Envelope solitons of acoustic plate modes and surface waves

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(Received 20 December 2002; published 9 June 2003)

The problem of the existence of evelope solitons in elastic plates and at solid surfaces covered by an elastic film is revisited with special attention paid to nonlinear long-wave short-wave interactions. Using asymptotic expansions and multiple scales, conditions for the existence of envelope solitons are established and it is shown how their parameters can be expressed in terms of the elastic moduli and mass densities of the materials involved. In addition to homogeneous plates, weak periodic modulation of the plate's material parameters are also considered. In the case of wave propagation in an elastic plate, modulations of weakly nonlinear carrier waves are governed by a coupled system of partial differential equations consisting of evolution equations for the complex amplitude of the carrier wave (the nonlinear Schrödinger equation for envelope solitons and the Mills-Trullinger equations for gap solitons), and the wave equation for long-wavelength acoustic plate modes. In contrast to this situation, envelope solitons of surface acoustic waves in a layered structure are normally described by the nonlinear Schrödinger equation alone. However, at higher orders of the carrier wave amplitude, the envelope soliton is found to be accompanied by a quasistatic long-wavelength strain field, which may be localized at the surface with penetration depth into the substrate of the order of the inverse amplitude or which may radiate energy into the bulk. A new set of modulation equations is derived for the resonant case of the carrier wave's group velocity being equal to the phase velocity of long-wavelength Rayleigh waves of the uncoated substrate.

DOI: 10.1103/PhysRevE.67.066603 PACS number(s): 43.25.+y, 05.45.Yv, 46.40.Cd

I. INTRODUCTION

Modulational instability of plane carrier waves, envelope solitons, and gap solitons have been discussed in various areas of physics, especially in nonlinear optics, where they have been generated and investigated in experiments. These phenomena have also been envisaged for guided acoustic waves in elastic plates and at solid surfaces ([1], for further references on surface acoustic envelope solitons see Ref. [2]), but have not yet been observed in experiments to our knowledge, while several experiments have been carried out on wave form evolution of an initially sinusoidal Rayleigh wave in the presence of dispersion [3–5] and recently on the evolution of short intense pulses on coated substrates [6,7]. An important feature of nonlinear acoustics in comparison to nonlinear optics is the unavoidable presence of second-order nonlinearity. (There is no acoustic analog of an optical Kerr medium.) In systems with third-order nonlinearity with second-order nonlinearity being absent, "slow" long-wavelength modulations of a sinusoidal carrier wave and the formation of envelope solitons are described by the nonlinear Schrödinger equation (NLS). The parameters occuring in the NLS follow from a nonlinear dispersion relation for weakly nonlinear sinusoidal waves [8].

However, in the presence of second-order nonlinearity, a nonlinear dispersion relation for plane waves is not always sufficient for the description of modulations of a weakly nonlinear carrier wave. Here, the carrier wave is coupled to quasistatic components of the wave field. It has been shown for nonlinear waves in various systems that the modulation equations are a NLS coupled to the wave equation or evolution equation for a long-wavelength low-frequency degree of freedom [9–14]. In the case of nonlinear waves described by the Benjamin-Ono equation, this additional degree of freedom has dramatic consequences as it gives rise to a cancellation of the dominant nonlinearity in the NLS [13].

In systems that have an additional periodic spatial variation of its properties, gaps may open up in the frequency spectrum of wave solutions of the corresponding linearized systems. In the presence of third-order nonlinearity, gap solitons may form that are described by a set of two coupled nonlinear evolution equations for the complex amplitudes of forward and backward propagating waves with wave vectors at the edge of the first Brillouin zone [17,18]. We shall call these evolution equations the Mills-Trullinger (MT) equations. It has recently been pointed out by Iizuka and Kivshar [19] that in the presence of second-order nonlinearity and nonresonant coupling between the carrier wave and its second harmonic, an extra quasistatic degree of freedom has to be accounted for explicitly, and the MT-equations are coupled to the wave equation for this additional degree of freedom. Iizuka and Kivshar have derived their modulation equations using an effectively one-dimensional treatment of the propagation of light in a medium with periodically varying dielectric constant. In Sec. IV, we shall derive the corresponding modulation equations for elastic plate modes.

The modulation equations for weakly nonlinear waves are usually derived by carrying out an asymptotic expansion of the wave field using multiple scales. The complex amplitude of the carrier wave is usually chosen to be of first order in the expansion parameter ε. In the following sections, we shall show that the quasistatic parts of the displacement field required by the second-order nonlinearity enter the expansion at differing orders of ε. For gap solitons in elastic plates,

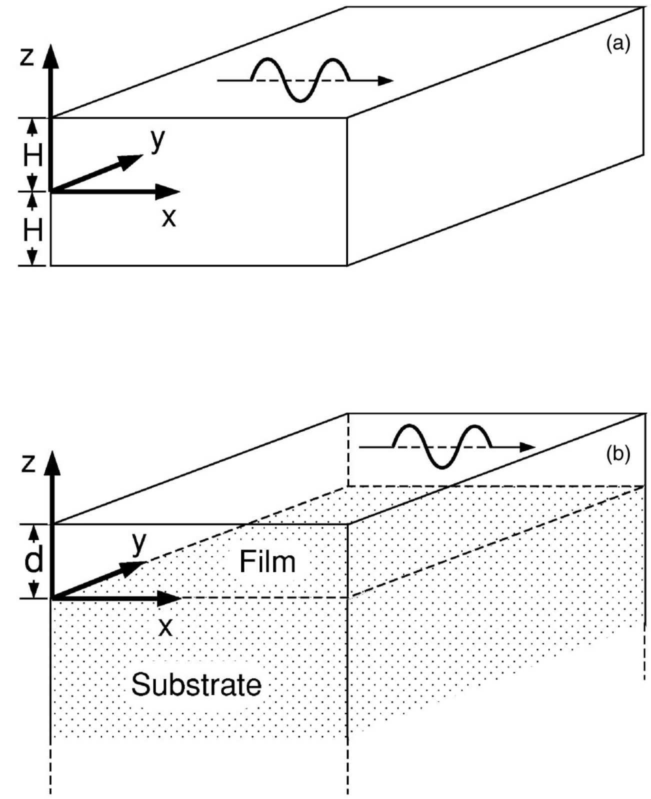

FIG. 1. Geometries: free-standing elastic plate (a), homogeneous elastic substrate covered by an elastic film (b).

they are of order $O(\varepsilon^0)$, for envelope solitons in elastic plates, they are of order $O(\varepsilon^1)$, and for envelope solitons of surface acoustic waves, they come into play at order $O(\varepsilon^2)$. This latter fact implies that they do not influence the spatiotemporal evolution of the complex amplitude of the carrier wave at leading order, and this evolution is still governed by the nonlinear Schrödinger equation.

However, there is a special case of resonant long-wave short-wave coupling of guided acoustic waves propagating at the surface of a coated elastic medium, namely, the group velocity associated with a short-wavelength guided carrier wave being equal to the velocity of the Rayleigh waves with wavelength very long such that the film only constitutes a small perturbation for their propagation characteristics. In the last section, we shall discuss this case in some detail and derive the corresponding modulation equations.

II. BASIC EQUATIONS

The propagation of acoustic waves is considered in elastic plates [Fig. 1(a)] and along the surface of a semi-infinte elastic medium covered by a film made of a material different from the substrate [Fig. 1(b)]. The coordinate system is chosen such that the z axis is normal to the surfaces. The elastic plate fills the spatial region $-H<z<H$, the substrate fills the halfspace $z<0$, and the coating film the region $0<z<d$. The displacement field $\mathbf{u}$ is assumed to depend on time t and on the spatial coordinates x and z, but to be independent of y. In the elastic media, it has to satisfy the equation of motion

$$\rho \frac{\partial^2}{\partial t^2} u_\alpha = T_{\alpha\beta,\beta}. \tag{2.1}$$

Here and in the following, Cartesian indices are denoted by lower-case Greek letters, and for derivatives with respect to the Cartesian coordinate x_β we use the short-hand notation $,\beta$. Also, we invoke the summation convention for repeated Cartesian indices. ρ is the mass density of the elastic medium, and the Kirchhoff-Piola stress tensor $(T_{\alpha\beta})$ may be expanded in powers of displacement gradients $u_{\alpha,\beta}$,

$$T_{\alpha\beta} = C_{\alpha\beta\,\mu\nu} u_{\mu,\nu} + \tfrac{1}{2} S_{\alpha\beta\,\mu\nu\,\zeta\xi} u_{\mu,\nu} u_{\zeta,\xi} + \tfrac{1}{6} S_{\alpha\beta\,\mu\nu\,\zeta\xi\,\lambda\sigma} u_{\mu,\nu} u_{\zeta,\xi} u_{\lambda,\sigma} + O(u^4_{\alpha,\beta}). \tag{2.2}$$

The elements of the sixth-rank tensor $(S_{\alpha\beta\,\mu\nu\,\zeta\xi})$ consist of linear combinations of second-order and third-order elastic moduli, while the elements of the eighth-rank tensor $(S_{\alpha\beta\,\mu\nu\,\zeta\xi\,\lambda\sigma})$ are linear combinations of second-order, third-order and fourth-order elastic moduli [16]. At a free surface (at $z=\pm H$ in the case of a plate and at $z=d$ in the second system under consideration), the boundary condition

$$T_{\alpha 3} = 0 \tag{2.3}$$

for $\alpha=1,2,3$ has to be satisfied. At the interface between film and substrate at $z=0$ [Fig. 1(b)], there are the two boundary conditions

$$T_{\alpha 3}|_{z=0_-} = T_{\alpha 3}|_{z=0_+}, \tag{2.4}$$

$$u_\alpha|_{z=0_-} = u_\alpha|_{z=0_+}. \tag{2.5}$$

When describing surface waves, we also require that the displacement field decays to zero for $z\to-\infty$. At higher orders of the expansion parameter ε in an asymptotic expansion of the displacement field, we may soften this latter requirement replacing it by Sommerfeld radiation conditions.

III. ENVELOPE SOLITONS OF PLATE MODES

To derive modulations of weakly nonlinear acoustic waves in an elastic plate, we follow usual practice and write the displacement field as an asymptotic expansion in powers of an expansion parameter $0<\varepsilon\ll 1$,

$$\mathbf{u}(x,z,t) = \varepsilon \mathbf{u}^{(1)}(x,z,t;X^{(1)},T^{(1)};\ldots) + \varepsilon^2 \mathbf{u}^{(2)}(x,z,t;X^{(1)},T^{(1)};\ldots) + O(\varepsilon^3) \tag{3.1}$$

with stretched coordinates $X^{(n)}=\varepsilon^n x$ and $T^{(n)}=\varepsilon^n t$. We take the first-order field to be of the form

$$\mathbf{u}^{(1)}(x,z,t;X^{(1)},T^{(1)};\ldots) = \{A(X^{(1)},T^{(1)};\ldots)\mathbf{W}(z)e^{i(qx-\omega t)} + \text{c.c.}\} + \mathbf{U}^{(1,0)}(X^{(1)},T^{(1)};\ldots), \tag{3.2}$$

consistent with the equation of motion and boundary conditions at first order of ε. Here, $\mathbf{W}(z)$ is the modal field of a linear plate mode with wave number q and frequency ω. This plate mode is the carrier wave and its modulations will be studied in the following. The quasistatic long-wavelength field contribution $\mathbf{U}^{(1,0)}$ does not appear in the first-order terms of the equations of motion and boundary conditions, but will be indispensible later to satisfy compatibility conditions at higher order.

The second-order field in Eq. (3.2) is conveniently decomposed as follows:

$$\begin{aligned}\mathbf{u}^{(2)}=&\{A^2(X^{(1)},T^{(1)};\dots)\mathbf{F}^{(2,2)}(z)e^{2i(qx-\omega t)}+\text{c.c.}\}\\&+\{\mathbf{U}^{(2,1)}(z;X^{(1)},T^{(1)};\dots)e^{i(qx-\omega t)}+\text{c.c.}\}\\&+|A(X^{(1)},T^{(1)};\dots)|^2\mathbf{F}^{(2,0)}(z)\\&+\mathbf{U}^{(2,0,0)}(X^{(1)},T^{(1)};\dots)+\mathbf{U}^{(2,0,1)}(X^{(1)},T^{(1)};\dots)z.\end{aligned} \tag{3.3}$$

When inserting Eq. (3.3) into the equation of motion and the boundary conditions at the two surfaces of the infinite plate, the following explicit solutions are obtained when making use of the linear independence of the factors $\exp[ni(qx-\omega t)]$ for $n=0,\pm1,\pm2$:

$$U_\alpha^{(2,0,1)}=-\Gamma_{\alpha\gamma}C_{\gamma3\beta1}\frac{\partial}{\partial X^{(1)}}U_\beta^{(1,0)}, \tag{3.4}$$

where $(\Gamma_{\alpha\beta})$ is the matrix inverse of the 3×3-matrix $(C_{\alpha3\,\beta3})$.

$$F_{\alpha,3}^{(2,0)}=-\Gamma_{\alpha\beta}S_{\beta3\,\mu\nu\,\zeta\xi}[D_\nu(iq)W_\mu][D_\xi(iq)W_\zeta]^*, \tag{3.5}$$

and we have defined the operator $D_\alpha(iq)=\delta_{\alpha1}iq+\delta_{\alpha3}\partial/\partial z$. A compatibility condition requires that A depends on the stretched coordinates $X^{(1)}$ and $T^{(1)}$ only via $\xi^{(1)}=X^{(1)}-V_gT^{(1)}$, where $V_g=\partial\omega/\partial q$ is the group velocity. (For simplicity, we assume here that the symmetry of the propagation geometry is such that the vector of the group velocity is pointing into the same direction as the wave vector of the carrier wave.) One then finds

$$U_\alpha^{(2,1)}=-i\frac{\partial}{\partial\xi^{(1)}}A\frac{\partial}{\partial q}W_\alpha \tag{3.6}$$

apart from a term that may be absorbed in $\mathbf{u}^{(1)}$.

If 2ω is not close to the frequency of a plate mode with wave number $2q$, the inhomogeneous linear boundary value problem for $\mathbf{F}^{(2,2)}$ has a unique solution that we shall not determine here explicitly.

At order $O(\varepsilon^3)$, we may decompose

$$\begin{aligned}\mathbf{u}^{(3)}=&\sum_{n=1}^{3}\{\mathbf{U}^{(3,n)}(z;\xi^{(1)},T^{(1)};\dots)e^{ni(qx-\omega t)}+\text{c.c.}\}\\&+\mathbf{U}^{(3,0)}(X^{(1)},T^{(1)};\dots).\end{aligned} \tag{3.7}$$

Inserting this into the equation of motion and boundary conditions up to third order in ε and making again use of the linear independence of the exponential factors $\exp[ni(qx-\omega t)]$, this time for $n=0,\pm1,\pm2,\pm3$, four inhomogeneous linear boundary value problems are obtained (for $n=0,1,2,3$). If there is no resonance with a plate mode at the third harmonic frequency and wave number, only two of these boundary value problems are singular, namely, that corresponding to $n=0$ and $n=1$ and require solvability conditions. These two solvability conditions are the desired modulation equations and can be brought into the following form:

$$\kappa\left\{i\frac{\partial}{\partial T^{(2)}}+\frac{1}{2}\frac{\partial^2\omega}{\partial q^2}\frac{\partial^2}{\partial\xi^{(1)2}}\right\}A-N|A|^2A-M_\alpha A\frac{\partial}{\partial X^{(1)}}U_\alpha^{(1,0)}=0, \tag{3.8}$$

$$2H\left\{\rho\delta_{\alpha\beta}\frac{\partial^2}{\partial T^{(1)2}}-[C_{\alpha1\,\beta1}-C_{\alpha1\,\mu3}\Gamma_{\mu\nu}C_{\nu3\,\beta1}]\times\frac{\partial^2}{\partial X^{(1)2}}\right\}U_\beta^{(1,0)}=M_\alpha\frac{\partial}{\partial\xi^{(1)}}|A|^2. \tag{3.9}$$

In Eqs. (3.8) and (3.9), we have introduced the real coefficients

$$M_\alpha=\{S_{\alpha1\,\mu\nu\,\zeta\xi}-C_{\alpha1\,\sigma3}\Gamma_{\sigma\lambda}S_{\lambda3\,\mu\nu\,\zeta\xi}\}\times\int_{-H}^{H}[D_\nu(iq)W_\mu(z)][D_\xi(iq)W_\zeta(z)]^*dz, \tag{3.10}$$

$$\begin{aligned}N=&S_{\alpha\beta\,\mu\nu\,\zeta\xi}\int_{-H}^{H}[D_\nu(iq)W_\mu(z)]^*\{[D_\xi(iq)W_\zeta(z)]^*\\&\times[D_\beta(2iq)F_\alpha^{(2,2)}(z)]+[D_\xi(iq)W_\zeta(z)]\\&\times[D_\beta(0)F_\alpha^{(2,0)}(z)]\}dz+\frac{1}{2}S_{\alpha\beta\,\mu\nu\,\zeta\xi\,\sigma\lambda}\\&\times\int_{-H}^{H}[D_\nu(iq)W_\mu(z)]^*[D_\xi(iq)W_\zeta(z)]^*\\&\times[D_\beta(iq)W_\alpha(z)][D_\lambda(iq)W_\sigma(z)]dz,\end{aligned} \tag{3.11}$$

$$\kappa=2\omega\int_{-H}^{H}\rho W_\alpha^*(z)W_\alpha(z)dz. \tag{3.12}$$

A strong simplification is achieved if the medium has cubic symmetry and the axes of the coordinate system are along the cubic axes. In this case, we may classify the carrier wave as sagittal or shear horizontal. In the latter case, modes corresponding to the lowest branch of the dispersion relation have to be excluded as this branch is nondispersive. The only nonzero component of the vector $\mathbf{M}$ is M_1 and, consequently, the carrier wave is only coupled to $U_1^{(1,0)}$, i.e., a long-wavelength longitudinal plate mode. This applies to both sagittal and shear-horizontal carrier waves. In terms of the variables A and $\partial U_1^{(1,0)}/\partial X^{(1)}$, Eqs. (3.8) and (3.9) become the Zakharov equations [12].

Letting now $U_1^{(1,0)}$ depend on $X^{(1)}$ and $T^{(1)}$ via $\xi^{(1)}$ like A, one may eliminate the variable $U_1^{(1,0)}$ from the modulation equations (3.8) and (3.9) by integrating Eq. (3.9) once and inserting the result for $\partial U_1^{(1,0)}/\partial \xi^{(1)}$ into Eq. (3.8) to obtain the NLS

$$\left\{ i\frac{\partial}{\partial T^{(2)}} + \frac{1}{2}\frac{\partial^2 \omega}{\partial q^2}\frac{\partial^2}{(\partial \xi^{(1)})^2} \right\} A - \overline{N}|A|^2 A = 0 \tag{3.13}$$

with effective nonlinear coupling coefficient

$$\overline{N} = \frac{1}{\kappa}\left\{ N + \frac{M_1^2}{2H\rho(V_g^2 - C_L^2)} \right\}, \tag{3.14}$$

where $C_L = \sqrt{(c_{11}^2 - c_{12}^2)/(c_{11}\rho)}$ is the phase velocity of the longitudinal plate mode. The second term in the curly bracket of Eq. (3.14) has been missing in Eq. (3.4) of Ref. [15]. Depending on the slope V_g of the plate mode dispersion curves, it can have either sign and influence the existence criterion (Lighthill criterion) for envelope solitons.

The case of a resonance at the second harmonic frequency has already been discussed in Refs. [15,20]. If there is a resonance at the third harmonic, i.e., if there is a plate mode having frequency 3ω and wave number $3q$, the complex amplitude B of this waveguide mode has to be taken as an extra independent degree of freedom. Since B may be chosen to be of second order in ε, the modulation equations (3.8) and (3.9) will not be affected.

IV. GAP SOLITONS OF PLATE MODES

We now consider periodic variations of the material properties of the plate. To keep the following derivations as simple as possible, we shall assume that it is only the mass density of the plate that varies, $\rho(x) = \rho_0 + 2\overline{\rho_1}\cos(2qx)$. A generalization to periodic variations of the elastic moduli or periodic corrugation of the plate's surfaces is straightforward.

The periodic variation of the density introduces frequency gaps in the dispersion relation of linear plate modes, and one may expect that nonlinearity leads to spatially localized excitations having frequencies in these gaps in the same way as have been found long ago in the field of nonlinear optics [17,18].

To have the effects of the periodic variation of the same order as those of the nonlinearity, we use the scaling $\overline{\rho_1} = \varepsilon^2 \rho_1$. Furthermore, we introduce stretched coordinates $\xi = \varepsilon^2 x$ and $\tau = \varepsilon^2 t$. The displacement field is then written as an asymptotic expansion of form (3.1) with, however, an additional term of order $O(\varepsilon^0)$, $\mathbf{U}^{(0)}$, which only depends on stretched coordinates. The first-order term in this expansion is chosen as a superposition of two counterpropagating linear plate modes with wave vectors at the edges of the first Brillouin zone introduced by the periodic density variation,

$$\mathbf{u}^{(1)}(x,z,t;\xi,\tau) = A_+(\xi,\tau)e^{i(qx-\omega t)}\mathbf{W}(z) + A_-(\xi,\tau)e^{i(-qx-\omega t)}\mathbf{W}^*(z) + \text{c.c.} \tag{4.1}$$

The second-order term in the asymptotic expansion may be decomposed as follows:

$$\begin{aligned}\mathbf{u}^{(2)}(x,z,t;\xi,\tau) = \{&\mathbf{F}^{(2,2+)}(z)A_+^2(\xi,\tau)e^{2i(qx-\omega t)} \\ &+ \mathbf{F}^{(2,2-)}(z)A_-^2(\xi,\tau)e^{2i(-qx-\omega t)} \\ &+ \mathbf{F}^{(2,20)}(z)A_+(\xi,\tau)A_-(\xi,\tau)e^{-2i\omega t} \\ &+ \mathbf{F}^{(2,0+)}(z)A_+(\xi,\tau)A_-^*(\xi,\tau)e^{2iqx} + \text{c.c.}\} \\ &+ \mathbf{U}^{(2,0)}(z;\xi,\tau).\end{aligned} \tag{4.2}$$

To simplify the derivations, we now consider the special case of an elastic medium of cubic symmetry with the cubic axes along the axes of the coordinate system. Furthermore, the first-order displacement field $\mathbf{u}^{(1)}$ is chosen to have shear-horizontal polarization. This means that $\mathbf{W}$ has the simple form $W_\alpha(z) = \delta_{\alpha 2} 2\cos(p[z-H])$ with p being a positive integer multiple of $\pi/(2H)$ and consequently $\rho_0\omega^2 = c_{44}(q^2 + p^2)$. Generalization to arbitrary symmetry and polarization is straightforward.

The second-order field $\mathbf{u}^{(2)}$ is of sagittal polarization. The functions $F_1^{(2,j\sigma)}$ and $F_3^{(2,j\sigma)}$, $j=0,2$, $\sigma = -,0,+$, are solutions of the ordinary differential equations

$$\left[\rho_0\Omega_{j\sigma}^2 - c_{11}Q_{j\sigma}^2 + c_{44}\frac{\partial^2}{\partial z^2}\right]F_1^{(2,j\sigma)}(z) + (c_{12}+c_{44})iQ_{j\sigma} \times \frac{\partial}{\partial z}F_3^{(2,j\sigma)}(z) = I_1^{(j\sigma)}(z), \tag{4.3a}$$

$$\left[\rho_0\Omega_{j\sigma}^2 - c_{44}Q_{j\sigma}^2 + c_{11}\frac{\partial^2}{\partial z^2}\right]F_3^{(2,j\sigma)}(z) + (c_{12}+c_{44})iQ_{j\sigma} \times \frac{\partial}{\partial z}F_1^{(2,j\sigma)}(z) = I_3^{(j\sigma)}(z), \tag{4.3b}$$

where $\Omega_{2\sigma} = 2\omega$, $\Omega_{0\sigma} = 0$, $Q_{2\pm} = \pm 2q$, $Q_{20} = 0$. [The case $(j\sigma) = (0,0)$ does not occur.] The inhomogeneities are linear combinations of $\cos^2(p[z-H])$, $\sin^2(p[z-H])$, $\sin(p[z-H])\cos(p[z-H])$ involving the coupling coefficients $S_{11\,21\,21} = S_{33\,23\,23} = c_{166} + c_{11}$, $S_{11\,23\,23} = S_{33\,21\,21} = c_{144} + c_{12}$, $S_{31\,21\,23} = S_{13\,21\,23} = c_{456} + c_{44}$.

At $z = \pm H$, the functions $F_1^{(2,j\sigma)}$ and $F_3^{(2,j\sigma)}$ have to satisfy the boundary conditions

$$c_{44}\left[\frac{\partial}{\partial z}F_1^{(2,j\sigma)} + iQ_{j\sigma}F_3^{(2,j\sigma)}\right]_{z=\pm H} = 0, \tag{4.4a}$$

$$\left[c_{11}\frac{\partial}{\partial z}F_3^{(2,j\sigma)} + c_{12}iQ_{j\sigma}F_1^{(2,j\sigma)}\right]_{z=\pm H} = J_3^{(j\sigma)}, \tag{4.4b}$$

where $J_3^{(j\sigma)}$ are constants. Apart from special resonant situations, which can be avoided by choosing q appropriately, the four boundary-value problems are nonsingular and may be solved in a straightforward way.

The functions $U_1^{(2,0)}(z;\xi,\tau)$, $U_3^{(2,0)}(z;\xi,\tau)$ have to satisfy

$$c_{44}\frac{\partial^2}{\partial z^2}U_1^{(2,0)}(z;\xi,\tau)=0, \tag{4.5a}$$

$$c_{11}\frac{\partial^2}{\partial z^2}U_3^{(2,0)}(z;\xi,\tau)=-\frac{\partial}{\partial z}\,4\{S_{33\,21\,21}q^2\cos^2(p[z-H]) + S_{33\,23\,23}p^2\sin^2(p[z-H])\}\times\{|A_+(\xi,\tau)|^2+|A_-(\xi,\tau)|^2\}, \tag{4.5b}$$

with boundary conditions

$$-c_{44}\left[\frac{\partial}{\partial z}U_1^{(2,0)}\right]_{z=\pm H}=c_{44}\frac{\partial}{\partial \xi}U_3^{(0)}, \tag{4.6a}$$

$$\left[-c_{11}\frac{\partial}{\partial z}U_3^{(2,0)}\right]_{z=\pm H} = c_{12}\frac{\partial}{\partial\xi}U_1^{(0)}+4\{S_{33\,21\,21}q^2\cos^2(p[z-H]) + S_{33\,23\,23}p^2\sin^2(p[z-H])\}_{z=\pm H}\{|A_+|^2+|A_-|^2\}. \tag{4.6b}$$

Integrating Eq. (4.5) and obeying Eq. (4.6) leads to

$$\frac{\partial}{\partial z}U_1^{(2,0)}=-\frac{\partial}{\partial\xi}U_3^{(0)}, \tag{4.7a}$$

$$\frac{\partial}{\partial z}U_3^{(2,0)}=-\frac{c_{12}}{c_{11}}\frac{\partial}{\partial\xi}U_1^{(0)}-\frac{4}{c_{11}}\{S_{33\,21\,21}q^2\cos^2(p[z-H]) + S_{33\,23\,23}p^2\sin^2(p[z-H])\}\{|A_+|^2+|A_-|^2\}. \tag{4.7b}$$

The third-order part of the displacement field is analogously decomposed as

$$\mathbf{u}^{(3)}=\sum_{j,\ell=-3}^{3}\mathbf{U}^{(3,j\ell)}(z;\xi,\tau)\exp[i(jqx-\ell\omega t)], \tag{4.8}$$

where $\mathbf{U}^{(3,j\ell)}=\mathbf{U}^{(3,-j-\ell)*}$.

Equating to zero the prefactors of $\exp[i(jqx-\ell\omega t)]$ in the equations of motion and boundary conditions, one may obtain the functions $\mathbf{U}^{(3,j\ell)}(z)$ as solutions of inhomogeneous linear boundary-value problems. These boundary-value problems are singular only for $j,\ell=\pm1$ and $j=\ell=0$. In the case $\ell=1$, $j=\pm1$, we obtain the ordinary differential equations

$$-\left\{\rho_0\omega^2-c_{44}q^2+c_{44}\frac{\partial^2}{\partial z^2}\right\}U_2^{(3,\pm1\,1)} = \left\{4i\omega\rho_0\frac{\partial}{\partial\tau}A_\pm\mp4iqc_{44}\frac{\partial}{\partial\xi}A_\pm+2\rho_1\omega^2A_\mp\right\}\times\cos(p[z-H])+\left\{iqG_1^{(1,\pm)}(z)+\frac{\partial}{\partial z}G_3^{(1,\pm)}(z)\right\}\times|A_\pm|^2A_\pm+\left\{iqG_1^{(2,\pm)}(z)+\frac{\partial}{\partial z}G_3^{(2,\pm)}(z)\right\}\times|A_\mp|^2A_\pm+(iq)^2 2\cos(p[z-H])\times\left\{S_{21\,21\,11}\frac{\partial}{\partial\xi}U_1^{(0)}+S_{21\,21\,33}\frac{\partial}{\partial z}U_3^{(2,0)}(z)\right\}\times A_\pm+\frac{\partial}{\partial z}\left\{-2p\sin(p[z-H])\left[S_{23\,23\,11}\frac{\partial}{\partial\xi}U_1^{(0)} + S_{23\,23\,33}\frac{\partial}{\partial z}U_3^{(2,0)}(z)\right]\right\}A_\pm. \tag{4.9}$$

The corresponding boundary conditions are

$$\left[-c_{44}\frac{\partial}{\partial z}U_2^{(3,\pm1\,1)}\right]_{z=\pm H} = [G_3^{(1,\pm)}|A_\pm|^2A_\pm+G_3^{(2,\pm)}|A_\mp|^2A_\pm]_{z=\pm H} -\left[2p\sin(p[z-H])\left(S_{23\,23\,11}\frac{\partial}{\partial\xi}U_1^{(0)} + S_{23\,23\,33}\frac{\partial}{\partial z}U_3^{(2,0)}\right)A_\pm\right]_{z=\pm H}. \tag{4.10}$$

In Eqs. (4.9) and (4.10), we have made use of the fact that the component $U_3^{(0)}$ does not couple to the amplitudes $A_\pm$ and may be chosen to be zero. The functions $G_\alpha^{(j,\pm)}(z), j=1,2,\ \alpha=1,3$ are not specified here.

By multiplying Eq. (4.9) with $\cos(p[z-H])$, integrating over z from $-H$ to $+H$ and making use of the boundary conditions, the following condition for the solvability of the inhomogeneous problem, Eqs. (4.9) and (4.10), is obtained:

$$2i\omega\rho_0\left\{\frac{\partial}{\partial\tau}\pm V_g\frac{\partial}{\partial\xi}\right\}A_\pm+\rho_1\omega^2A_\mp = [N_1|A_\pm|^2+N_2|A_\mp|^2]A_\pm+MA_\pm\frac{\partial}{\partial\xi}U_1^{(0)}. \tag{4.11}$$

The coefficients N_1 and N_2 depend on cubic anharmonic coupling coefficients $S_{2\alpha\,2\beta\,\alpha\beta}$, $\alpha,\beta\in\{1,3\}$ in a bilinear way and linearly on the quartic anharmonic coefficient $S_{21\,21\,21\,21}$. The coefficient M is explicitly given by

$$M=q^2\left[S_{21\,21\,11}-\frac{c_{12}}{c_{11}}S_{21\,21\,33}\right]+p^2\left[S_{23\,23\,11}-\frac{c_{12}}{c_{11}}S_{23\,23\,33}\right]. \tag{4.12}$$

In the boundary-value problem for $\mathbf{U}^{(3,0)}$, the inhomogeneity vanishes and $\mathbf{U}^{(3,0)}$ remains undetermined at this stage.

The fourth-order field $\mathbf{u}^{(4)}$ may be decomposed into Fourier components in a way similar to Eq. (4.8),

$$\mathbf{u}^{(4)}=\sum_{j,\ell=-4}^{4}\mathbf{U}^{(4,j\ell)}(z;\xi,\tau)\exp[i(jqx-\ell\omega t)]. \tag{4.13}$$

Special attention has to be paid to the equations that have to be satisfied by the sagittal components of $\mathbf{U}^{(4,00)}$,

$$-c_{44}\frac{\partial^2}{\partial z^2}U_1^{(4,00)}(z)=(c_{12}+c_{44})\frac{\partial}{\partial\xi}\frac{\partial}{\partial z}U_3^{(2,0)}(z)-\left\{\rho_0\frac{\partial^2}{\partial\tau^2}-c_{11}\frac{\partial^2}{\partial\xi^2}\right\}U_1^{(0)}+\frac{\partial}{\partial z}R_1(z)+4\{S_{11\,21\,21}q^2\cos^2(p[z-H])+S_{11\,23\,23}p^2\sin^2(p[z-H])\}\frac{\partial}{\partial\xi}[|A_+|^2+|A_-|^2], \tag{4.14a}$$

$$-c_{11}\frac{\partial^2}{\partial z^2}U_3^{(4,00)}(z)=(c_{12}+c_{44})\frac{\partial}{\partial\xi}\frac{\partial}{\partial z}U_1^{(2,0)}(z)+\frac{\partial}{\partial z}R_3(z). \tag{4.14b}$$

The corresponding boundary conditions are

$$\left[-c_{44}\frac{\partial}{\partial z}U_1^{(4,00)}\right]_{z=\pm H}=\left[c_{44}\frac{\partial}{\partial\xi}U_3^{(2,0)}+R_1\right]_{z=\pm H}, \tag{4.15a}$$

$$\left[-c_{11}\frac{\partial}{\partial z}U_3^{(4,00)}\right]_{z=\pm H}=\left[c_{12}\frac{\partial}{\partial\xi}U_1^{(2,0)}+R_3\right]_{z=\pm H}. \tag{4.15b}$$

Explicit expressions for the functions R_1 and R_3 are not needed here. The compatibility conditions of this boundary-value problem are obtained by integrating Eq. (4.14) over the cross section of the plate and using the boundary conditions (4.15). When doing this for Eq. (4.14a), we obtain

$$\left\{\rho_0\frac{\partial^2}{\partial\tau^2}-\left(c_{11}-\frac{c_{12}^2}{c_{11}}\right)\frac{\partial^2}{\partial\xi^2}\right\}U_1^{(0)}=2M\frac{\partial}{\partial\xi}[|A_+|^2+|A_-|^2]. \tag{4.16}$$

It is the last term on the right-hand side of Eq. (4.14a) that requires the presence of the quasistatic zero-order field $U_1^{(0)}$, while the second compatibility condition contains $U_3^{(0)}$, which may then be chosen to be zero.

The coupled modulation equations (4.11) and (4.16) are the plate mode analogs of Eqs. (4)–(6) in Ref. [19] that have been derived for an optical system. After rescaling, they take the simple form

$$i\left(\frac{\partial}{\partial t}\pm\frac{\partial}{\partial x}\right)B_\pm+B_\mp=(|B_\pm|^2+h|B_\mp|^2)B_\pm+mB_\pm\frac{\partial}{\partial x}U, \tag{4.17}$$

$$\left\{\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right\}U=m\frac{\partial}{\partial x}[|B_+|^2+|B_-|^2] \tag{4.18}$$

with real parameters h,m,c. Stationary or moving solitary solutions can be derived in the same way as demonstrated by Iizuka and Kivshar for their equations, namely, with the ansatz

$$B_\pm(x,t)=\Delta^{\mp1/2}f(\zeta)\exp[i(\theta_{1,2}(\zeta)-\Omega t+\theta_0)] \tag{4.19}$$

and $U(x,t)=u(\zeta)$, where $\zeta=x-Vt$ and V is the velocity of the solitary wave. Making use of the spatial localization of the solitary solution, one obtains from Eq. (4.18)

$$\frac{\partial}{\partial\zeta}u=\frac{m[\Delta+(1/\Delta)]}{V^2-c^2}f^2(\zeta). \tag{4.20}$$

Inserting Eqs. (4.19) and (4.20) into Eq. (4.17), one is led to the same ordinary differential equations for $\theta_1\pm\theta_2$ and f as have been solved in Ref. [19].

The modulation equations (4.17) and (4.18) can be derived from the Lagrangian density

$$\ell=i\left\{B_+^*\frac{\partial}{\partial\tau}B_++B_-^*\frac{\partial}{\partial\tau}B_-+B_+^*\frac{\partial}{\partial\xi}B_+-B_-^*\frac{\partial}{\partial\xi}B_-\right\}+B_-^*B_++B_+^*B_--\frac{1}{2}(|B_+|^4+|B_-|^4)-h|B_+|^2|B_-|^2-m(|B_+|^2+|B_-|^2)\frac{\partial}{\partial\xi}U+\frac{1}{2}\left(\frac{\partial}{\partial\tau}U\right)^2-c^2\frac{1}{2}\left(\frac{\partial}{\partial\xi}U\right)^2. \tag{4.21}$$

Identifying continuous symmetries and using Noether's theorem, the following four conserved quantities may be derived:

$$\text{energy}\quad E=\int\left\{\ell-iB_+^*\frac{\partial}{\partial\tau}B_+-iB_-^*\frac{\partial}{\partial\tau}B_--\left(\frac{\partial}{\partial\tau}U\right)^2\right\}d\xi,$$

$$\text{momentum}\quad P=\int\left\{iB_+^*\frac{\partial}{\partial\xi}B_++iB_-^*\frac{\partial}{\partial\xi}B_-+\left(\frac{\partial}{\partial\tau}U\right)\times\left(\frac{\partial}{\partial\xi}U\right)\right\}d\xi,$$

$$\text{the ``mass''}\quad N=\int\{|B_+|^2+|B_-|^2\}d\xi,$$

and the quantity

$$\int \frac{\partial}{\partial \tau} U \, d\xi.$$

This fourth conserved quantity is associated with translational invariance of the plate along the x direction and follows immediately from Eq. (4.18), while the corresponding conservation law for the system of modulation equations studied in Ref. [19] is more difficult to derive.

When looking for stationary solutions of the system of Eqs. (4.17) and (4.18), the second time derivative in Eq. (4.18) vanishes, and the variable U may be eliminated to yield the Mills-Trullinger equations

$$i\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right) B_\pm + B_\mp = \left[\left(1 - \frac{m^2}{c^2}\right)|B_\pm|^2 + \left(h - \frac{m^2}{c^2}\right)|B_\mp|^2\right]. \tag{4.22}$$

However, the time-derivative term in Eq. (4.18) influences the stability properties of the stationary solitary solutions. This becomes evident when Eqs. (4.17) and (4.18) are linearized around a solitary solution leading to a non-Hermitian eigenvalue problem. The dominant instability of the solitary wave solutions of the Mills-Trullinger equations (4.22) is of oscillatory character, which is associated with eigenvalues that have a small real part (λ', the growth rate) and a large imaginary part (λ'', a frequency) [21]. At distances far away from the solitary wave, the eigenvector is a linear combination of almost plane waves with complex frequency $\lambda'' - i\lambda'$ and wave vectors having a small imaginary part, too. This may be interpreted as the solitary wave decaying via the emission of radiation. In the presence of coupling to the degree of freedom U, radiation is expected to occur via this channel, too, i.e., via sound waves with wave vector $k \approx \lambda''/c$. We note, in addition, that translational instabilities can occur only in exceptional cases, if at all, since there is no gap in the spectrum of the acoustic field U.

In the derivation of the modulation equations (4.17) and (4.18), we have assumed that there is no resonance of the Brillouin zone center modes and higher harmonics. This means, in particular, that there is no linear plate mode of sagittal polarization that has frequency $\approx 2\omega$ and (1) wave vector $\approx \pm 2q$ or (2) wave vector ≈ 0. The latter assumption means that the frequency of the homogeneous thickness vibration of the plate is not close to 2ω. The case of assumption (1) not being satisfied has been discussed in Ref. [22] for nonlinear acoustic waves propagating in an elastic film covering a semi-infinite substrate and earlier in optical contexts [23–26]. When assumption (2) is not satisfied, one is led to the three coupled evolution equations analyzed by Mak, Malomed, and Chu [27] with parameter $D=0$. (In order to obtain the second-derivative term ($D\neq 0$), an additional stretched coordinate would have to be introduced.)

V. SURFACE ACOUSTIC ENVELOPE SOLITONS

A. General case

After having discussed modulations of weakly nonlinear waves in a free-standing elastic plate, we now consider nonlinear guided acoustic waves in a plate in contact with a semi-infinite elastic medium [Fig. 1(b)].

In order to describe long-wavelength modulations of surface acoustic waves in a layered structure, we start with the asymptotic expansion (3.1) of the displacement field, where now

$$\mathbf{u}^{(1)}(x,z,t) = A\mathbf{W}(z|q)e^{i(qx-\omega t)} + \text{c.c.} \tag{5.1}$$

The complex amplitude A depends on stretched coordinates $X^{(1)}, T^{(1)}, \ldots$ and $\mathbf{W}(z|q)\exp[i(qx-\omega t)]$ is the displacement field of a linear surface acoustic wave with wave vector $\mathbf{q} = (q,0)$ and frequency ω. The depth profile of a surface acoustic wave, $\mathbf{W}(z|q)$, is chosen such that $\mathbf{W}(z|-q) = \mathbf{W}^*(z|q)$ and $\mathbf{W}(0|q)$ does not depend on the modulus of q.

For the second-order field $\mathbf{u}^{(2)}$, we use the decomposition (3.3) with $\mathbf{U}^{(2,0,1)}=0$. When using this decomposition, we have assumed that there is no waveguide mode with wave vector $(2q,0)$ and frequency 2ω. Then, $\mathbf{F}^{(2,2)}$ is the solution of a nonsingular linear boundary value problem satisfying the boundary condition $\mathbf{F}^{(2,2)}(z)\to 0$ as $z\to -\infty$. For $\mathbf{F}^{(2,0)}(z)$ we obtain again (3.5). With the requirement $\mathbf{F}^{(2,0)}(z)\to 0$ as $z\to -\infty$, $\mathbf{F}^{(2,0)}$ is then uniquely determined. Explicitly,

$$F_\alpha^{(2,0)}(z) = -\int_{-\infty}^{z} \Gamma_{\alpha\beta}(z') S_{\beta 3\, \mu\nu\, \zeta\xi}(z') [D_\nu(iq) W_\mu(z')] \times [D_\xi(iq) W_\zeta(z')]^*, \tag{5.2}$$

where $\Gamma_{\alpha\beta}(z)$ and $S_{\beta 3\, \mu\nu\, \zeta\xi}(z)$ are step functions taking their corresponding values in the film and substrate regions.

In the same way as in the case of plate modes, the compatibility condition in the linear inhomogeneous boundary value problem for $\mathbf{U}^{(2,1)}$ requires that A depends on $X^{(1)}$ and $T^{(1)}$ only via $\xi^{(1)} = X^{(1)} - V_g T^{(1)}$, where V_g is the group velocity of the linear surface wave with wave number q, and the explicit solution for $\mathbf{U}^{(2,1)}$ is Eq. (3.6).

The quantity $\mathbf{U}^{(2,0)}$ depends on stretched coordinates only. In order to be able to satisfy the boundary conditions at $z\to -\infty$, we have to introduce an additional stretched coordinate $Z^{(1)} = \varepsilon z$. The dependence of $\mathbf{U}^{(2,0)}$ on the stretched coordinates will emerge at higher orders of ε, as we shall show below. Only the displacement field in the substrate region depends on $Z^{(1)}$, while in the film, terms proportional to positive powers of z are allowed as has been the case in a free-standing plate.

The third-order field $\mathbf{u}^{(3)}$ has form (3.7) with $\mathbf{U}^{(3,n)}(z)$ for $n = \pm 2, \pm 3$ being solutions of nonsingular linear boundary-value problems, assuming that there is no resonance of the second and third harmonic with a linear waveguide mode. The solvability condition for $\mathbf{U}^{(3,1)}$ yields the nonlinear Schrödinger equation (3.13), where now $\bar{N} = N/(2\kappa)$ and N and κ are defined in Eqs. (3.11) and (3.12), respectively. In the latter two expressions, the integrals over z have now to be extended from $-\infty$ to d rather than from $-H$ to H.

We now turn to the quasistatic contribution $\mathbf{U}^{(3,0)}$ of the third-order part of the displacement field. It has to satisfy the differential equation

$$-C_{\alpha 3\ \beta 3}U^{(3,0)}_{\beta,33} = \frac{\partial}{\partial z}J_{\alpha} + \{S_{\alpha 1\ \mu\nu\ \zeta\xi}[D_{\nu}(iq)W_{\mu}][D_{\xi}(iq)W_{\zeta}]^{*} + (C_{\alpha 1\ \beta 3}+C_{\alpha 3\ \beta 1})F^{(2,0)}_{\beta,3}\}\frac{\partial}{\partial X^{(1)}}|A|^2, \quad (5.3)$$

subject to the following boundary conditions:

$$\left[C_{\alpha 3\ \beta 3}U^{(3,0)}_{\beta,3}+J_{\alpha}+C_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(1)}}\{F^{(2,0)}_{\beta}|A|^2+U^{(2,0,0)}_{\beta}\} + C_{\alpha 3\ \beta 3}\frac{\partial}{\partial Z^{(1)}}U^{(2,0,0)}_{\beta}\right]_{z=0_-} = \left[C_{\alpha 3\ \beta 3}U^{(3,0)}_{\beta,3}+J_{\alpha}+C_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(1)}} \times\{F^{(2,0)}_{\beta}|A|^2+U^{(2,0,0)}_{\beta}\}\right]_{z=0_+}, \quad (5.4a)$$

$$\left[C_{\alpha 3\ \beta 3}U^{(3,0)}_{\beta,3}+J_{\alpha}+C_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(1)}}\{F^{(2,0)}_{\beta}|A|^2+U^{(2,0,0)}_{\beta}\}\right]_{z=d} = 0. \quad (5.4b)$$

In addition, the three components of $\mathbf{U}^{(3,0)}$ have to be continuous at the interface. The quantities $J_{\alpha}, \alpha=1,2,3$ are not given here explicitly. They decay into the substrate exponentially as functions of z.

We now decompose $\mathbf{U}^{(3,0)}(z)=\mathbf{U}^{(3,0,0)}+\mathbf{U}^{(3,0,1)}z\,\theta(z)+\mathbf{U}^{(3,0,E)}(z)$, where $\theta(z)$ is the Heavyside step function and $\mathbf{U}^{(3,0,E)}(z)$ decays into the substrate exponentially and is defined below. The quantities $\mathbf{U}^{(3,0,0)}$ and $\mathbf{U}^{(3,0,1)}$ depend on stretched coordinates only, and

$$\frac{\partial}{\partial z}U^{(3,0,E)}_{\alpha}(z) = -\Gamma_{\alpha\beta}(z)\Bigg\{J_{\beta}(z)+\int_{-\infty}^{z}dz'\{S_{\beta 1\ \mu\nu\ \zeta\xi}(z') \times[D_{\nu}(iq)W_{\mu}(z')][D_{\xi}(iq)W_{\zeta}(z')]^{*} + [C_{\alpha 1\ \beta 3}(z')+C_{\alpha 3\ \beta 1}(z')] \times F^{(2,0)}_{\beta,3}(z')\}\frac{\partial}{\partial X^{(1)}}|A|^2\Bigg\}. \quad (5.5)$$

Inserting this into the boundary conditions, we are led to the following solvability conditions:

$$\left[C^{(S)}_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(1)}}U^{(2,0,0)}_{\beta}+C^{(S)}_{\alpha 3\ \beta 3}\frac{\partial}{\partial Z^{(1)}}U^{(2,0,0)}_{\beta}\right]_{Z^{(1)}=0} = \lambda_{\alpha}\frac{\partial}{\partial X^{(1)}}|A|^2 \quad (5.6)$$

with real coefficients

$$\lambda_{\alpha} = \int_{-\infty}^{d}dz\,[S_{\alpha 1\ \mu\nu\ \zeta\xi}(z)-C_{\alpha 1\ \gamma 3}(z)\Gamma_{\gamma\beta}(z)S_{\beta 3\ \mu\nu\ \zeta\xi}(z)] \times[D_{\nu}(iq)W_{\mu}(z)][D_{\xi}(iq)W_{\zeta}(z)]^{*}. \quad (5.7)$$

The superscript (S) refers to the substrate. Since Eq. (5.6) follows from the boundary conditions at the surface and interface, the argument $Z^{(1)}$ in $U^{(2,0,0)}_{\beta}$ on the left-hand side of Eq. (5.6) has to be set equal to zero.

At fourth order of the expansion parameter ε, the quasistatic part of the equation of motion for the displacement field in the substrate yields for depths much larger than the carrier wavelength:

$$\left\{\delta_{\alpha\beta}\rho^{(S)}\frac{\partial^2}{\partial T^{(1)2}}-C^{(S)}_{\alpha 1\ \beta 1}\frac{\partial^2}{\partial X^{(1)2}}-C^{(S)}_{\alpha 3\ \beta 3}\frac{\partial^2}{\partial Z^{(1)2}} -(C^{(S)}_{\alpha 1\ \beta 3}+C^{(S)}_{\alpha 3\ \beta 1})\frac{\partial^2}{\partial X^{(1)}\partial Z^{(1)}}\right\}U^{(2,0,0)}_{\beta}=0. \quad (5.8)$$

Obviously, Eq. (5.8) are the linear equations of motion in the substrate involving only stretched coordinates. Equation (5.6) has to be regarded as boundary conditions for the variables $U^{(2,0,0)}_{\alpha}$, $\alpha=1,2,3$, at the substrate surface, $Z^{(1)}=0$. A further boundary condition that has to be imposed is that $\mathbf{U}^{(2,0,0)}$ either decays exponentially as $Z^{(1)}\to-\infty$ or energy is radiated into the substrate. Since the amplitude of the carrier wave, A, depends on $X^{(1)}$ and $T^{(1)}$ only through $\xi^{(1)}=X^{(1)}-V_gT^{(1)}$ with V_g being the group velocity, the solution of the linear boundary-value problem posed by Eqs. (5.8) and (5.6) may be written in the form

$$\mathbf{U}^{(2,0,0)}(X^{(1)},Z^{(1)},T^{(1)}) = \int_{-\infty}^{\infty}\frac{dK}{2\pi}\exp[iK\xi^{(1)}]\sum_{r=1}^{3}\mathbf{b}(r)\exp[K\hat{\alpha}(r)Z^{(1)}]I(K), \quad (5.9)$$

where

$$I(K)=\int_{-\infty}^{\infty}d\xi^{(1)}|A(\xi^{(1)})|^2\exp(-iK\xi^{(1)}). \quad (5.10)$$

The vectors $\mathbf{b}(r)$ and coefficients $\hat{\alpha}(r)$ depend on V_g and the sign of K in Eq. (5.9). The coefficients $\hat{\alpha}(r)$ either have a positive real part or are purely imaginary. In the latter case, the acoustic Poynting vector associated with the plane wave $\exp[iK(X^{(1)}-i\hat{\alpha}(r)Z^{(1)}-V_gT^{(1)})]\mathbf{b}(r)$ is directed into the substrate. The physical interpretation of these findings is as fol-

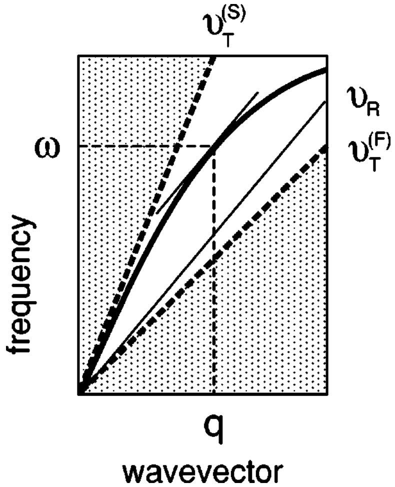

FIG. 2. Illustration of the resonance condition of long-wavelength Rayleigh waves and short-wavelength Love waves belonging to the lowest branch. Dispersion curve of Love waves (thick solid) and Rayleigh waves of the substrate (thin solid). $v_T^{(S)}, v_T^{(F)}$: velocity of transverse sound waves of the substrate and film material, respectively.

lows: An envelope pulse of a carrier wave with short wavelength $2\pi/q$ is accompanied by a wave packet consisting of long-wavelength Fourier components. Depending on whether the group velocity of the carrier wave is smaller or larger than a limiting value (the velocity of transversely polarized bulk waves in the case of isotropic substrates), this wave packet is either localized at the surface or corresponds to radiation into the bulk of the substrate. As this long-wavelength component appears at second order of the expansion parameter ε in our asymptotic expansion of the displacement field, it does not enter the evolution equation for the envelope of the carrier wave, and the latter is the nonlinear Schrödinger equation, as has been found in earlier works (for a review see Ref. [2]).

When V_g approaches the phase velocity v_R of the Rayleigh waves propagating on the uncoated substrate, $|\mathbf{b}(r)|$ diverge and the boundary-value problem, Eqs. (5.8) and (5.6), becomes singular. We note that the case $V_g = v_R$ is rather special, but can occur even in a highly dispersive acoustic slab waveguide without inducing a resonance at the second harmonic of the carrier wave's frequency. For example, such a resonance occurs for a certain value of qd on the lowest branch of the Love waves, if substrate and film are both isotropic materials and the velocity of transverse bulk waves in the film material is smaller than that of the Rayleigh waves propagating at the surface of the uncoated substrate (Fig. 2). For the higher-order Love wave branches, this resonance occurs if the Rayleigh wave velocity of the substrate is smaller than the velocity of transverse bulk waves of the film material. Nonlinear interactions of long waves and short waves with the group velocity of the short wave being equal to the phase velocity of the long wave has been discussed from a general point of view and for different examples by Benney [10,11]. We shall now give a detailed derivation of the corresponding modulation equations for the case of surface acoustic waves, which are found to exhibit a specific feature.

In order to avoid secular terms in $X^{(1)}$ or $T^{(1)}$, a term $U^{(1,0)}$ has to be added to the right-hand side of Eq. (5.1). It is continuous across the interface. In the film, it depends only on stretched coordinates $X^{(n)}$, $T^{(n)}$, $n=1,2,\ldots$, and in the substrate, it also depends on the stretched depth coordinates $Z^{(n)}$. The second-order part of the displacement field is of form (3.3) with the modification that $\mathbf{U}^{(2,0,1)}$ is nonzero only in the film region, while in the spatial region filled by the substrate, $\mathbf{U}^{(2,0,0)}$ depends on $Z^{(n)}$ in addition to $X^{(n)}$ and $T^{(n)}$, $n=1,2,\ldots$. The quantities $\mathbf{U}^{(2,1)}$, $\mathbf{F}^{(2,2)}$, and $\mathbf{F}^{(2,0)}$ are determined in the same way as before. In particular, $\mathbf{F}^{(2,0)}$ is given by Eq. (5.2). Inserting Eq. (5.2) into the second-order boundary conditions, we obtain at the interface

$$\left[C^{(S)}_{\alpha 3\,\beta 1}\frac{\partial}{\partial X^{(1)}}U^{(1,0)}_\beta + C^{(S)}_{\alpha 3\,\beta 3}\frac{\partial}{\partial Z^{(1)}}U^{(1,0)}_\beta\right]_{Z^{(1)}=0} = \left[C^{(F)}_{\alpha 3\,\beta 1}\frac{\partial}{\partial X^{(1)}}U^{(1,0)}_\beta + C^{(F)}_{\alpha 3\,\beta 3}U^{(2,0,1)}_\beta\right]_{Z^{(1)}=0}, \tag{5.11}$$

and at the surface

$$\left[C^{(F)}_{\alpha 3\,\beta 1}\frac{\partial}{\partial X^{(1)}}U^{(1,0)}_\beta + C^{(F)}_{\alpha 3\,\beta 3}U^{(2,0,1)}_\beta\right]_{Z^{(1)}=0}, \tag{5.12}$$

which implies

$$\left[C^{(S)}_{\alpha 3\,\beta 1}\frac{\partial}{\partial X^{(1)}}U^{(1,0)}_\beta + C^{(S)}_{\alpha 3\,\beta 3}\frac{\partial}{\partial Z^{(1)}}U^{(1,0)}_\beta\right]_{Z^{(1)}=0} = 0 \tag{5.13}$$

and

$$U^{(2,0,1)}_\alpha = -\Gamma^{(F)}_{\alpha\beta}C^{(F)}_{\beta 3\,\gamma 1}\frac{\partial}{\partial X^{(1)}}U^{(1,0)}_\gamma . \tag{5.14}$$

At depths $|z| \gg 1/q$ the equations of motion at third order of ε lead to

$$\left\{\delta_{\alpha\beta}\rho^{(S)}\frac{\partial^2}{\partial T^{(1)2}} - C^{(S)}_{\alpha 1\,\beta 1}\frac{\partial^2}{\partial X^{(1)2}} - C^{(S)}_{\alpha 3\,\beta 3}\frac{\partial^2}{\partial Z^{(1)2}} - (C^{(S)}_{\alpha 1\,\beta 3} + C^{(S)}_{\alpha 3\,\beta 1})\frac{\partial^2}{\partial X^{(1)}\partial Z^{(1)}}\right\}U^{(1,0)}_\beta = 0. \tag{5.15}$$

Equation (5.15) together with Eq. (5.13) are the equations of motion and boundary conditions for a displacement field in a homogeneous substrate with uncoated stress-free surface. Their general solution localized at the surface is a superposition of (generalized) Rayleigh waves:

$$\mathbf{U}^{(1,0)}=\int\frac{dK}{2\pi}\exp(iK\xi^{(1)})\mathbf{W}(Z^{(1)}|K)\times C(K;T^{(2)},X^{(2)},Z^{(2)};\ldots). \tag{5.16}$$

Here, $\mathbf{W}(Z^{(1)}|K)$ is the depth profile of a linear Rayleigh wave with two-dimensional wave vector $(K,0)$ and the amplitudes $C(K)$ depend on stretched coordinates $T^{(n)},X^{(n)},Z^{(n)}$ with $n\geqslant 2$.

The third-order contribution to the displacement field may again be decomposed as $\mathbf{u}^{(3)}=\Sigma_{n=1}^{3}\{\mathbf{U}^{(3,n)}(z)\exp[ni(qx-\omega t)]+\text{c.c.}\}+\mathbf{U}^{(3,0)}(z)$. In addition to z, the functions $\mathbf{U}^{(3,n)}$, $n=0,1,2,3$, depend on the stretched coordinates $T^{(m)},X^{(m)}$, $m=1,2,\ldots$, and, in the substrate, also on $Z^{(m)}$. In the spatial region filled by the film material, we decompose

$$\begin{aligned}\mathbf{U}^{(3,0)}(z;T^{(1)},X^{(1)};\ldots)&=\mathbf{U}^{(3,0,0)}(T^{(1)},X^{(1)};\ldots)\\&\quad+z\mathbf{U}^{(3,0,1)}(T^{(1)},X^{(1)};\ldots)\\&\quad+\tfrac{1}{2}z^2\mathbf{U}^{(3,0,2)}(T^{(1)},X^{(1)};\ldots)\\&\quad+\mathbf{U}^{(3,0,E)}(z).\end{aligned}\tag{5.17}$$

In the substrate region, $\mathbf{U}^{(3,0)}=\mathbf{U}^{(3,0,0)}(T^{(1)},X^{(1)},Z^{(1)};\ldots)+\mathbf{U}^{(3,0,E)}(z)$, i.e., $\mathbf{U}^{(3,0,0)}$ depends on the stretched depth coordinates $Z^{(m)}$. The quantity $\partial\mathbf{U}^{(3,0,E)}(z)/\partial z$ is given by Eq. (5.5). $\mathbf{U}^{(3,0,E)}(z)$ is uniquely determined by requiring that it decays exponentially for $z\to-\infty$ and is continuous at the interface.

We now examine the quasistatic part of the boundary conditions at third order of ε, which is similar to Eq. (5.4) but contains some additional terms. On the left-hand side of Eq. (5.4a) (the boundary condition at the interface), we have to add

$$\left[C^{(S)}_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(2)}}U^{(1,0)}_{\beta}+C^{(S)}_{\alpha 3\ \beta 3}\frac{\partial}{\partial Z^{(2)}}U^{(1,0)}_{\beta}\right]_{z=0_-}\tag{5.18a}$$

and on the right-hand side of Eq. (5.4a),

$$\left[C^{(F)}_{\alpha 3\ \beta 1}\frac{\partial}{\partial X^{(2)}}U^{(1,0)}_{\beta}\right]_{z=0_+}.\tag{5.18b}$$

To the left-hand side of Eq. (5.4b) (the boundary condition at the surface) the following terms have to be added:

$$C^{(F)}_{\alpha 3\ \beta 1}\left\{\frac{\partial}{\partial X^{(2)}}U^{(1,0)}_{\beta}+d\frac{\partial}{\partial X^{(1)}}U^{(2,0,1)}_{\beta}\right\}.\tag{5.18c}$$

From the equation of motion in the film region we obtain

$$C^{(F)}_{\alpha 3\ \beta 3}U^{(3,0)}_{\beta,33}=\rho^{(F)}\frac{\partial^2}{\partial T^{(1)2}}U^{(1,0)}_{\alpha}-C^{(F)}_{\alpha 1\ \beta 1}\frac{\partial}{\partial X^{(1)2}}U^{(1,0)}_{\beta}-(C^{(F)}_{\alpha 3\ \beta 1}+C^{(F)}_{\alpha 1\ \beta 3})\frac{\partial}{\partial X^{(1)}}U^{(2,0,1)}_{\beta}.\tag{5.19}$$

Combining the boundary conditions (5.4) with Eqs. (5.18), making use of Eqs. (5.14) and (5.19), the following relation is obtained, which plays the role of a boundary condition for the fourth-order equation of motion:

$$\begin{aligned}&\left[C^{(S)}_{\alpha 3\ \beta 1}\left\{\frac{\partial}{\partial X^{(2)}}U^{(1,0)}_{\beta}+\frac{\partial}{\partial X^{(1)}}U^{(2,0,0)}_{\beta}\right\}+C^{(S)}_{\alpha 3\ \beta 3}\left\{\frac{\partial}{\partial Z^{(2)}}U^{(1,0)}_{\beta}+\frac{\partial}{\partial Z^{(1)}}U^{(2,0,0)}_{\beta}\right\}\right]_{Z^{(1)}=0}\\&=d\left[(C^{(F)}_{\alpha 1\ \beta 1}-C^{(F)}_{\alpha 1\ \mu 3}\Gamma^{(F)}_{\mu\nu}C^{(F)}_{\nu 3\ \beta 1})\frac{\partial^2}{\partial X^{(1)2}}U^{(1,0)}_{\beta}-\rho^{(F)}\times\frac{\partial^2}{\partial T^{(1)2}}U^{(1,0)}_{\alpha}\right]_{Z^{(1)}=0}+\lambda_{\alpha}\frac{\partial}{\partial X^{(1)}}|A|^2.\end{aligned}\tag{5.20}$$

In the absence of $\mathbf{U}^{(2,0,0)}$ and A, Eq. (5.20) is easily recognized as the well-known effective boundary conditions from which the influence of the film on the dispersion relation of the Rayleigh waves can be calculated to leading order in the ratio of film thickness and wavelength [29,30]. This boundary condition together with the equation of motion at fourth order of ε will provide an equation for the evolution of the amplitudes $C(K)$ of the long-wavelength Rayleigh waves. At depths $|z|\gg 1/q$, the fourth-order equation of motion yields

$$\begin{aligned}&\left\{\delta_{\alpha\beta}\rho^{(S)}\frac{\partial^2}{\partial T^{(1)2}}-C^{(S)}_{\alpha 1\ \beta 1}\frac{\partial^2}{\partial X^{(1)2}}-C^{(S)}_{\alpha 3\ \beta 3}\frac{\partial^2}{\partial Z^{(1)2}}-(C^{(S)}_{\alpha 3\ \beta 1}+C^{(S)}_{\alpha 1\ \beta 3})\frac{\partial^2}{\partial X^{(1)}\partial Z^{(1)}}\right\}U^{(2,0,0)}_{\beta}\\&+2\left\{\delta_{\alpha\beta}\rho^{(S)}\frac{\partial^2}{\partial T^{(1)}\partial T^{(2)}}-C^{(S)}_{\alpha 1\ \beta 1}\frac{\partial^2}{\partial X^{(1)}\partial X^{(2)}}-C^{(S)}_{\alpha 3\ \beta 3}\frac{\partial^2}{\partial Z^{(1)}\partial Z^{(2)}}-(C^{(S)}_{\alpha 3\ \beta 1}+C^{(S)}_{\alpha 1\ \beta 3})\times\frac{1}{2}\left(\frac{\partial^2}{\partial X^{(1)}\partial Z^{(2)}}+\frac{\partial^2}{\partial Z^{(1)}\partial X^{(2)}}\right)\right\}U^{(1,0)}_{\beta}=0.\end{aligned}\tag{5.21}$$

We are looking for solutions $\mathbf{U}^{(2,0,0)}$ that depend on $X^{(1)},T^{(1)}$ via the combination $\xi^{(1)}=X^{(1)}-V_gT^{(1)}$. Consequently, we may represent $\mathbf{U}^{(2,0,0)}$ in the substrate region as a Fourier integral of the form

$$\mathbf{U}^{(2,0,0)}=\int\frac{dK}{2\pi}\exp(iK\xi^{(1)})\mathbf{g}(Z^{(1)}|K).\tag{5.22}$$

For $\mathbf{U}^{(1,0)}$ and $|A|^2$, we use the Fourier representations (5.10) and (5.16).

We now apply the projection method [28] to the system of equations of motion (5.21) and boundary conditions (5.20): We multiply Eq. (5.21) by $W^{*}_{\alpha}(Z^{(1)}|K)\exp(-iK\xi^{(1)})$, sum over α, and integrate over $\xi^{(1)}$ and $Z^{(1)}$. Integrating twice by parts and making use of Eq. (5.20) and the boundary conditions satisfied by $\mathbf{U}^{(1,0)}$ at $Z^{(1)}=0$, one may eliminate the field $\mathbf{U}^{(2,0,0)}$ and obtain the following result:

$$i\left\{\kappa_R\,\mathrm{sgn}(K)\left[\frac{\partial}{\partial T^{(2)}}+V_g\frac{\partial}{\partial X^{(2)}}\right]+\sigma\frac{\partial}{\partial Z^{(2)}}\right\}C(K)$$
$$=-dMK^2C(K)-W_\alpha^*(0|K)\lambda_\alpha iKI(K). \tag{5.23}$$

The coefficient κ_R is an integral similar to Eq. (3.12),

$$\kappa_R=2\rho^{(S)}v_R|K|\int_{-\infty}^0 W_\alpha^*(Z|K)W_\alpha(Z|K)\,dZ. \tag{5.24}$$

The coefficient M,

$$M=W_\alpha^*(0|K)\{\delta_{\alpha\beta}\rho^{(S)}V_g^2-C^{(S)}_{\alpha 1\ \beta 1}$$
$$+C^{(S)}_{\alpha 1\ \mu 3}\Gamma^{(S)}_{\mu\nu}C^{(S)}_{\nu 3\ \beta 1}\}W_\beta(0|K), \tag{5.25}$$

does not depend on K and enters the dispersion relation between frequency ω and wave vector K of the linear Rayleigh waves of the coated substrate for wavelengths much longer than the film thickness,

$$\omega(K)=v_RK\left[1-\frac{Md}{v_R\kappa_R}|K|+O([Kd]^2)\right]. \tag{5.26}$$

The coefficient σ in front of the derivative with respect to $Z^{(2)}$,

$$\sigma=2\int_{-\infty}^0\left\{C^{(S)}_{\alpha 3\ \beta 3}\ \mathrm{Im}\left[W_\alpha^*(Z^{(1)}|K)\frac{\partial}{\partial Z^{(1)}}W_\beta(Z^{(1)}|K)\right]\right.$$
$$\left.+C^{(S)}_{\alpha 3\ \beta 1}\mathrm{Im}[W_\alpha^*(Z^{(1)}|K)iKW_\beta(Z^{(1)}|K)]\right\}dZ, \tag{5.27}$$

vanishes, since the integrand is proportional to the three-component of the time-averaged energy flux associated with a Rayleigh wave with wave vector $(K,0)$. Defining the quantity

$$U_0(\xi^{(1)};T^{(2)},\xi^{(2)};\ldots)$$
$$=\int\frac{dK}{2\pi}iK\lambda_\alpha W_\alpha(0|K)C(K;T^{(2)},X^{(2)},0;\ldots)$$
$$\times\exp(iK\xi^{(1)}), \tag{5.28}$$

multiplying Eq. (5.23) by $|K|\lambda_\alpha W_\alpha(0|K)\exp(iK\xi^{(1)})$ and integrating over K yields

$$\kappa_R\frac{\partial}{\partial T^{(2)}}U_0+dM\frac{\partial^2}{\partial\xi^{(1)2}}\hat{H}[U_0]=\bar{F}\frac{\partial^2}{\partial\xi^{(1)2}}\hat{H}[A^2], \tag{5.29}$$

where $\bar{F}=|\lambda_\alpha W_\alpha(0|K)|^2$, which is independent of K.

$$\hat{H}[U](\xi)=\frac{P}{\pi}\int\frac{U(\xi')}{\xi'-\xi}d\xi' \tag{5.30}$$

is the Hilbert transform, P indicating the Cauchy principle value. As has become evident from the above derivation, the appearance of the Hilbert transform is due to the peculiar depth structure of the long-wavelength Rayleigh wave.

The quasistatic long-wavelength field $\mathbf{U}^{(1,0)}$ enters the contribution to the third-order equation of motion and boundary conditions through the derivatives $[\partial\mathbf{U}^{(1,0)}/\partial X^{(1)}]_{Z^{(1)}=0}$ and $[\partial\mathbf{U}^{(1,0)}/\partial Z^{(1)}]_{Z^{(1)}=0}$. Eliminating the latter in favor of the former by using Eq. (5.13) and proceeding in the usual way, we finally obtain

$$\left\{i\frac{\partial}{\partial T^{(2)}}+\frac{1}{2}\frac{\partial^2\omega}{\partial q^2}\frac{\partial^2}{(\partial\xi^{(1)})^2}\right\}A-(N/\kappa)|A|^2A-(1/\kappa)AU_0$$
$$=0. \tag{5.31}$$

The coefficients N and κ are again defined in Eqs. (3.11) and (3.12), where the integrals have to be extended from $-\infty$ to d.

The coupled system of Eqs. (5.31) and (5.29) are the modulation equations for a surface acoustic carrier wave with group velocity equal to that of long-wavelength Rayleigh waves. Obviously, this system of equations has solitary wave solutions corresponding to envelope pulses of sech^2 form that travel with the group velocity V_g and have $U_0=0$. Whether there are pulse solutions traveling with a different velocity and whether these are stable remains to be clarified in subsequent work.

To our knowledge, Maugin, Hadouaj, and Malomed [31] were the first to derive coupled evolution equations for long and short waves in the context of surface acoustic waves. They considered shear-horizontal carrier waves and found their modulations governed by the Zakharov equations, which differ from the modulation equations (5.29) and (5.31).

B. Thin-film/small-mismatch limit for Rayleigh waves

In a frequency regime where the film thickness is smaller than the typical wavelengths, nonlinear evolution of the Rayleigh waves is governed by an evolution equation of the following form:

$$i\frac{\partial}{\partial\tau}B(k,\tau)=k\left[\int_0^k F(k'/k)B(k',\tau)B(k-k',\tau)\frac{dk'}{2\pi}\right.$$
$$+2\int_k^\infty(k/k')F^*(k/k')B(k',\tau)$$
$$\left.\times B^*(k'-k,\tau)\frac{dk'}{2\pi}\right]+D(k)B(k,\tau) \tag{5.32}$$

with $k>0$ (see, for example, Refs. [32,7] and corresponding earlier work [33–38]) and $B(k)$ is the Fourier transform of a displacement gradient component at the surface, e.g., $u_{3,1}$:

$$u_{3,1}(x,0,t)=\int_0^\infty B(k,\tau)\exp[ik(x-v_Rt)]\frac{dk}{2\pi}+\text{c.c.} \tag{5.33}$$

and τ is a stretched time coordinate. Normally, $D(k) = \alpha_R k^2$ with coefficient α_R determined by the linear acoustic mismatch between film and substrate. In special cases of acoustic mismatch, $D(k) \propto k^3$. If the second-order elastic moduli and/or the mass density are varying continuously near the surface with only small relative deviations from their bulk values, Eq. (5.32) is also applicable. The function $D(k)$ may then have a more complicated dependence on k with $D(k) \propto k^2$ as $k \to 0$.

The continuous and bounded function F depends on the second-order and third-order elastic moduli of the substrate only. [In deriving Eq. (5.32), the nonlinearity of the film material has been neglected.] In terms of the real-space variable $u_{3,1}$, the evolution equation (5.32) would exhibit a highly nonlocal nonlinearity. Equation (5.32) may be compared to the KdV equation, which takes on the form

$$i\frac{\partial}{\partial \tau}B(k,\tau) = kF_0\left[\int_0^k B(k',\tau)B(k-k',\tau)\frac{dk'}{2\pi} + 2\int_k^\infty B(k',\tau)B^*(k'-k,\tau)\frac{dk'}{2\pi}\right] + \alpha_0 k^3 B(k,\tau) \tag{5.34}$$

with real constant F_0 after Fourier-transform with respect to the spatial coordinate.

In order to highlight the difference between the modulation equations for the KdV equation (5.34) and the Rayleigh wave evolution equation (5.32), we sketch the derivation of the modulation equations for a carrier wave with wave number q in the weakly nonlinear regime of Eq. (5.34). We expand the Fourier amplitudes $B(k)$ in powers of a expansion parameter η ($0 < \eta \ll 1$),

$$B(k) = \eta B^{(1)}(k) + \eta^2 B^{(2)}(k) + O(\eta^3), \tag{5.35}$$

and introduce further stretched coordinates $T^{(m)} = \eta^m \tau$. "Slow" variations in real space are associated with small wave numbers $k^{(m)} = \eta^m k_m$ [39,13], where k_m/q is of order $O(\eta^0)$. Consequently, we write

$$B^{(m)}(k,\tau) = \sum_{n=-\infty}^{\infty} B^{(m,n)}(k_1,T^{(1)};k_2,T^{(2)};\ldots) \times \exp(-in\omega_0\tau) \tag{5.36}$$

and define $\omega_0 = \alpha_0 q^3$, $V_g = 3\alpha_0 q^2$. Collecting terms of equal order in the expansion parameter η, we find

$$B^{(1,1)}(\tau;k_1,T^{(1)};\ldots) = \tilde{A}(k_1;T^{(2)};\ldots) \times \exp(-iV_g k_1 T^{(1)} - iV_g k_2 T^{(2)}), \tag{5.37}$$

$$B^{(2,2)}(k_1,T^{(1)};\ldots) = -\frac{F_0}{3\alpha_0 q^2}\int B^{(1,1)}(k_1',T^{(1)};\ldots) \times B^{(1,1)}(k_1-k_1',T^{(1)};\ldots)\frac{dk_1'}{2\pi}. \tag{5.38}$$

At third order of η, the following system of coupled equations is obtained:

$$i\frac{\partial}{\partial T^{(2)}}\tilde{A}(k_1;T^{(2)}) = 3\alpha_0 q k_1^2 \tilde{A}(k_1;T^{(2)}) - \frac{2F_0^2}{3\alpha_0 q}\int\int \tilde{A}^*(k_1'-k_1;T^{(2)}) \times \tilde{A}(k_1'';T^{(2)})\tilde{A}(k_1'-k_1'';T^{(2)})\frac{dk_1'}{2\pi}\frac{dk_1''}{2\pi} + 2qF_0\int \tilde{A}(k_1-k_1';T^{(2)})\tilde{B}^{(2,0)}(k_1',T^{(1)};T^{(2)})\frac{dk_1'}{2\pi}, \tag{5.39}$$

$$\left\{i\frac{\partial}{\partial T^{(1)}} + V_g k_1\right\}\tilde{B}^{(2,0)}(k_1,T^{(1)};T^{(2)}) = 2k_1F_0\int \tilde{A}^*(k_1';T^{(2)})\tilde{A}(k_1'+k_1;T^{(2)})\frac{dk_1'}{2\pi}, \tag{5.40}$$

where

$$\tilde{B}^{(n,0)}(k_1,T^{(1)};T^{(2)}) = B^{(n,0)}(k_1,T^{(1)};T^{(2)}) \times \exp(iV_g k_1 T^{(1)} + iV_g k_2 T^{(2)}).$$

Transforming these two equations back into real space,

$$A(\xi^{(1)},T^{(2)}) = \int \frac{dk_1}{2\pi}\tilde{A}(k_1;T^{(2)})\exp(ik_1\xi^{(1)}), \tag{5.41}$$

$$U_0(\xi^{(1)},T^{(1)};T^{(2)}) = \int \frac{dk_1}{2\pi}\tilde{B}^{(2,0)}(k_1,T^{(1)};T^{(2)})\exp(ik_1\xi^{(1)}), \tag{5.42}$$

one gets the familiar set of coupled modulation equations [13]:

$$0 = i\frac{\partial}{\partial T^{(2)}}A + 3\alpha_0 q\frac{\partial^2}{\partial \xi^{(1)2}}A + \frac{2F_0^2}{3\alpha_0 q}|A|^2 A - 2qF_0 A U_0, \tag{5.43}$$

$$\frac{\partial}{\partial T^{(1)}}U_0 = -2F_0\frac{\partial}{\partial \xi^{(1)}}|A|^2. \tag{5.44}$$

Going now through the same arguments for the evolution equation (5.32) instead of the KdV equation, we find that due to factor k/k' in the second nonlinear term in Eq. (5.32), the nonlinearity in the quasistatic part of the evolution equation enters only at fourth order of the expansion parameter η. Consequently, one obtains instead of Eq. (5.40),

$$\left\{ i\frac{\partial}{\partial T^{(1)}} + V_g k_1 \right\} \tilde{B}^{(2,0)}(k_1, T^{(1)}; T^{(2)}) = 0 \qquad (5.45)$$

and may set $\tilde{B}^{(2,0)} = 0$. Here V_g is the deviation of the group velocity associated with the carrier wave from the Rayleigh wave velocity v_R. At the next order of η,

$$\left\{ i\frac{\partial}{\partial T^{(1)}} + V_g k_1 \right\} \tilde{B}^{(3,0)}(k_1, T^{(1)}; T^{(2)}) = 2\frac{k_1^2}{q}F^*(0)\int \tilde{A}^*(k_1'; T^{(2)})\tilde{A}(k_1' + k_1; T^{(2)})\frac{dk_1'}{2\pi}. \qquad (5.46)$$

As long as $V_g \neq 0$, Eq. (5.46) may be solved for $\tilde{B}^{(3,0)}$ without involving secular terms in $T^{(1)}$. However, $\tilde{B}^{(3,0)}$ does not enter the third-order equation corresponding to Eq. (5.39), which now takes the form

$$i\frac{\partial}{\partial T^{(2)}}\tilde{A}(k_1; T^{(2)}) = \frac{1}{2}\left[\frac{\partial^2 D(k)}{\partial k^2}\right]_{k=q} k_1^2 \tilde{A}(k_1; T^{(2)}), - \frac{2|F(1/2)|^2}{D(2q) - 2D(q)}\int\int \tilde{A}^*(k_1'; T^{(2)})\tilde{A}(k_1''; T^{(2)}) \times \tilde{A}(k_1' - k_1''; T^{(2)})\frac{dk_1'}{2\pi}\frac{dk_1''}{2\pi}. \qquad (5.47)$$

After Fourier transform, this becomes the nonlinear Schrödinger equation (3.13) with ω replaced by D and

$$\bar{N} = \frac{2|F(1/2)|^2}{2D(q) - D(2q)}. \qquad (5.48)$$

It was shown in Ref. [2] that the expression for N/κ in Eq. (5.31) derived in the previous subsection converges to Eq. (5.48) in the limit of small dispersion.

The special case $V_g = 0$ corresponds to the equality of group velocity of the carrier wave and phase velocity of the long-wavelength Rayleigh waves. In order to avoid secular terms that would arise in Eq. (5.46), $B^{(2,0)}$ is again needed, and one obtains as a compatibility condition for $k_1 > 0$:

$$0 = -i\frac{\partial}{\partial T^{(2)}}B^{(2,0)}(k_1; T^{(2)}) + \frac{1}{2}\left[\frac{\partial^2 D(k)}{\partial k^2}\right]_{k=0} k_1^2 B^{(2,0)}(k_1; T^{(2)}) + 2F^*(0)\frac{k_1^2}{q}\int \frac{dk_1'}{2\pi}\tilde{A}(k_1'; T^{(2)})\tilde{A}^*(k_1' - k_1; T^{(2)}). \qquad (5.49)$$

On the right-hand side of Eq. (5.47), the following term has to be added:

$$+2q\int_0^{\cdots}\frac{dk_1'}{2\pi}\{F(0)B^{(2,0)}(k_1'; T^{(2)})\tilde{A}(k_1 - k_1'; T^{(2)}) + F^*(0)B^{(2,0)*}(k_1'; T^{(2)})\tilde{A}(k_1 + k_1'; T^{(2)})\}. \qquad (5.50)$$

Equations (5.49) and (5.47) with the additional term (5.50) lead to the set of coupled modulation equations (5.29) and (5.31), which are thus obtained in an independent way in the limit of a thin film or small acoustic mismatch between substrate and film.

VI. CONCLUSION

The goal of this paper was to derive envelope equations that govern gradual modulations of weakly nonlinear waves in acoustic waveguides. Particular attention has been paid to the interaction of a short-wavelength carrier wave with long-wavelength components of the displacement field. Three example systems have been treated in detail: (1) a homogeneous elastic plate, (2) an elastic plate with certain material properties such as its mass density varying periodically along the direction of wave propagation, and (3) surface acoustic waves on a substrate coated by a film consisting of a material different from that of the substrate. In case (1), the nonlinear Schrödinger equation for the complex amplitude of the carrier wave is nonlinearly coupled to the wave equation for the real amplitude of the quasilongitudinal plate mode. This system of modulation equations (3.8) and (3.9) is the well-known Zakharov system [9,12,31] (or an extended version of it in the general case). In case (2), we have derived a system of three coupled evolution equations (4.17) and (4.18) that are extensions of the well-known gap soliton equations [18] and the acoustic analogs of corresponding equations derived recently by Iizuka and Kivshar [19] for an optical system. For weakly nonlinear surface acoustic waves (system 3), we have shown that their modulations are normally governed by a single nonlinear Schrödinger equation. However, there is a resonant long-wave short-wave interaction, when the group velocity of the carrier wave is equal to the phase velocity of long-wavelength Rayleigh waves in the absence of the film. For this resonant situation, the coupled modulation equations (5.29) and (5.31) have been derived. These involve Hilbert transforms that typically occur in the context of surface acoustic waves.

ACKNOWLEDGMENT

Financial support from the INTAS program (Grant No. INTAS-99 N 0167) is gratefully acknowledged.

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