum_{m=1}^{n}\ln(1-p^{-2m})+o(q)\\&=-\ln[(-2q)^n n!]-\frac{1}{2}n(n+2)q+o(q),\end{aligned} \tag{4.8}$$

where we used relation $1-1/p^{2m}\simeq -2mq[1+(m+\frac{1}{2})q]$, and $o(q^k)$ denotes terms which are negligible with respect to q^k. Note that complex number $B_n^{(1)}$ is only determined modulo $\{2\pi i\}$, but $\exp(B_n^{(1)})$ is single valued. Furthermore, we observe that $B_n^{(2)}=O(nq^2)$. The analysis of $B_n^{(3)}$ in Eq. (4.7) is more involved and given in Appendix B. The result is

$$B_n^{(3)}=\frac{\pi^2}{12q}+\frac{1}{2}\ln q-K_0+\frac{1}{2}\left(n+\frac{13}{12}-\frac{\pi^2}{72}\right)q+o(q), \tag{4.9}$$

where

$$K_0=\frac{3}{4}+\frac{\pi^2}{24}-\frac{1}{2}\ln 2-R\simeq 0.733\,598. \tag{4.10}$$

Combining the small-q results (4.8) and (4.9) for $B_n^{(1)}$ and $B_n^{(3)}$ with Eq. (4.6), yields, for A_n,

$$\begin{aligned}A_n&=\frac{a}{2p^n}\exp[B_n^{(1)}+B_n^{(3)}+o(q)]=\frac{(-1)^n}{2n!(2q)^n}\\&\quad\times\exp\left[\frac{\pi^2}{12q}-K_0-\frac{1}{2}n(n-1)q+K_1q+o(q)\right],\end{aligned} \tag{4.11}$$

where

$$K_1=\frac{25}{24}-\frac{\pi^2}{144}\simeq 0.973\,127\,8. \tag{4.12}$$

To describe the crossover between the two different limiting behaviors, i.e., Eq. (4.4) with first $q\to 0$, next $c\to\infty$, and Eqs. (4.5) and (4.11) with first $c\to\infty$, next $q\to 0$ we need to couple these limits, which will be discussed next.

By an extension of the steps followed in Appendix B, it can be verified that the terms denoted by $o(q)$ in Eq. (4.11) have form $n^{k_1}q^{k_2}$ with $k_1\leqslant k_2+1$ and $k_2\geqslant 2$. Therefore, those terms can be neglected against the terms of order q if $n\ll q^{-1}$.

Ratio $R(c)$ between distribution function $f(c)$ in Eq. (4.5) and its asymptotic high-energy form $A_0e^{-a|c|}$ define a *crossover* function

$$R(c)\equiv f(c)/A_0e^{-a|c|}=\sum_{n=0}^{\infty} b_n r_n, \tag{4.13}$$

where r_n and $b_n=A_n/A_0$ follow from Eq. (4.5) and (4.11) as

$$\begin{aligned}r_n&=\exp[-a|c|(p^{-n}-1)],\\b_n&=\frac{(-1)^n}{n!(2q)^n}\exp\left[-\frac{1}{2}n(n-1)q+o(n^2q)\right].\end{aligned} \tag{4.14}$$

Here we have written $o(q)\to o(n^2q)$ to emphasize the fact that Eq. (4.14) remains valid if $n\ll q^{-1}$. So, there is a crossover behavior in $R(c)$ from a large-c behavior of $O(\exp[-c^2/2])\simeq 0$ in the small-q Sonine polynomial expansion (3.4), to the small-q behavior of $R(c)$ of $O(1)$ in Eq. (4.13). The transition region is characterized by a crossover velocity c_0 such that $R(c_0)\approx\frac{1}{2}$. The interesting questions are how does c_0 scale with q in the quasielastic limit and what is the width of the crossover region? To address these questions, note that series (4.13) converges for all velocities and the signs of the terms are alternating. Therefore, when breaking off the infinite sum at $n=N$, the maximum error is $|b_{N+1}|r_{N+1}$:

$$\begin{aligned}R(c)&=\sum_{n=0}^{N} b_n r_n+\Delta^{(N)}(c)\equiv R^{(N)}(c)+\Delta^{(N)}(c),\\|\Delta^{(N)}(c)|&\leqslant|b_{N+1}|r_{N+1}.\end{aligned} \tag{4.15}$$

This suggests that the pure exponential high-energy tail $A_0 e^{-a|c|}$ qualitatively describes the large-c behavior of $f(c)$ if

$$|b_1| r_1 = \frac{e^{-a|c|q/p}}{2q} \simeq \frac{e^{-\sqrt{q}|c|}}{2q} \leqslant \frac{1}{2}. \tag{4.16}$$

Of course, the bound $\frac{1}{2}$ may be replaced by any number of the order of 1 in this estimate. Equation (4.16) implies that $w \equiv |c|\sqrt{q}/\ln q^{-1} \geqslant 1$. Therefore, we can estimate the crossover velocity to be $c_0 = (\ln q^{-1})/\sqrt{q}$ or equivalently, $w_0 = 1$. To confirm this and get a closed form for crossover function $R(c)$, consider a value of w in range $0.5 < w < 1$ and take $N = \beta q^{w-1}$, where $\beta \gtrsim 1$. In that case, $N \gg 1$ but $N^2 q \ll 1$, so that $r_n \simeq q^{nw}$ and $b_n \simeq (-1)^n/n!(2q)^n$ for $n \lesssim N$ and $|b_{N+1}| r_{N+1} \simeq (2\beta/e)^{-N}/2\beta\sqrt{2\pi N}$. Therefore, with this choice of N,

$$R^{(N)}(c) \simeq \sum_{n=0}^{N} \frac{(-1)^n}{n!} \left(\frac{q^{w-1}}{2} \right)^n,$$

$$|\Delta^{(N)}(c)| \leqslant \frac{1}{2\beta\sqrt{2\pi N}} \left(\frac{2\beta}{e} \right)^{-N}. \tag{4.17}$$

If $\beta > e/2 \simeq 1.36$ then $\Delta^{(N)}(c) \ll 1$ and $R(c)$ can be approximated by $R^{(N)}(c)$. By the same arguments, the upper limit in the summation of Eq. (4.17) can be replaced by infinity. Choice $N = \beta q^{w-1}$ is justified by the fact that for $w < 1$, term $|b_n| r_n$ reaches a high maximum value $|b_{n_0}| r_{n_0} \simeq \exp(n_0 + \frac{1}{2})/\sqrt{2\pi n_0}$ at $n_0 \simeq \frac{1}{2}(q^{w-1} - 1)$ and then decays rapidly. If $w > 1$, however, $|b_n| r_n$ decreases monotonically and thus $\Delta^{(N)}(c) \ll 1$ for any choice of N. In conclusion, the crossover function for $w > 0.5$ in the quasi-elastic limit becomes

$$R(c) \simeq \exp(-q^{w-1}/2), \quad w \equiv |c|\sqrt{q}/\ln q^{-1}. \tag{4.18}$$

At $w = 1$ we have $R(c = c_0) \simeq 1/\sqrt{e} \simeq 0.6$, thus confirming the estimate of crossover velocity c_0 made below Eq. (4.16). Figure 3 represents crossover function $R(c)$ versus scaled velocity w for $q = 0.01$, 0.001, and 0.0001. To measure the width of the crossover region, let w_1 and w_2 denote the values of w at which $R = 0.1$ and $R = 0.9$, respectively. From Eq. (4.18) we obtain $w_1 \simeq 1 - 1.5/\ln q^{-1}$ and $w_2 \simeq 1 + 1.6/\ln q^{-1}$, so the width scales as $w_2 - w_1 \sim 1/\ln q^{-1}$. Going back to unscaled velocities, the crossover takes place between $c_1 = c_0 - 1.5/\sqrt{q}$ and $c_2 = c_0 + 1.6/\sqrt{q}$ with a width $c_2 - c_1 \sim 1/\sqrt{q}$. For $q = 0.01$, 0.001, and 0.0001, one has $c_0 \simeq 46$, 218, and 921 and $c_2 - c_1 \simeq 31$, 98, and 310, respectively. For these high values of the velocity, the distribution function is extremely small. For instance, at $c = c_0$, $f(c_0) \simeq \frac{1}{2}\exp[(q^{-1} + \frac{1}{2})\ln q + \pi^2/12q - K_0 - \frac{1}{2}]$. This yields $f(c_0) \sim 10^{-166}$, 10^{-2645}, and $10^{-36\,431}$ for $q = 0.01$, 0.001, and 0.0001, respectively. These values are beyond the accuracy of any numerical or simulation method, so the high-energy tail in the quasielastic limit would look like a Maxwellian for the domain of velocities numerically accessible. On the other hand, our asymptotic analysis of the exact solution shows that the true tail is actually exponential.

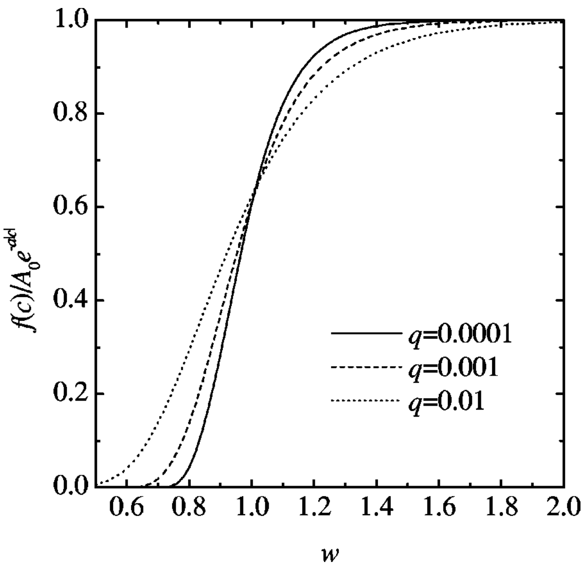

FIG. 3. Plot of the ratio between velocity distribution function $f(c)$ and its high-energy tail $A_0 e^{-a|c|}$ as a function of scaled velocity $w \equiv |c|\sqrt{q}/\ln q^{-1}$ in the quasielastic limit for $q = 0.01$, 0.001, and 0.0001.

V. CONCLUSION

The exact nonequilibrium steady-state solution of the nonlinear Boltzmann equation for a driven one-dimensional inelastic Maxwell gas was obtained in Ref. [15] in the form of an infinite product for Fourier transform $\phi(k)$ of distribution function $f(c)$. The main goal of this paper has been to show that this relatively simple exact solution in the one-dimensional case also possesses the generic properties of overpopulation of high-energy tails and exhibits a rich mathematical structure, especially in the different limiting cases.

We have inverted the Fourier transform to express $f(c)$ in the form of an infinite series of exponentially decaying terms, as given by Eq. (3.3) with velocity c measured in units of the rms velocity (i.e., $\langle c^2 \rangle^{1/2} = 1$). For all values of the coefficient of restitution $0 \leqslant \alpha < 1$, the high-energy tail is exponential, namely $f(c) \simeq A_0 \exp(-a|c|)$, where $a \equiv 1/\sqrt{pq} = 2/\sqrt{1-\alpha^2}$ and amplitude A_0 is given by Eq. (3.8) and plotted in Fig. 1.

Special attention has been paid to two complementary limiting cases: the totally inelastic limit ($\alpha \to 0$) and the quasielastic limit ($\alpha \to 1$). In the former case some poles coalesce and the dominant high-energy term is still exponential but the subdominant term becomes an exponential times a linear function of the velocity, where the numerical value of the associated amplitudes is given by Eq. (3.16).

The quasielastic limit is much more delicate and requires some care. If we first take $\alpha \to 1$ at fixed $|c|$ and next $|c| \to \infty$ (order A), the high-energy tail has a Maxwellian form. On the other hand, if the limits are taken in the reverse order, i.e., first $|c| \to \infty$ at fixed $\alpha < 1$ and then $\alpha \to 1$ (order B), the asymptotic high-energy tail is exponential. The crossover be-

TABLE I. Asymptotic behavior of the distribution function $f(c)$ for one-dimensional systems in the quasielastic limit. In general, the result depends on the order of limits. Order A corresponds to take first $\alpha \to 1^-$ and then $|c| \to \infty$, whereas order B refers to the reverse order, i.e., first $|c| \to \infty$ and then $\alpha \to 1^-$. The first/second footnote in the second column gives the reference where the result for order A/B was obtained.

State	System	Order A	Order B
Free cooling	Hard spheres[a,b]	$\frac{1}{2}[\delta(c-1)+\delta(c+1)]$	$e^{-a\|c\|}$
	Maxwell model[c,c]	c^{-4}	c^{-4}
White noise	Hard spheres[a,d]	$e^{-a\|c\|^3}$	$e^{-a\|c\|^{3/2}}$
	Maxwell model[e,f]	e^{-ac^2}	$e^{-a\|c\|}$
Gravity thermostat	Hard spheres[e,g]	$\frac{1}{2}[\delta(c-1)+\delta(c+1)]$	e^{-ac^2}
	Maxwell model[e,h]	$\frac{1}{2}[\delta(c-1)+\delta(c+1)]$	$e^{-a\|c\|}$

[a]Reference [8].
[b]Reference [5].
[c]Reference [13,15,22].
[d]Reference [5,8].
[e]This work.
[f]Reference [8,23].
[g]Reference [10].
[h]Reference [23].

tween both limiting behaviors is described by the coupled limit $c \to \infty$ and $q \to 0$ with scaling variable $w = |c| \sqrt{q} / \ln q^{-1}$ fixed with $q \equiv \frac{1}{2}(1-\alpha) \ll 1$, and occurs at $w \simeq 1$. If $w < 1$ (more specifically, $1 - w \gtrsim 1/\ln q^{-1}$), the distribution function is essentially a Maxwellian while the true exponential high-energy tail is reached if $w > 1$ (more specifically, $w - 1 \gtrsim 1/\ln q^{-1}$).

It is of interest to emphasize that the results for the scaling form in the quasielastic limit not only depend sensitively on the order in which both limits are taken but also depend strongly on the collisional interaction, i.e., on the energy dependence of the collisional frequency, as well as on the mode of energy supply to the system. To illustrate this, we have collected in Table I what is known for the different inelastic models in one dimension, i.e., (i) hard spheres and (ii) Maxwell models, and for different modes of energy supply, i.e., (i) no energy input or free cooling, (ii) energy input or driving through Gaussian white noise, represented by forcing term $-D\partial^2 F(v,t)/\partial v^2$ in the Boltzmann equation, and (iii) energy input through a *negative* friction force $\propto gv/|v|$ acting in the direction of the particle's velocity but independent of its speed. This driving, referred to as gravity thermostat, can be represented as the forcing term $g(\partial/\partial v)[(v/|v|)F(v,t)]$ in the Boltzmann equation. The results corresponding to order A with the gravity thermostat have been obtained by the same method as followed in Ref. [8]. It is worthwhile noting that in the quasielastic limit a bimodal distribution $\frac{1}{2}[\delta(c+1)+\delta(c-1)]$ is observed in inelastic hard sphere systems, both for free cooling and for driving through the gravity thermostat whereas in inelastic Maxwell models this bimodal distribution is only observed for the gravity thermostat.

It is important to note that in the normalization where velocities are measured in units of the rms velocity, the high-energy tail in the driven inelastic Maxwell model is only observable for very large velocities, as illustrated in Fig. 2 for strong $(\alpha \to 0)$ and intermediate $(\alpha = \frac{1}{2})$ inelasticity. In the quasielastic limit, where $(\alpha \to 1)$, the tail is even pushed further out towards infinity, as analyzed at the end of Sec. IV. This also explains how to reconcile the paradoxical results of exponential large-c behavior with the very accurate representation (4.4) of the distribution function in the thermal range, in the form of a Maxwellian multiplied by a polynomial expansion in Hermite or Sonine polynomials with coefficients related to the cumulants. The validity of these polynomial expansions, over a large range of inelasticities $(0 \leqslant \alpha < 1)$ had been observed before, in Ref. [5], for inelastic hard spheres and in Ref. [18] for inelastic Maxwell models. On the other hand, the high-energy tail is $\propto e^{-a|c|}$, and not $\propto c^N e^{-c^2/2}$, where N is some large number and yields diverging moments $M_{2n} = \langle c^{2n} \rangle$ and cumulants C_{2n} in limit $n \to \infty$, as shown in Sec. II.

The exact solutions of the nonlinear Boltzmann equation for the freely evolving [13] and the driven [15] inelastic Maxwell models (extended in this paper) as well as the rigorous proof of Ref. [24] for the long time approach of the distribution function to a scaling form validate the self-consistent method developed in Ref. [5] for analytical studies of possible over- or underpopulations of the high-energy tail of velocity distributions, not only for inelastic Maxwell models but, more importantly, also for inelastic hard sphere, where exact solutions are not known. This possibility of assessing the validity of general kinetic theory methods by means of exact solutions of the nonlinear Boltzmann equation is one of the main reasons why the study of inelastic Maxwell models is of interest.

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APPENDIX A: LARGE-k EXPANSION

The asymptotic behavior of ψ for large k can be obtained by inserting ansatz $\psi = -\lambda|k| + \ln(Ak^2) + \Sigma_{n=1}^{\infty} a_n k^{-2n}$ with unknown coefficients $\{A, a_n\}$ into Eq. (2.2) and equating the coefficients of equal powers of $\ln k$ and k^n with result

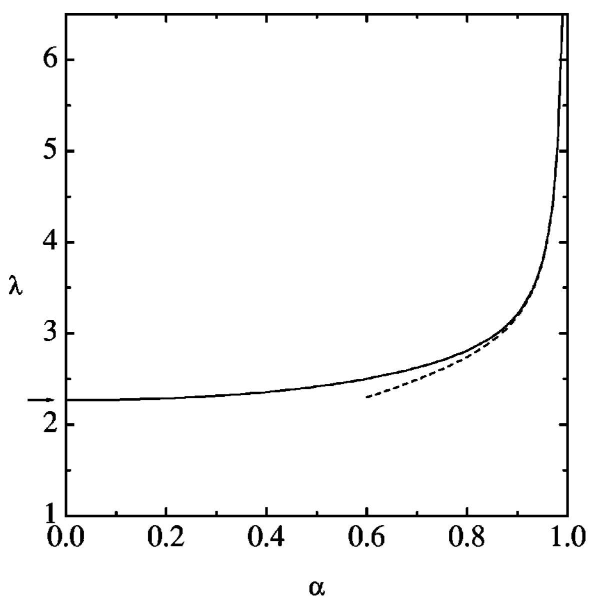

FIG. 4. Coefficient λ as a function of the coefficient of restitution. The arrow indicates the value $\lambda=\pi/2\ln 2$ at $\alpha=0$. The dotted line represents the asymptotic form Eq. (A7) for small q.

$$\psi(k)=-\lambda|k|+\ln(k^2/pq)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{(k^2pq)^{-n}}{p^{-2n}+q^{-2n}-1}, \tag{A1}$$

where λ is as yet undetermined. The series converges for $k^2\geqslant q/p$. However, for $\psi(k)$ mentioned above to qualify as a solution of Eq. (2.2), the radius of convergence is further restricted to $(qk)^2\geqslant q/p$ or $pqk^2\geqslant 1$. Constant λ must be chosen such that $\psi(k)$ satisfies boundary condition $\psi\simeq-\frac{1}{2}k^2$ at small k. This can be done by matching Eq. (A1) with Eq. (2.4). The latter satisfies already the small-k boundary condition. Matching at $pqk^2=1$ then yields

$$\frac{\lambda}{\sqrt{pq}}=-2\ln(pq)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{1-p^{2n}-q^{2n}}+\frac{1}{p^{-2n}+q^{-2n}-1}\right). \tag{A2}$$

Both terms can be combined into a single n-sum with $n=\pm1,\pm2,\ldots$. The above result is not only convenient for numerical evaluation, as shown in Fig. 4, but also for analytic evaluation in two limiting cases. We first consider the *totally inelastic* limit ($\alpha\to 0$ or $p=q=\frac{1}{2}$). There, expansion (A1) can be cast into a simpler form:

$$\begin{aligned}\psi(k)&=-\lambda|k|+\ln(4k^2)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{k^{-2n}}{1-2^{-(2n+1)}}\\&=-\lambda|k|+\ln(4k^2)+\frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{2^m}\ln\left(1+\frac{1}{2^{2m}k^2}\right)\\&=-\lambda|k|+\frac{1}{2}\sum_{m=0}^{\infty}\frac{1}{2^m}\ln(1+2^{2m}k^2).\end{aligned} \tag{A3}$$

Matching this expression in $k^2=1/pq=4$ with the exact solution in Eq. (2.3), $\psi(2)=-\Sigma_{m=0}^{\infty}2^m\ln(1+2^{-2m})$, yields the following nice result:

$$\lambda=\frac{1}{2}\sum_{m=-\infty}^{\infty}2^m\ln(1+2^{-2m})=\frac{1}{4\ln 2}\int_0^{\infty}dx\,x^{-3/2}\ln(1+x)=\frac{\pi}{2\ln 2}. \tag{A4}$$

One can verify using the Euler-MacLaurin summation formula [see Eq. (23.1.30) of Ref. [29]] that all correction terms to the integral are vanishing, and the integral is listed in Eq. (4.293.3) of Ref. [30].

In the quasielastic limit ($\alpha\to 1$ or $q\to 0$) the sum originating from the second term inside $(\cdots)$ in Eq. (A2) is of order of $\mathcal{O}(q^2)$ and will be neglected. To evaluate first term T for small q, we expand it as follows:

$$\begin{aligned}T&=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\frac{1}{1-p^{2k}}\left[1+\frac{q^{2k}}{1-p^{2k}}+O(q^{2(2k-1)})\right]\\&=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left[\frac{1}{2kq}+\frac{2k-1}{4k}+\frac{q^{2k}}{(2kq)^2}+O(q)\right]\\&=-\frac{1}{2q}\mathrm{Li}_2(-1)-\frac{1}{2}\mathrm{Li}_1(-1)+\frac{1}{4}\mathrm{Li}_2(-1)+\frac{1}{4}+O(q),\end{aligned} \tag{A5}$$

where the polylogarithmic functions are defined as

$$\mathrm{Li}_k(x)=\sum_{n=1}^{\infty}x^n/n^k \tag{A6}$$

with $\mathrm{Li}_2(-1)=-\frac{1}{12}\pi^2$ and $\mathrm{Li}_1(-1)=-\ln 2$ [31]. The final result for λ at small q is then

$$\lambda=\frac{1}{\sqrt{q}}\left[\frac{\pi^2}{24}-2q\ln q+\frac{1}{2}q\left(\ln 2+\frac{1}{2}-\frac{\pi^2}{12}\right)+q^2\ln q+\mathcal{O}(q^2)\right]. \tag{A7}$$

APPENDIX B: ASYMPTOTICS IN QUASIELASTIC LIMIT

To calculate $B_n^{(3)}$ in Eq. (4.7) for small q we expand the logarithm and perform the (m,ℓ) summation. The result is,

$$\begin{aligned}B_n^{(3)}&=\sum_{k=1}^{\infty}\frac{p^{2k}}{k(1-p^{2k})}\frac{(1+q^{2k}/p^{2k})^{n+1}}{1-q^{2k}/(1-p^{2k})}\\&=\sum_{k=1}^{\infty}\frac{p^{2k}}{k(1-p^{2k})}\left[1+\frac{q^{2k}}{1-p^{2k}}+\frac{q^{4k}}{(1-p^{2k})^2}+\frac{(n+1)q^{2k}}{p^{2k}}\right]+o(q)\end{aligned}$$

$$\equiv S(x)+\delta S(n,x)+o(q), \tag{B1}$$

where all contributions $\propto q^0$ and $\propto q$ have been included. The dominant term is

$$S(x)=\sum_{k=1}^{\infty}\frac{e^{-kx}}{k(1-e^{-kx})}\quad (x=-2\ln p). \tag{B2}$$

In the remaining contributions to Eq. (B1) only term $k=1$ needs to be taken into account and yields

$$\delta S(x,n)=\frac{1}{4}+\frac{1}{2}q\left(n+\frac{3}{4}\right)\simeq\frac{1}{4}+\frac{1}{4}x\left(n+\frac{3}{4}\right). \tag{B3}$$

To study the small-x behavior of Eq. (B2) we construct an asymptotic series for $S(x)$ by expanding $1/(1-e^{-kx})$ in powers of x. This can be done most conveniently by using the small-x expansion of $x\coth x$ or equivalently the generating function for the Bernoulli numbers $B_{2k}=(-1)^{k+1}|B_{2k}|$ [see Eqs. (23.1.1-2) of Ref. [29]], which we write as

$$\frac{1}{1-e^{-x}}=\frac{1}{x}+\frac{1}{2}+\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}x^{2m-1}. \tag{B4}$$

Substitution of Eq. (B4) with $x\to kx$ into Eq. (B2) yields

$$S(x)=\frac{1}{x}\mathrm{Li}_2(e^{-x})-\frac{1}{2}\ln(1-e^{-x})+\sum_{m=1}^{\infty}\frac{x^{2m-1}}{(2m)!}B_{2m}\mathrm{Li}_{2-2m}(e^{-x}). \tag{B5}$$

We have used definition (A6) of the polylogarithmic functions, which are all singular in $x=0$. To determine the behavior of dilogarithm $\mathrm{Li}_2(e^{-x})$, we use functional relation [see Eq. (5) of Ref. [31]]

$$\mathrm{Li}_2(e^{-x})=\frac{\pi^2}{6}-\ln(e^{-x})\ln(1-e^{-x})-\mathrm{Li}_2(1-e^{-x})$$
$$=\frac{\pi^2}{6}+x\left(\ln x-\frac{1}{2}x\right)-\left(x-\frac{1}{4}x^2\right)+O(x^3). \tag{B6}$$

Here, the small-x expansion of the sum in Eq. (B5) can be obtained from the relation

$$\mathrm{Li}_{-n}(e^{-x})=\sum_{k=1}^{\infty}k^n e^{-kx}=\left(-\frac{d}{dx}\right)^n\mathrm{Li}_0(e^{-x})$$
$$=\left(-\frac{d}{dx}\right)^n(e^x-1)^{-1}$$
$$=\left(-\frac{d}{dx}\right)^n\left\{\frac{1}{x}-\frac{1}{2}+\sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}x^{2m-1}\right\}$$
$$=\frac{n!}{x^{n+1}}\left[1-\frac{1}{2}x\,\delta_{n0}+o(x)\right]. \tag{B7}$$

The small-x expansion of $1/(e^x-1)$ has been obtained from Eq. (B4) with $x\to -x$.

By combining relations (B6) and (B7) with the small-x expansion of $\ln(1-e^{-x})$, we obtain from Eq. (B5),

$$S(x)=\frac{\pi^2}{6x}+\frac{1}{2}\ln(1-e^{-x})-\frac{1}{x}\mathrm{Li}_2(1-e^{-x})+\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)}\left(1-\frac{1}{2}x\,\delta_{m1}\right)$$
$$=\frac{\pi^2}{6x}+\frac{1}{2}\ln x-1+R_0-\frac{x}{24}+O(x^2), \tag{B8}$$

where $R_0\equiv\sum_{m=1}^{\infty}B_{2m}/[2m(2m-1)]$. As $|B_{2k}|\sim 2(2k)!/(2\pi)^{2k}$ for $k\to\infty$ [see Eqs. (23.2.16) and (23.2.18) of Ref. [29]], the series is a divergent asymptotic series with *alternating* signs. One obtains the greatest accuracy, denoted by $R_0^{(m_0)}$, if one breaks off the series just before the smallest term in the series, which is defined to be the (m_0+1)th term. Then the maximum error is $|B_{2m_0+2}/(2m_0+1)(2m_0+2)|$ [30]. In the present case, one can simply verify that $m_0=3$, and the best possible estimate for the remainder in the limit where $q\to 0$ is given by,

$$R_0=R_0^{(3)}\pm\frac{1}{56}|B_8|\simeq 0.081\,349\,2\pm 0.000\,595\,2. \tag{B9}$$

The inaccuracy in $S(x)$, caused by the inaccuracy in asymptotic series R_0, can be substantially reduced, if so desired, by restricting the m-sum in Eq. (B5) to $m_0=3$ terms and calculating the difference

$$\Delta(x)=\sum_{k=1}^{\infty}\frac{e^{-kx}}{k}\left[\frac{1}{1-e^{-kx}}-\frac{1}{kx}-\frac{1}{2}-\sum_{m=1}^{3}\frac{B_{2m}}{(2m)!}(kx)^{2m-1}\right], \tag{B10}$$

in the small-x limit as an integral. The result, for instance, to seven decimal points is $\Delta=-0.000\,287\,7$. Hence,

$$R_0=R_0^{(3)}+\Delta\simeq 0.081\,061\,5. \tag{B11}$$

Combination of results (B1), (B3), and (B8) gives the dominant small-x behavior of $B_n^{(3)}$ in the form

$$B_n^{(3)}=\frac{\pi^2}{6x}+\frac{1}{2}\ln x-\frac{3}{4}+R_0+\frac{1}{4}x\left(n+\frac{7}{12}\right). \tag{B12}$$

Final elimination of $x=2q(1+\frac{1}{2}q+\frac{1}{3}q^2+\cdots)$ in favor of q gives Eq. (4.9) in the main text.

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