Large-aspect-ratio limit of neoclassical transport theory

S. K. Wong[1] and V. S. Chan[2]
[1]*San Diego Mesa College, San Diego, California 92111, USA*
[2]*General Atomics, P.O. Box 85608, San Diego, California 92186-5608, USA*
(Received 15 January 2003; published 23 June 2003)

This paper presents a comprehensive description of neoclassical transport theory in the banana regime for large-aspect-ratio flux surfaces of arbitrary shapes. The method of matched-asymptotic expansions is used to obtain analytical solutions for plasma distribution functions and to compute transport coefficients. The method provides justification for retaining only the part of the Fokker-Planck operator that involves the second derivative with respect to the cosine of the pitch angle for the trapped and barely circulating particles. It leads to a simple equation for the freely circulating particles with boundary conditions that embody a discontinuity separating particles moving in opposite directions. Corrections to the transport coefficients are obtained by generalizing an existing boundary layer analysis. The system of moment and field equations is consistently taken in the cylinder limit, which facilitates the discussion of the treatment of dynamical constraints. It is shown that the nonlocal nature of Ohm's law in neoclassical theory renders the mathematical problem of plasma transport with changing flux surfaces nonstandard.

DOI: 10.1103/PhysRevE.67.066406 PACS number(s): 52.25.Fi, 52.25.Xz, 52.55.Fa

I. INTRODUCTION

A long time has passed since Galeev and Sagdeev [1–3] discussed a transport theory for collisionless tokamak plasmas, which has come to be known as neoclassical theory, followed shortly by a series of papers by Rosenbluth, Hazeltine, and Hinton [4–6] which put the theory on firm mathematical foundations and calculate a comprehensive set of transport coefficients. Ensuing works that widen the domain of the theory by treating more general magnetic geometry, incorporating more plasma species, allowing for faster plasma rotations, are too numerous to list. The state of the theory as of the mid 1970s is extensively documented in the review paper of Hinton and Hazeltine [7]. The later review by Hirshman and Sigmar [8] on the subject of impurity transport relies in large measures on a fluid description that has since enjoyed wide acceptance. Two books have also been written on neoclassical theory [9,10]. Under the circumstances, to write on the topics revealed by the title is an undertaking that requires justification.

A prerequisite for the justification is easily satisfied: the theory remains as relevant today as when it was first introduced. Indeed, it might even be more so as there is now experimental evidence that ion thermal conductivity could be comparable or even less than neoclassical values in some cases [11]. Also important is the fact that many aspects of the theory that have to do with the interplay between guiding-center motion and Coulomb collisions continue to influence thinking on transport in toroidal devices, of which the tokamak is but one example. In the end, however, justification must rest on what this paper has to offer.

The present work is an outgrowth of an attempt to better understand the mathematical nature and justification for the approximation based on a ‘‘localized’’ distribution function introduced in Ref. [4], which makes possible analytic calculations in the banana regime in the limit of large aspect ratios, and is implicit in many of the works to follow. The approximation allows the Fokker-Planck operator to be replaced by a much simpler operator, thus rendering the problem analytically tractable. In Ref. [4], the replacement is made in the quadratic functional of entropy production. The simpler operator retains only the second derivative of the distribution function with respect to the cosine of the pitch angle. The variation of the functional leads to a solvable equation for the trial function, which is then used to compute the transport coefficients. However, the variation is performed on a derivative of the trial function rather than the trial function itself as is customarily done. If the latter course were followed, the trial function would not be completely determined as we shall demonstrate. In Ref. [5], the approximate operator is taken to be the pitch-angle-scattering operator or the full angular part of the Laplacian in spherical coordinates. Its use in the entropy production functional together with the variation of the trial function reproduces the results in Ref. [4]. The pitch-angle-scattering operator has been adopted in all subsequent works that produce analytic results in the banana regime. In attempting to justify the use of this operator based on a large-aspect-ratio expansion of the distribution function, we realized that the operator actually contains terms formally of the same order as terms that have been neglected. In this sense, the approximation does not appear to be consistent.

The work of Galeev and Sagdeev is based on a direct solution of the drift kinetic equations. The approximate collision operator they use also contains unjustified extra terms. Their results for transport fluxes are also different from those of Ref. [4], presumably due to confusion in the evaluation of certain integrals by integration by parts.

We have found a resolution to the issue of approximate collision operator by a consistent expansion in the inverse aspect ratio. The approximation to the linearized drift kinetic equation (LDKE) is made differently for two groups of particles which will be called the ‘‘freely circulating particles’’ and the ‘‘slow particles.’’ The first group consists of the majority of particles that are only slightly influenced by the mirroring effects of the inhomogeneous magnetic field. The second group consists of the trapped and barely circulating particles that are greatly influenced by the magnetic mirror. The matching of the distribution functions for these two classes of particles in the sense of matched-asymptotic expansions [12] completely determines the distribution function for the slow particles, which turns out to be the same as the trial function in Ref. [4]. In the region of the freely circulating particles, the distribution function is shown to be annihilated by the Fokker-Planck operator. This has the consequence that the transport fluxes can be evaluated using the distribution function of the slow particles alone if the flux-friction relations are used. The equation for the freely circulating particles is invoked in Ref. [2], where it is solved to account for the effect of self-collisions, although this is not necessary, as we will show. However, one of our results is that the distribution function for the freely circulating particles exhibits a discontinuity across the plane $v_\parallel = 0$. Since this discontinuity is not taken into account in Ref. [2], the accuracy of their results is in doubt.

The procedure just described can be readily applied to noncircular flux surfaces provided the magnetic well along the surface is shallow. This leads to a common geometry factor for all neoclassical transport coefficients in the banana regime. Formulas for transport coefficients have been given in the literature for finite aspect ratio and general geometry [8,13,14]. As a rule, these formulas are hard to justify because they stem from the use of simplified Fokker-Planck operators. The geometry factor we found represents the asymptotic limit as the inverse aspect ratio approaches zero and is, in this sense, exact.

Hinton and Rosenbluth [5] have obtained a correction to the diffusion coefficient in the banana regime from a boundary layer analysis applied to the region delineating the trapped and circulating particles. We have found that the Wiener-Hopf technique they used can be generalized to noncircular flux surfaces and to the full matrix of transport coefficients. A single geometry factor is again found to be present in all corrections.

Besides transport coefficients, another element of neoclassical transport theory is the moment equations and field equations which, taken together, provide a closed description of the plasma in macroscopic variables. Generally, when the shape of flux surfaces changes in the course of plasma transport, the forms of the equations and how they should be consistently advanced in time have always been a nontrivial matter. The extensive literature on this problem from the point of view of resistive magnetohydrodynamic and, to a lesser extent, neoclassical theory, is well documented by Blum and Le Foll [15]. We have found that by consistently taking the cylinder limit, which is appropriate for large-aspect-ratio flux surfaces, the equations assume much simpler forms that have not been presented as a whole in the past. In these forms, the question of consistency and the construction of numerical procedure for time advance are much easier to discuss. The equations are further simplified by an explicit elimination of the toroidal components. Examining the remaining poloidal components of the system reveals that the root of the mathematical difficulty lies in the nonlocal nature of Ohm's law: the parallel current density is a flux function and is related to the parallel inductive electric field averaged over a flux surface.

In view of these developments, we feel that there is a need for a document that offers a critical and comprehensive presentation of neoclassical theory for large-aspect-ratio flux surfaces in the banana regime, stating clearly the various assumptions and approximations that have been made, offering justifications as much as possible. Besides acting as a collection of firmly established results in their simplest forms, it is hoped that this paper serves some pedagogical purpose by shedding light on the precautions that need to be taken to extend the theory to useful parameter ranges such as finite aspect ratios and less extreme collisionality. Many have made important contributions to neoclassical theory, which are used in this paper as a matter of common knowledge. It is hoped that we are not being remiss in referring the readers to Refs. [7–10] for the extensive references in the literature.

The balance of this paper is organized as follows. Section II is concerned with the formulation of neoclassical transport theory in general magnetic geometry, with the only restriction that the poloidal magnetic field be much less than the toroidal field. Containing few important results, they are included mainly for the purpose of establishing notations and identifying relevant quantities. In Sec. II A, the LDKEs are derived by an expansion in the ratio of poloidal gyroradius over the plasma scale length. In Sec. II B, the forms of the moment equations and the definition of transport fluxes under the same expansion are obtained. The main results of our work are presented in Sec. III, where restriction to large-aspect-ratio flux surfaces is made. The flux-friction relations are first derived in Sec. III A. The electron LDKE is simplified and solved in Sec. III B using the method of matched-asymptotic expansions. This is followed in Sec. III C by a similar discussion for the ion LDKE. Section III D shows how the transport fluxes are calculated and presents the transport coefficients, comparing them with existing works. Section III E presents the field equations and moment equations in the cylinder limit, the explicit elimination of the toroidal components, and the role played by Ohm's law. Section IV provides a summary of our work. The Appendix provides a streamlined description of the Hinton-Rosenbluth boundary layer analysis, leading to corrections for all the transport coefficients.

II. GENERAL GEOMETRY FORMULATION

A. Linearized drift kinetic equation

Our starting point is the drift kinetic equation for each plasma species. Using as velocity space variables the unit mass kinetic energy $w = v^2/2$ and the magnetic moment $\mu = v_\perp^2/2B$, the equation for a species of mass m and charge q is

$$\frac{\partial f}{\partial t} + \left(v_\parallel \hat{b} + \vec{v}_D + \frac{\vec{E}\times\vec{B}}{B^2} \right) \cdot \vec{\nabla} f + \frac{q}{m} (v_\parallel \hat{b} + \vec{v}_D) \cdot \vec{E} \frac{\partial f}{\partial w} = C(f,f), \tag{1}$$

where $C(f,f)$ represents Coulomb collisions. The magnetic field is axisymmetric and is given by

$$\vec{B}=I\vec{\nabla}\zeta+\vec{\nabla}\zeta\times\vec{\nabla}\psi, \tag{2}$$

where ζ is toroidal angle, ψ is the poloidal flux, and I is a flux function. The magnitude of the parallel velocity is given by $|v_\parallel|=\sqrt{2(w-\mu B)}$, and the curvature and grad-B drift $\vec{v}_D$ can be written in the form

$$\vec{v}_D=-\frac{m}{q}v_\parallel\hat{b}\times\vec{\nabla}\frac{v_\parallel}{B}, \tag{3}$$

in the low-β approximation. The electric field consists of an inductive and an electrostatic part:

$$\vec{E}=-\vec{\nabla}\phi+\vec{E}_A. \tag{4}$$

For simplicity of presentation, the poloidal variation of the electrostatic field is neglected. It can be shown that its inclusion does not change the final forms of the transport equations [5].

The transport phenomena are described by a reduction of the drift kinetic equation together with the Maxwell equations, to a closed set of equations involving only fluid variables. Following Ref. [7], such a closed set can be obtained using the Chapman-Enskog approach to expand the drift kinetic equations order by order. It will be necessary to assume that the poloidal component of the magnetic field is much smaller than the toroidal component. Choosing the ordering parameter Δ to be the ion poloidal gyroradius over the plasma scale length ℓ, which includes the magnetic field scale length and the radial electrostatic potential scale length, it is then justified [16] to retain only first-order guiding-center drifts as in Eq. (1). Specifically, the following orderings are assumed:

$$\frac{|\vec{\nabla}\psi|}{I}\sim\Delta,\quad \sqrt{\frac{m_e}{m_i}}\sim\Delta^2,\quad \frac{eE_A\ell}{T_e}\sim\Delta^4,\quad \frac{e\phi}{T_e}\sim 1. \tag{5}$$

The expansion is implemented by assigning frequency scales to the various terms in the drift kinetic equation, with the ion transit frequency ω_0, the reciprocal of the time taken by a typical ion to move once around the poloidal direction, chosen conveniently as a reference. The ion-ion collision frequency is taken to be of the same order as ω_0, so that the different regimes of collisionality will be distinguished by subsidiary expansions in solving the LDKEs that follow. The time derivatives are assigned orders consistent with the slowest possible variation.

It is readily shown that the zeroth-order distribution functions are stationary Maxwellians $f_0=n(m/2\pi T)^{3/2}e^{-mw/T}$, where both the density n and the temperature T, which can be different for electrons and ions, are constant on each flux surface and can be taken as functions of the poloidal flux ψ at each given time. The first-order distribution functions satisfy the LDKEs

$$\vec{v}_\parallel\cdot\vec{\nabla}f_{e1}-\nu_{ei}\left(Lf_{e1}+\frac{m_e v_\parallel u_{\parallel i}}{T_e}f_{e0}\right)-C^\ell_{ee}f_{e1}$$
$$=\frac{m_eI}{e}\vec{v}_\parallel\cdot\vec{\nabla}\frac{v_\parallel}{B}\left(\frac{n_e'}{n_e}-\frac{e\phi'}{T_e}+\overline{\frac{m_ew}{T_e}-\frac{3}{2}}\frac{T_e'}{T_e}\right)f_{e0}$$
$$-\frac{e}{T_e}v_\parallel E_\parallel f_{e0}, \tag{6}$$

$$\vec{v}_\parallel\cdot\vec{\nabla}_{fi1}-C^\ell_{ii}f_{i1}=-\frac{m_iI}{Z_ie}\vec{v}_\parallel\cdot\vec{\nabla}\frac{v_\parallel}{B}\left(\frac{n_i'}{n_i}+\frac{Z_ie\phi'}{T_i}\right.$$
$$\left.+\overline{\frac{m_iw}{T_i}-\frac{3}{2}}\frac{T_i'}{T_i}\right)f_{i0}, \tag{7}$$

for the electrons and the ions, respectively, in a pure plasma. Here a prime denotes differentiation with respect to ψ. As a result of mass-ratio expansion, ion-electron collisions are neglected in Eq. (7), while electron-ion collisions are described in Eq. (6) by pitch-angle scattering in the rest frame of the ions, modeled by the operator

$$L=\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\mu v_\parallel\frac{\partial}{\partial\mu}=\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}, \tag{8}$$

depending on whether $[v,\mu,\mathrm{sgn}(v_\parallel)]$ or $(v,\xi=v_\parallel/v)$ are chosen as the velocity space variables. The parallel ion flow is calculated from $u_{\parallel i}=\int d^3v v_\parallel f_{i1}/n_i$. The energy-dependent collision frequency ν_{ei} is given by $\nu_{ei}=(3\sqrt{\pi}/4\tau_{ei})(\overline{v_e}/v)^3$ in terms of the notations $\overline{v_a}=\sqrt{2T_a/m_a}$ for the thermal velocity of species a and $\tau_{ab}=(3\sqrt{m_a}T_a^{3/2})/(4\sqrt{2\pi}Z_a^2Z_b^2e^4n_b\ell n\Lambda)$ for the collision time between species a and b. The linearized like-particle collision operator C^ℓ_{aa} has also been used.

More convenient forms for the LDKEs can be obtained for the shifted distribution functions f_{e1}' and f_{i1}' defined as follows:

$$f_{e1}=f_{e1}'-\frac{m_ev_\parallel}{T_e}\frac{I}{B}\frac{\partial\phi}{\partial\psi}f_{e0}+\frac{m_ev_\parallel u_{\parallel i}'}{T_e}f_{e0}-\frac{eW}{T_e}f_{e0}$$
$$-\frac{eD}{T_e}v_\parallel\frac{\langle E_\parallel B\rangle}{\langle B\rangle}f_{e0}, \tag{9}$$

$$f_{i1}=f_{i1}'-\frac{m_iv_\parallel}{T_i}\frac{I}{B}\frac{\partial\phi}{\partial\psi}f_{i0}, \tag{10}$$

where

$$u_{\parallel i}'=\frac{1}{n_i}\int d^3v v_\parallel f_{i1}'=u_{\parallel i}+\frac{I}{B}\frac{\partial\phi}{\partial\psi}, \tag{11}$$

W is a solution of the equation

$$\hat{b}\cdot\vec{\nabla}W=E_\parallel-\frac{\langle E_\parallel B\rangle}{\langle B\rangle}, \tag{12}$$

where $\langle\ \rangle$ denotes flux average, $D(v)$ is the solution of the Spitzer problem

$$C_e(v_\parallel D f_{e0}) = -v_\parallel f_{e0}, \tag{13}$$

where $C_e = \nu_{ei} L + C_{ee}^{\ell}$. As a result, Eqs. (6) and (7) are replaced by

$$\begin{aligned}\vec{v}_\parallel\cdot\vec{\nabla} f'_{e1} - C_e f'_{e1} = {}& \frac{m_e I}{e}\vec{v}_\parallel\cdot\vec{\nabla}\frac{v_\parallel}{B}\left(\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}\right.\\ &\left.+\overline{\frac{m_e w}{T_e}-\frac{3}{2}}\frac{1}{T_e}\frac{\partial T_e}{\partial\psi}\right) f_{e0}\\ &+\frac{eD}{T_e}\vec{v}_\parallel\cdot\vec{\nabla} v_\parallel\frac{\langle E_\parallel B\rangle}{\langle B\rangle} f_{e0}\\ &-\frac{m_e}{T_e}\vec{v}_\parallel\cdot\vec{\nabla}(v_\parallel u'_{\parallel i}) f_{e0},\end{aligned} \tag{14}$$

$$\begin{aligned}\vec{v}_\parallel\cdot\vec{\nabla} f'_{i1} - C_{ii}^{\ell} f'_{i1} = {}& -\frac{m_i I}{Z_i e}\vec{v}_\parallel\cdot\vec{\nabla}\frac{v_\parallel}{B}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial\psi}\right.\\ &\left.+\overline{\frac{m_i w}{T_i}-\frac{3}{2}}\frac{1}{T_i}\frac{\partial T_i}{\partial\psi}\right) f_{i0},\end{aligned} \tag{15}$$

which do not involve $\partial\phi/\partial\psi$. The removal of the unknown poloidal variation of $E_\parallel$ has motivated the separation of the term involving W in Eq. (9). [Our choice for W differs from the customary one [7] in which the term $\langle E_\parallel B\rangle B/\langle B^2\rangle$ appears instead of $\langle E_\parallel B\rangle/\langle B\rangle$. The results are the same in the limit of large aspect ratio, which is pursued in this work. For general geometry, to the best of our knowledge, the motivation for the customary choice is unclear.] The shift of the distribution function by the term proportional to $u'_{\parallel i}$ allows the left side of Eq. (14) to involve f'_{e1} only with no contribution from f'_{i1}. This shift and the one involving Spitzer's function in Eq. (9) allow all terms on the right side of Eq. (14) to have the form $\vec{v}_\parallel\cdot\vec{\nabla} v_\parallel A$, where A is a function of energy and spatial coordinates.

In the limit where collision frequencies are much less than transit frequencies, Eqs. (14) and (15) can be further reduced. Neglecting the collision term in the electron equation in a first approximation, it follows that f'_{e1} is of the form

$$\begin{aligned}f'_{e1} = {}& \frac{m_e I}{e}\frac{v_\parallel}{B}\left(\frac{1}{n_e}\frac{\partial n_e}{\partial\psi}+\overline{\frac{m_e w}{T_e}-\frac{3}{2}}\frac{1}{T_e}\frac{\partial T_e}{\partial\psi}\right) f_{e0}\\ &+\frac{eD}{T_e} v_\parallel\frac{\langle E_\parallel B\rangle}{\langle B\rangle} f_{e0} - \frac{m_e v_\parallel u'_{\parallel i}}{T_e} f_{e0} + g_e,\end{aligned} \tag{16}$$

where g_e is only a function of the invariants μ, v for trapped electrons and μ, v plus $\mathrm{sgn}(v_\parallel)$ for the circulating electrons. Perturbation on the collision term leads to the equation

$$\oint \frac{dl_\parallel}{v_\parallel} C_e f'_{e1} = 0, \tag{17}$$

for the determination of g_e, where the integration is over one closed orbit in the case of the trapped electrons and over one turn in the poloidal direction in the case of the circulating electrons. It follows immediately that g_e vanishes for trapped electrons because the round-trip integration annihilates the contribution of the first three terms in Eqs. (16) and (17). Thus, Eq. (17) needs to be solved only for the circulating electrons.

By an identical argument, the ion distribution function can be written as

$$f'_{i1} = -\frac{m_i I}{Z_i e}\frac{v_\parallel}{B}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial\psi}+\overline{\frac{m_i w}{T_i}-\frac{3}{2}}\frac{1}{T_i}\frac{\partial T_i}{\partial\psi}\right) f_{i0} + g_i, \tag{18}$$

where g_i is independent of the poloidal coordinate, vanishes for trapped ions, and is determined from

$$\oint \frac{dl_\parallel}{v_\parallel} C_{ii}^{\ell} f'_{i1} = 0, \tag{19}$$

for circulating ions.

In general, the solutions to Eqs. (17) and (19) can only be obtained by numerical methods because of the complexity of the Fokker-Planck operator. It will be shown that in the limit of large aspect ratio, analytical results can be achieved. Solutions to Eqs. (17) and (19) also exhibit discontinuities at the boundary separating the trapped and circulating particles. These can be smoothed over through a boundary layer analysis [5] leading to corrections in the fluxes in higher collisionality order.

B. Moment equations and field equations

The solutions of the LDKEs are used to achieve the closure of the moment equations, which will now be derived. We begin with the following moment equations for each species, which follow directly from the drift kinetic equation (1):

$$\left\langle\frac{\partial}{\partial t}\int d^3v f\right\rangle + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\left(\vec{v}_D+\frac{\vec{E}\times\vec{B}}{B^2}\right)\cdot\vec{\nabla}\psi f\right\rangle = 0, \tag{20}$$

$$\begin{aligned}&\left\langle\frac{\partial}{\partial t}\int d^3v\, mwf\right\rangle + \frac{1}{V'}\frac{\partial}{\partial\psi}V'\left\langle\int d^3v\, mw\right.\\ &\quad\left.\times\left(\vec{v}_D+\frac{\vec{E}\times\vec{B}}{B^2}\right)\cdot\vec{\nabla}\psi f\right\rangle - q\left\langle\int d^3v(\vec{v}_\parallel+\vec{v}_D) f\cdot\vec{E}\right\rangle\\ &\quad=\left\langle\int d^3v\, mwC(f,f)\right\rangle,\end{aligned} \tag{21}$$

where V is the volume enclosed by the flux surface, and $V' = \partial V/\partial\psi$. We proceed to transform these equations into equations for the flux functions $n(\psi,t)$ and $T(\psi,t)$ by substituting the expansions $f = f_0 + f_1$ and evaluating the various terms to leading order in Δ. In doing so, we shall use the approximation $\vec{E}_A = E_\parallel\hat{b}$, valid when the poloidal field is much smaller than the toroidal field. The resulting density equations become, to leading orders,

$$\left\langle \frac{\partial n_e}{\partial t} \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Gamma_e = 0, \tag{22}$$

$$\left\langle \frac{\partial n_i}{\partial t} \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' \Gamma_i = 0, \tag{23}$$

with

$$\Gamma_e = \left\langle \int d^3 v \vec{v}_{De} \cdot \vec{\nabla} \psi f'_{e1} \right\rangle + n_e I \left(\frac{\langle E_\parallel B \rangle}{\langle B \rangle} \left\langle \frac{1}{B} \right\rangle - \left\langle \frac{E_\parallel}{B} \right\rangle \right), \tag{24}$$

$$\Gamma_i = \left\langle \int d^3 v \vec{v}_{Di} \cdot \vec{\nabla} \psi f'_{i1} \right\rangle, \tag{25}$$

where we have used Eqs. (9) and (10) to express f_{e1} and f_{i1} in terms of f'_{e1} and f'_{i1}, respectively. Taken at face values, they imply that the ion density would vary in the frequency scale $\Delta^2 \omega_0$ and electron density in $\Delta^4 \omega_0$.

Alternate expressions can be derived for the particle fluxes. Multiplying the linearized drift kinetic equations (14) and (15) by the factor $(I/qB) m v_\parallel$, integrating over velocity space, and flux averaging, using also the identity

$$\vec{v}_D \cdot \vec{\nabla} \psi = \frac{mI}{q} \vec{v}_\parallel \cdot \vec{\nabla} \frac{v_\parallel}{B}, \tag{26}$$

it follows that $\Gamma_i = 0$, and

$$\Gamma_e = \frac{I}{e} \left\langle \frac{R'_{\parallel e}}{B} \right\rangle + n_e I \left[\frac{\langle E_\parallel B \rangle}{\langle B \rangle} \left\langle \frac{1}{B} \right\rangle - \left\langle \frac{E_\parallel}{B} \right\rangle \right], \tag{27}$$

where $R'_{\parallel e} = \int d^3 v m_e v_\parallel C_e f'_{e1}$. There is thus no ion flux in the frequency scale $\Delta^2 \omega_0$. Because of ambipolarity, the ion flux cannot be of lower order than Δ^4, which is the order of the electron flux implied by Eq. (24). We should, therefore, set the ion flux to zero in the order Δ^3. When this is done in the toroidal angular momentum moment of the ion drift kinetic equation, an equation for the time derivative of the radial electric field arises, as demonstrated in Ref. [16]. In this manner, the transport of the angular momentum can be discussed. It is also possible [17] to abandon the restrictive assumption on the radial scale length of the electrostatic potential that leads to the ratio of toroidal velocity over ion thermal velocity being of the order of Δ in the original neoclassical theory. In the resulting theory for an arbitrarily rotating plasma, angular momentum transport can be discussed on the same footing as energy transport. However, this is beyond the scope of the present work. In any case, particle transport is solely determined by the electron flux in Eq. (24).

For the ion energy equation valid in the frequency scale $\Delta^2 \omega_0$, the term with inductive electric field need not be kept, and the collision term can be evaluated using the zeroth-order distribution functions. The result is

$$\left\langle \frac{3}{2} \frac{\partial}{\partial t} n_i T_i \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' Q_i = Q_\Delta, \tag{28}$$

where

$$Q_i = \left\langle \int d^3 v m_i w \vec{v}_{Di} \cdot \vec{\nabla} \psi f'_{i1} \right\rangle, \tag{29}$$

and $Q_\Delta = 3(m_e/m_i)(n_e/\tau_{ee})(T_e - T_i)$.

The processing of the electron energy equation is more complicated. It involves evaluating the collision term in a frame comoving with the ions and combining with the term representing the work done by the electric field. We simply present the result as follows:

$$\left\langle \frac{3}{2} \frac{\partial}{\partial t} n_e T_e \right\rangle + \frac{1}{V'} \frac{\partial}{\partial \psi} V' Q_e = -Q_\Delta + \langle J_\parallel E_\parallel \rangle + \langle R'_{\parallel e} u'_{\parallel i} \rangle - I \frac{\partial p_i}{\partial \psi} \left(\frac{\langle E_\parallel B \rangle}{\langle B \rangle} \left\langle \frac{1}{B} \right\rangle - \left\langle \frac{E_\parallel}{B} \right\rangle \right), \tag{30}$$

where

$$Q_e = \left\langle \int d^3 v m_e w \vec{v}_{De} \cdot \vec{\nabla} \psi f'_{e1} \right\rangle + \frac{5}{2} n_e T_e I \left(\frac{\langle E_\parallel B \rangle}{\langle B \rangle} \left\langle \frac{1}{B} \right\rangle - \left\langle \frac{E_\parallel}{B} \right\rangle \right), \tag{31}$$

$$J_\parallel = n_e e u_{\parallel i} - e \int d^3 v v_\parallel f_{e1} = \sigma_S \frac{\langle E_\parallel B \rangle}{\langle B \rangle} + J_{\mathrm{nc}}, \tag{32}$$

with the introduction of Spitzer's conductivity

$$\sigma_S = \frac{e^2}{T_e} \int d^3 v v_\parallel D f_{e0}, \tag{33}$$

and the neoclassical current

$$J_{\mathrm{nc}} = -e \int d^3 v v_\parallel f'_{e1}. \tag{34}$$

Equation (30) is valid up to the frequency scale $\Delta^4 \omega_0$ which is the scale associated with the energy flux. Collisional energy exchange, however, occurs in the scale $\Delta^2 \omega_0$. It is noteworthy that the radial electric field does not occur in the moment equations (23), (28), (30) or the expressions for the fluxes given by Eqs. (24), (29), and (31). It only appears in Eq. (11) where it contributes to the ion parallel flow $u_{\parallel i}$.

To complete the transport description, we shall now present the field equations. While many of the equations to follow have appeared in the literature, here they are consistently simplified using the approximation $|\vec{\nabla} \psi|/I \ll 1$, and are written in forms that facilitate the discussion of the mathematical nature of the transport problem.

First, Ampere's law $\vec{\nabla} \times \vec{B} = 4\pi \vec{J}$ is decomposed into its parallel and perpendicular components. Using guiding-center drifts and magnetizations for a Maxwellian plasma, the perpendicular current becomes the diamagnetic current, which, when substituted into the perpendicular component of Ampere's law, leads to the Grad-Shafranov equation

$$\Delta^*\psi \equiv R^2\vec{\nabla}\cdot\frac{\vec{\nabla}\psi}{R^2} = -I\frac{\partial I}{\partial\psi} - 4\pi R^2\frac{\partial p}{\partial\psi}, \tag{35}$$

where $p = p_e + p_i$ is the total pressure. Using the approximation $\vec{B}\cdot\vec{\nabla}\times\vec{B} \approx I\Delta^*\psi/R^2$, the parallel component of Ampere's law becomes

$$\frac{1}{R}\Delta^*\psi = 4\pi J_\parallel . \tag{36}$$

It is more convenient to replace the Grad-Shafranov equation by

$$\frac{1}{8\pi}\frac{\partial I^2}{\partial\psi} + \frac{I^2}{\langle B^2\rangle}\frac{\partial p}{\partial\psi} + \frac{I\langle J_\parallel B\rangle}{\langle B^2\rangle} = 0, \tag{37}$$

which is obtained using Eqs. (35) and (36) and

$$J_\parallel = -\frac{I}{B}\frac{\partial p}{\partial\psi}\left(1 - \frac{B^2}{\langle B^2\rangle}\right) + \frac{\langle J_\parallel B\rangle B}{\langle B^2\rangle}, \tag{38}$$

a result that follows from $\vec{\nabla}\cdot\vec{J} = 0$. Equation (37) reduces to a transparent form when the large aspect ratio is taken in Sec. III E.

Faraday's law $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}_A$ is next decomposed into its toroidal and poloidal components using, for the inductive electric field, the representation

$$\vec{E}_A = E_\zeta R\vec{\nabla}\zeta + \vec{\nabla}\zeta\times\vec{\nabla}\psi_E , \tag{39}$$

valid for any axisymmetric solenoidal field. Then the poloidal component takes the form

$$\frac{\partial\psi}{\partial t} = \frac{I}{B}E_\parallel , \tag{40}$$

when the approximation $E_\zeta \approx E_\parallel$ is invoked. After using Eq. (40), the toroidal component can be written as

$$\Delta^*\psi_E = -\frac{\partial I}{\partial t} - \frac{I}{B}E_\parallel\frac{\partial I}{\partial\psi}. \tag{41}$$

Finally, returning to the moment equations (23), (28), and (30), the time differentiation and the flux average operation can be commuted using the relation

$$\left\langle\frac{\partial A}{\partial t}\right\rangle = \left(\frac{\partial A}{\partial t}\right)_\psi + I\left\langle\frac{E_\parallel}{B}\right\rangle\frac{\partial A}{\partial\psi}, \tag{42}$$

for any flux function $A(\psi,t)$, a result that follows from Eq. (40).

A complete set of equations for plasma transport comprises the four field equations (36), (37), (40), and (41); the three moment equations (23), (28), and (30); together with the expressions for the transport fluxes Γ_e, Q_e, Q_i, $J_\parallel$ and the quantities $R'_{e\parallel}$, $u'_{\parallel i}$ obtained from the solutions f'_{e1} and f'_{i1} of the LDKEs. The system thus involves four field variables ψ, I, $E_\parallel$, ψ_E and three plasma variables n_e, T_e, T_i. We can regard Eqs. (37) and (41) as equations to determine I and ψ_E, respectively, when appropriate boundary conditions are imposed. This amounts to a separation of the toroidal components of the system. In the remaining poloidal components, there are five variables n_e, T_e, T_i, $E_\parallel$, ψ to be determined from Eqs. (23), (28), (30) and Eqs. (36), (40). We postpone a discussion of the mathematical nature of the problem until the large-aspect-ratio limit is taken.

III. LARGE-ASPECT-RATIO APPROXIMATION

A. Flux-friction relations

The LDKEs and the bounced averaged Fokker-Planck equations in Sec. II A depend on the poloidal variation of the magnetic field. It is unlikely that accurate explicit formulas for the transport fluxes and other quantities that enter in the moment equations can be obtained for an arbitrary variation. Fortunately, this can be accomplished if the flux surfaces have very large aspect ratio, while not restricted to the circular shape. Furthermore, the moment and field equations also assume simple forms, which facilitates the discussion of consistency and the devising of schemes for their solution in cases where plasma diffusion and shape change of flux surfaces occur at the same time.

We shall first introduce the cylinder limit of the toroidal configuration. An appropriate geometrical center is first chosen for the vacuum vessel wall on a poloidal cross section. Its distance from the central axis is denoted by R_0. Defining $\hat{z} = R\vec{\nabla}\zeta$, $B_z = I/R_0$, and reinterpreting from this section on ψ to mean ψ/R_0, the cylinder limit of the magnetic field can be written as

$$\vec{B} = B_z\hat{z} + \hat{z}\times\vec{\nabla}\psi. \tag{43}$$

The magnetic field on a flux surface varies inversely with the distance from the central axis. The small strength of this variation is described by the quantity

$$2\delta = \frac{B_{\max} - B_{\min}}{B_{\max}} \approx \frac{R_{\max} - R_{\min}}{R_{\min}}, \tag{44}$$

where the subscripts "max" and "min" refer to values on each flux surface. The variation itself can be expressed in terms of the normalized field strength

$$\hat{B} = \frac{B - B_{\min}}{B_{\max} - B_{\min}} = \frac{R_{\max} - R}{R_{\max} - R_{\min}}. \tag{45}$$

For circular flux surfaces, $\hat{B} = \sin^2\theta/2$, where θ is the poloidal angle measured from the outboard side.

Introducing the derivative of the area included within a poloidal flux contour with respect to ψ,

$$A' = \oint\frac{d\theta}{|\vec{\nabla}\psi\times\vec{\nabla}\theta|}, \tag{46}$$

where θ is the poloidal coordinate, the average of any quantity Q over a toroidal flux surface can be approximated by

$$\langle Q\rangle = \oint \frac{Q\,d\theta}{|\vec{\nabla}\psi\times\vec{\nabla}\theta|} \Bigg/ \oint \frac{d\theta}{|\vec{\nabla}\psi\times\vec{\nabla}\theta|}. \tag{47}$$

The particle and heat fluxes defined in Sec. II B are also reinterpreted so that Γ and Q, from this section on, mean $R_0\Gamma$ and $R_0 Q$, respectively. The expressions of these fluxes given by Eqs. (24), (29), and (31) are first simplified by dropping the terms containing the factor $\langle E_\parallel B\rangle\langle B^{-1}\rangle/\langle B\rangle - \langle E_\parallel/B\rangle$, which is of the order of δ while the terms retained will be demonstrated to be of the order of $\sqrt{\delta}$. The terms retained are now transformed into expressions which might be known as flux-friction relations, in which the collisional changes of distribution functions are involved.

Multiplying Eq. (14) by $(I/eB)m_e v_\parallel$, integrating over velocity space, averaging over a flux surface, and then going over to the large-aspect-ratio limit, it follows that

$$\Gamma_e = \frac{m_e}{e}\left\langle \int d^3v\, v_\parallel C_e f'_{e1}\right\rangle. \tag{48}$$

Similarly, the heat fluxes are given by

$$Q_e = \frac{m_e}{e}\left\langle \int d^3v\, m_e w\, v_\parallel C_e f'_{e1}\right\rangle, \tag{49}$$

$$Q_i = -\frac{m_i}{Z_i e}\left\langle \int d^3v\, m_i w\, v_\parallel C^{\ell}_{ii} f'_{i1}\right\rangle. \tag{50}$$

Finally, the neoclassical current defined by Eq. (34) can be transformed into

$$J_{\rm nc} = e\int d^3v\, D\, v_\parallel C_e f'_{e1}, \tag{51}$$

which follows after the replacement $v_\parallel = -C_e(v_\parallel D f_{e0})/f_{e0}$ and use of the hermiticity of the operator C_e. The four flux-friction relations in the above equations explicitly display the essential role of collisions on neoclassical transport. They are used to calculate the fluxes in the banana regime, using the analytic solutions for the distribution functions obtained in the following sections.

B. Solution of electron equation

In seeking to solve Eq. (17) in the large-aspect-ratio limit, we distinguish between two classes of electrons. The distinction is made when (ξ, v) rather than $[\mu, v, \mathrm{sgn}(v_\parallel)]$ are chosen as velocity space variables where $\xi = v_\parallel/v$ is the cosine of the pitch angle. The first class consists of electrons for which ξ is not small and will be called the *freely circulating electrons*. The second refers to those with $\xi\sim\sqrt{\delta}$, which will be called the *slow electrons*. They include the trapped and the barely circulating electrons.

For the freely circulating electrons, the poloidal variation of $v_\parallel$ can be neglected. Equation (17) then states that the average of f'_{e1} over a flux surface is annihilated by the collision operator. To leading order in δ, which turns out to be $\sqrt{\delta}$, it proves possible to seek f'_{e1} in a form that is independent of the poloidal coordinate when ξ and v are used as velocity space variables instead of μ, w, and $\mathrm{sgn}(v_\parallel)$. Thus, writing

$$f'_{e1} = \sqrt{\delta} f_C(\xi, v) + \cdots, \tag{52}$$

for the freely circulating electrons, the following equation is obtained:

$$C_e f_C = 0. \tag{53}$$

Two remarks should be made regarding the above equation. The first is that even though it is a good approximation only when ξ is not small, we shall seek its solution down to $\xi=0$, where boundary conditions will be obtained by matching with the solution for slow electrons. This is in accord with the method of matched-asymptotic expansions [12]. Second, the operator C_e in the equation depends also on the distribution function f'_{e1} for the slow electrons through the integral part of the linearized Fokker-Planck operator. Therefore, Eq. (53) is an inhomogeneous equation for f_C. It is possible to obtain the inhomogeneous term explicitly using the distribution function for the slow electrons found later in this section, and show that it contributes to the order in δ required in Eq. (53). But the resulting equation is hard to solve. Fortunately, the equation itself renders it unnecessary to seek a solution for the purpose of the calculation of fluxes.

In considering the slow electrons, we go back to Eq. (14) in lieu of Eq. (17). Keeping only the lowest-order terms in δ, the right hand side of Eq. (14) simplifies, and the equation becomes

$$\vec{v}_\parallel\cdot\vec{\nabla} f'_{e1} - C_e f'_{e1} = \vec{v}_\parallel\cdot\vec{\nabla}_\parallel A_e f_{e0}, \tag{54}$$

where

$$A_e = \frac{m_e}{e}\left(\frac{1}{n_e}\frac{\partial n_e}{\partial\psi} + \overline{\frac{m_e w}{T_e} - \frac{3}{2}}\,\frac{1}{T_e}\frac{\partial T_e}{\partial\psi}\right) - \frac{m_e U'}{T_e} + \frac{eD}{T_e}\langle E_\parallel\rangle, \tag{55}$$

and U' denotes the leading order term of $u'_{\parallel i}$ in δ, and is independent of the poloidal coordinate as demonstrated in Sec. III C.

The following crucial approximation is now made:

$$C_e f'_{e1} \approx \nu_e \frac{1}{2}\frac{\partial^2 f'_{e1}}{\partial\xi^2}, \tag{56}$$

where f'_{e1} is expressed in the variable ξ, v, θ instead of μ, w, $\mathrm{sgn}(v_\parallel)$, θ, and $\nu_e = \nu_{ei} + \nu_{ee}$, with the notation $\nu_{aa} = (3\sqrt{\pi}/4\tau_{aa})(G'/x^3)$ for the energy-dependent self-collision frequency, where $x = v/\overline{v_a}$ and

$$G'(x) = \left(\frac{1}{2x} + x\right)\mathrm{erf}(x) + \frac{1}{\sqrt{\pi}}e^{-x^2}. \tag{57}$$

To motivate this approximation, it is noted that if Eq. (54) is expressed in the variables (ξ, w, θ), and if both the collision

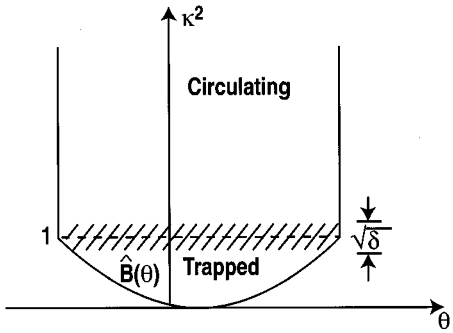

FIG. 1. The phase space for Eq. (61) at a fixed energy.

term and the resulting mirror force term are neglected, f'_{e1} can be solved for and the following asymptotic behavior would be obtained:

$$f'_{e1} - \frac{1}{\xi}, \quad \xi \to 0. \tag{58}$$

This suggests the dominance of the second derivative $\partial^2 f'_{e1}/\partial \xi^2$ in the collision operator for slow electrons, for which $\xi \sim \sqrt{\delta}$.

To proceed, it proves advantageous to introduce the variables ξ_* and κ^2 through the definitions

$$\xi = \sqrt{\delta}\xi_* = \sqrt{2\delta}\sqrt{\kappa^2 - \hat{B}(\theta)}, \tag{59}$$

$$\kappa^2 = 1 + \frac{w - \mu B_{\max}}{2\epsilon w}. \tag{60}$$

The variable κ^2, which is adopted in the work of Galeev and Sagdeev [1–3], represents the cosine of the pitch angle at the location of minimum magnetic field, scaled by the factor $\sqrt{2\delta}$. As shown in Fig. 1, the region $\kappa^2 > 1$ and $1 > \kappa^2 > \hat{B}(\theta)$ correspond to the circulating and trapped electrons, respectively. When κ^2, ν, θ are used as variables, and the approximation, Eq. (56), is made, Eq. (59) becomes

$$\xi_* \nu \hat{b} \cdot \vec{\nabla} f'_{e1} - \frac{\nu_e^*}{2} \xi_* \frac{\partial}{\partial \kappa^2} \xi_* \frac{\partial}{\partial \kappa^2} f'_{e1} = \sqrt{\delta}\xi_* \hat{b} \cdot \vec{\nabla} \xi_* \nu^2 A_e f_{e0}, \tag{61}$$

where $\nu_e^* = \nu_e \delta^{-3/2}$.

The banana regime is characterized by $\nu_e^* \ll 1$, which allows the collision term to be neglected in finding the leading order solution in collisionality. This leading order solution can be written in the form

$$f'_{e1} = \sqrt{\delta} f_S + \cdots, \tag{62}$$

$$f_S = \xi_* \nu A_e f_{e0} + g_S, \tag{63}$$

where g_S is a function of κ^2, ν, sgn(ξ). For similar reasons that apply to the function g_e in Eq. (16), the function g_S vanishes for the trapped electrons. For the circulating electrons, it is determined from the condition

$$\oint \frac{d\theta}{|\vec{\nabla}\psi \times \vec{\nabla}\theta|} \frac{\partial}{\partial \kappa^2} \xi_* \frac{\partial}{\partial \kappa^2} f_S = 0, \tag{64}$$

which follows from treating the collision term as a perturbation. It is noteworthy that the above is actually a homogeneous equation for g_S because the first term on the right of Eq. (63) makes no contribution in Eq. (64). The solution for the circulating electrons is an odd function in sgn(ξ) which for sgn(ξ) $= +1$ is given by

$$g_S = K_1 \int_1^{\kappa^2} \frac{d\kappa'^2}{\langle \sqrt{2(\kappa'^2 - \hat{B})} \rangle} + K_2, \tag{65}$$

where K_1 and K_2 are arbitrary constants. In actual fact, Eq. (17) does not apply in a thin layer separating the trapped and circulating electrons shown in Fig. 1, where the perturbation treatment of collisions fails. An analysis of this boundary layer is given in Ref. [4] and is adapted in the Appendix for noncircular flux surfaces. It shows that the constant K_2 is of the order of $\sqrt{\nu_e^*}$ and can be neglected in a first approximation. The analysis also gives corrections in this order to the fluxes to be derived in Sec. III D.

The constant K_1 cannot be determined from the consideration of slow electrons alone. It is, instead, determined by matching the solution for the slow electrons to that of the freely circulating electrons in the sense of matched-asymptotic expansions [12]. It is first noted that the following asymptotic behavior can be established:

$$\int_1^{\kappa^2} \frac{d\kappa'^2}{\langle \sqrt{2(\kappa'^2 - \hat{B})} \rangle} = \sqrt{2}\kappa - a + O\left(\frac{1}{\kappa}\right), \quad \kappa \to \infty, \tag{66}$$

where

$$a = \sqrt{2}\left[1 - \int_0^1 \frac{dk}{k^2}\left(\frac{1}{\langle \sqrt{1 - k^2 \hat{B}} \rangle} - 1\right)\right]. \tag{67}$$

For circular flux surfaces, we have $\langle \sqrt{1 - k^2 \hat{B}} \rangle = 2E(k^2)/\pi$, where $E(k^2)$ is the complete elliptic integral of the second kind, and the factor a in this form has been obtained in Ref. [4], where it has been numerically evaluated to be $\sqrt{2}(0.69)$. As a result of Eq. (65), when we change over to ξ_* as the independent variable instead of κ^2, and allow ξ_* to range between $-\infty$ to $+\infty$, we find, for slow electrons,

$$\begin{aligned} f'_{e1} &= \sqrt{\delta}\xi_* \nu A_e f_{e0} + \sqrt{\delta} K_1 (\xi_* - a + \cdots) \\ &= \xi(\nu A_e f_{e0} + K_1) - \sqrt{\delta} K_1 a + \cdots, \end{aligned} \tag{68}$$

as $\xi_* \to \infty$. Matching this to the Taylor expansion

$$f'_{e1} = \sqrt{\delta} f_C(0,\nu) + \cdots, \quad \xi \to 0, \tag{69}$$

for the freely circulating electrons, we require

$$K_1 = -\nu A_e f_{e0}, \tag{70}$$

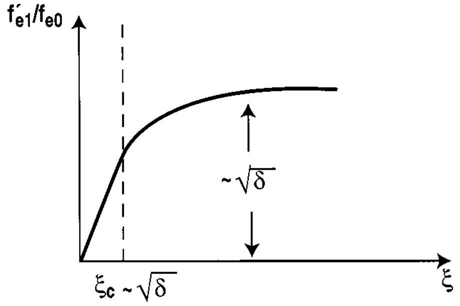

FIG. 2. The distribution function f'_{el} as a function of ξ at fixed values of energy w and the poloidal coordinate θ. The derivative $\partial f'_{el}/\partial\xi$ is localized to a region of the order of $\sqrt{\delta}$.

$$f_C(0,\nu) = -K_1 a. \tag{71}$$

It is thus established that for slow electrons,

$$f_S = \left[\xi_* - H(\kappa^2 - 1)\int_1^{\kappa^2} \frac{d\kappa'^2}{\langle\sqrt{2(\kappa'^2 - \hat{B})}\rangle}\right] \nu A_e f_{e0}, \tag{72}$$

where H is the Heaviside function, with the asymptotic behavior

$$f_S \to \pm a\nu A_e f_{e0} + O(1/\xi_*), \quad \xi_* \to \pm\infty. \tag{73}$$

For the freely circulating electrons, Eq. (53) holds with the boundary conditions

$$f_C = \pm a\nu A_e f_{e0}, \quad \xi = \pm 0. \tag{74}$$

Thus, regarded as a function over the range $-1 < \xi < 1$, f_C is an odd function in ξ that undergoes a finite jump at $\xi = 0$.

In mathematical terminology, the electron distribution function has been obtained from a singular perturbation technique using δ as a small parameter. The solution [Eq. (72)] for slow electrons represents a boundary layer, or inner, solution. The thickness of the boundary layer is of the order of $\sqrt{\delta}$ in the variable ξ. The distribution function for the freely circulating electrons corresponds to the outer solution. At fixed values of ν and θ, the distribution function f'_{e1} increases linearly with ξ, starting from zero. At the location $\xi_C = \sqrt{2\delta(1-\hat{B})}$ that separates the trapped and circulating electrons, the derivative $\partial f'_{e1}/\partial\xi$ suffers a discontinuity. Above ξ_C, the function f'_{e1} increases more slowly than linear or even decreases, leveling off to values of the order of $\sqrt{\delta}$ as ξ reaches the range beyond $\sqrt{\delta}$. The function f'_{e1} remains of the order of $\sqrt{\delta}$ throughout. The derivative $\partial f'_{e1}/\partial\xi$ is of the order of unity in the boundary layer and $\sqrt{\delta}$ elsewhere. In this sense, the function is "localized" [4,7]. The behavior of f'_{e1} as a function of ξ is sketched in Fig. 2.

The solution [Eq. (72)] for slow electrons is also obtained in Refs. [4], [5]. In Ref. [4], it is found from the variation of the entropy production functional, in which the approximation [Eq. (56)] is substituted for the Fokker-Planck operator. After integration by parts, the quadratic functional takes the form $\langle\int d\mu\, \xi(\partial f'_{e1}/\partial\mu)^2\rangle$, where only the parts essential to our argument are retained. Then the derivative $\partial f'_{e1}/\partial\mu$ is varied to yield the equation $\langle\xi\partial f'_{e1}/\partial\mu\rangle = 0$, which has the solution given by Eq. (72). In a conventional approach that considers variations of the function f'_{e1} itself, the Euler-Lagrange equation thus obtained would have been $\langle\partial/\partial\mu(\xi\partial f'_{e1}/\partial\mu)\rangle = 0$, and the solution would be given by Eq. (65) with indeterminate K_1 and K_2. In Ref. [5], the pitch-angle-scattering operator is used in the variational principle, and the resulting Euler-Lagrange equation does have the solution, Eq. (72). In the variable ξ, the pitch-angle-scattering operator L differs from the second-derivative operator [Eq. (56)] by the addition of a term of the form $-\xi\partial/\partial\xi$. This term is formally of the same order in δ as others that have been neglected in the Fokker-Planck operator. The use of the pitch-angle-scattering operator, therefore, cannot be justified beforehand.

Galeev and Sagdeev have also obtained the solution [Eq. (72)] for slow electrons. (Their solution presented as Eq. (30) in Ref. [1] apparently contains typographical errors. The correct form is to be found in Eq. (II-31) of Ref. [3].) They used an approximate collision operator directly in the drift kinetic equation for such electrons. The approximate operator, which can be found in Eq. (28) of Ref. [1] and Eq. (II-28) in Ref. [3], also appears to contain terms in addition to the second-derivative operator, which are of higher order in δ and are, therefore, hard to justify. Equation (53) for the freely circulating electrons also appears in the work of these authors [2,3], where it is apparently needed for the inclusion of self-collisions in the calculation of fluxes. They have not obtained the boundary conditions [Eq. (74)], and their solutions in terms of an expansion in Legendre polynomials, in fact, violate these boundary conditions. As we shall demonstrate, the calculation of fluxes to leading order in δ does not require the solution of Eq. (53), although its existence plays a crucial role.

C. Solution of ion equation

The solution of Eq. (19) for ions closely parallels that of the electron, Eq. (17), with one major change. It turns out that because ion self-collisions conserve momentum, it is not consistent to seek a "localized" behavior for f'_{i1} in the same sense as for f'_{e1}. Instead, such behavior can only be imposed on f''_{i1}, a shifted distribution function defined by

$$f'_{i1} = \frac{m_i v_\parallel U'}{T_i} f_{i0} + f''_{i1}. \tag{75}$$

This has the immediate consequence that $u'_{\parallel i} = U'$ to leading order in δ, as the contribution to $u'_{\parallel i}$ from f''_{i1} is of the order of $\sqrt{\delta}$. For the freely circulating ions, we now have

$$f''_{i1} = \sqrt{\delta} f_C + \cdots, \tag{76}$$

and

$$C^{\ell}_{ii} f_C = 0. \tag{77}$$

For the slow ions, the analog of Eq. (54) in the cylinder limit is

$$\vec{v}_{\parallel}\cdot\vec{\nabla} f''_{i1} - C^{\ell}_{ii} f''_{i1} = \vec{v}_{\parallel}\cdot\vec{\nabla} v_{\parallel} A_i f_{i0}, \tag{78}$$

where

$$A_i = -\frac{m_i}{Z_i e}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial\psi} + \overline{\frac{m_i w}{T_i}} - \frac{3}{2}\frac{1}{T_i}\frac{\partial T_i}{\partial\psi}\right) - \frac{m_i U'}{T_i}. \tag{79}$$

Its solution can be obtained in the form

$$f''_{i1} = \sqrt{\delta} f_S + \cdots, \tag{80}$$

where f_S is given by Eqs. (63) and (65) with the replacement of $A_e f_{e0}$ by $A_i f_{i0}$. An identical matching procedure as for electrons lead to the solution for f_S in the form of Eq. (72) with the replacement of $A_e f_{e0}$ by $A_i f_{i0}$. Also, the function $f_C(\xi,v)$ obeys the boundary condition [Eq. (74)] with the same replacement.

At this stage, the quantity U' remains undetermined. We note that as before, Eq. (77) is an inhomogeneous equation for f_C because there is contribution to the integral part of C^{ℓ}_{ii} from the slow ions. The inhomogeneous nature can be explicitly displayed as follows:

$$C_{ii}(f_C, f_{i0}) + C_{ii}(f_{i0}, f_C) = -\langle C_{ii}(f_{i0}, f_S)\rangle, \tag{81}$$

where the first and the last terms on the left correspond to the differential and the integral parts of the Fokker-Planck operator on f_c. Unlike the case for the electrons, the conservation of momentum imposes the following solvability constraint:

$$\left\langle \int d^3v\, m_i v_{\parallel} C_{ii}(f_{i0}, f_S)\right\rangle = 0. \tag{82}$$

Appealing again to momentum conservation, the above equation implies

$$\left\langle \int d^3v\, m_i v_{\parallel} C_{ii}(f_S, f_{i0})\right\rangle = 0, \tag{83}$$

in which the differential part of the Fokker-Planck operator occurs, which can be approximated by the retention of only the term involving $\partial^2/\partial\xi^2$. Changing over to the variable ξ_* instead of ξ, the integral in Eq. (83) can be simplified as follows:

$$\int_0^{\infty} dv\, 2\pi v^2 m_i v \nu_{ii} \int_{-\infty}^{+\infty} d\xi_* \frac{\xi_*}{2}\frac{\partial^2 f_S}{\partial\xi_*^2}$$

$$= -\int_0^{\infty} dv\, 2\pi v^2 m_i v \nu_{ii} \frac{f_S(+\infty) - f_S(-\infty)}{2}$$

$$= -a\int_0^{\infty} dv\, 2\pi v^2 m_i v \nu_{ii} v A_i f_{i0},$$

where we have performed integration by parts and used the asymptotic behavior [Eq. (73)] to neglect the surface term. Evaluating the integral and setting it to zero gives an equation for U',

$$U' = -\frac{T_i}{Z_i e}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial\psi} + \overline{y - \frac{3}{2}}\frac{1}{T_i}\frac{\partial T_i}{\partial\psi}\right), \tag{84}$$

in which the constant y, first introduced in Ref. [5], is given by

$$y = \int_0^{\infty} dx\, e^{-x^2} x^3 G' \Bigg/ \int_0^{\infty} dx\, e^{-x^2} x G'. \tag{85}$$

Using the simplified version of Eq. (11) in the large-aspect-ratio limit, it follows that the leading order parallel ion flow is $U = U' + \partial\phi/\partial\psi$.

D. Calculation of fluxes

The transport fluxes in the banana regime can be evaluated with the knowledge of the distribution functions f'_{e1} and f'_{i1} obtained in the previous sections. For this purpose, the flux-friction relations in Sec. III A will be used.

Consider first the electron flux given by Eq. (48). We divide the integral into contributions from the regions of the slow and the freely circulating electrons so that to leading order in δ,

$$\Gamma_e = \sqrt{\delta}\frac{m_e}{e}\int_S d^3v\, v_{\parallel} C_e f_S + \sqrt{\delta}\frac{m_e}{e}\int_C d^3v\, v_{\parallel} C_e f_C. \tag{86}$$

Formally, the second term is of the order of $\sqrt{\delta}$ and so is the first as will be presently demonstrated. However, in view of Eq. (53), the second term actually vanishes, making it unnecessary to solve for the distribution functions for the freely circulating electrons, which is otherwise a daunting task. (On p. 197 of Ref. [10], a justification for neglecting the contribution from the freely circulating particles in the evaluation of ion energy flux is given by asserting that $C^{\ell}_{ii} f''_{i1}$ is of the order of ν_{ii}/δ for the slow particles. This estimate neglects the fact that $f''_{i1}/f_{i0} \sim \sqrt{\delta}$, as can be seen from the actual solution. Taking this into account results in the estimate $C^{\ell}_{ii} f''_{i1} \sim \nu_{ii}/\sqrt{\delta}$, so that the contributions to the ion energy flux from the slow and the freely circulating ions are formally comparable. It is then necessary to invoke Eq. (77) or, analogously, Eq. (53) for the electron fluxes, to justify the neglect of the freely circulating particles as we have done.) In the first term, the collision operator can be approximated by the second-derivative operator $\partial^2/\partial\xi^2$. The integral can then be evaluated in the same way as that leading to the expression in Eq. (84), with the result

$$\Gamma_e = \frac{\sqrt{\delta}}{e}\int_0^{\infty} dv\, 2\pi v^2 m_e v \nu_e \int_{-\infty}^{+\infty} d\xi_* \frac{\xi_*}{2}\frac{\partial^2 f_S}{\partial\xi_*^2}$$

$$= -\frac{\sqrt{\delta}}{e}\int_0^{\infty} dv\, 2\pi v^2 m_e v \nu_e \frac{f_S(+\infty) - f_S(-\infty)}{2}$$

$$= -\frac{\sqrt{\delta} a}{e}\int_0^{\infty} dv\, 2\pi v^2 m_e v^2 \nu_e A_e f_{e0}.$$

Upon the elimination of U' from Eq. (55) using Eq. (84), we can write

$$A_e = \frac{m_e}{e} A_1 + \frac{m_e}{e}\frac{m_e \nu^2}{2T_e} A_2 + \frac{eD}{T_e}\langle E_\parallel \rangle, \tag{87}$$

where

$$A_1 = \frac{1}{n_e}\frac{\partial n_e}{\partial \psi} - \frac{3}{2}\frac{1}{T_e}\frac{\partial T_e}{\partial \psi} + \frac{T_i}{Z_i T_e}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial \psi} + \overline{y} - \frac{3}{2}\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\right), \quad A_2 = \frac{1}{T_e}\frac{\partial T_e}{\partial \psi}. \tag{88}$$

The quantities A_1, A_2, and $\langle E_\parallel \rangle$ are considered to be the forces that drive the electron transport fluxes and correspond to the same choice in Ref. [7] for circular cross-section flux surfaces. Using Eq. (87), the electron flux can be written out explicitly as

$$\Gamma_e = -3\sqrt{\delta} a \frac{n_e T_e}{\tau_{ee}} \int_0^\infty dx\, e^{-x^2} x (Z_i + G') \left[\frac{m_e}{e^2}(A_1 + x^2 A_2) + \frac{D}{T_e}\langle E_\parallel \rangle \right]. \tag{89}$$

Other electron fluxes are similarly evaluated.

The results of the flux evaluations can be cast into the form

$$\Gamma_e = -\sqrt{\delta} a n_e \left[\frac{m_e T_e}{e^2 \tau_{ee}} (L_{11} A_1 + L_{12} A_2) + L_{13}\langle E_\parallel \rangle \right], \tag{90a}$$

$$Q_e = -\sqrt{\delta} a n_e T_e \left[\frac{m_e T_e}{e^2 \tau_{ee}} (L_{21} A_1 + L_{22} A_2) + L_{23}\langle E_\parallel \rangle \right], \tag{90b}$$

$$J_{\mathrm{nc}} = -\sqrt{\delta} a n_e \left[T_e (L_{31} A_1 + L_{32} A_2) + \frac{e^2 \tau_{ee}}{m_e} L_{33} (E_\parallel) \right], \tag{90c}$$

with the introduction of the dimensionless transport coefficients

$$L_{ij} = 3\int_0^\infty dx\, e^{-x^2} x (Z_i + G') \alpha_i \alpha_j, \tag{91}$$

where $\alpha_1 = 1$, $\alpha_2 = x^2$, and $\alpha_3 = D(x)/\tau_{ee}$.

Turning to the ion energy flux, after eliminating U' from Eq. (79) to obtain

$$A_i = -\frac{m_i}{Z_i e}\left(\frac{m_i \nu^2}{2T_i} - y\right)\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}, \tag{92}$$

a similar evaluation of the integral in Eq. (50) leads to

$$Q_i = -\sqrt{\delta} a n_i T_i \frac{m_i T_i}{Z_i^2 e^2 \tau_{ii}} L_i \frac{T_i'}{T_i}, \tag{93}$$

where

$$L_i = 3\int_0^\infty dx\, e^{-x^2} x^3 (x^2 - y) G'. \tag{94}$$

Our results show that the transport fluxes share a common geometry factor, which is a feature not transparent in earlier works on general geometry [7,13,14], but clearly stated in Ref. [10] in the limit of large aspect ratio. Unlike Ref. [10], the factor a has been obtained in the exact asymptotic limit and is rigorously justified. (In Chap. 11 of Ref. [10], the common geometry factor is taken to be the "effective trapped particle fraction" $f_t = 1 - 3/4\langle B^2 \rangle \int_0^{1/B_{\max}} \lambda\, d\lambda / \langle \sqrt{1-\lambda B} \rangle$, which has the asymptotic limit $3/2\sqrt{\delta} a$ as δ goes to zero.) The fluxes in the electron channel obey Onsager's symmetry. However, it is unclear if such symmetry holds without the large-aspect-ratio approximation, as the identification of conjugate pairs of fluxes and forces presents some difficulty. One of the difficulties simply has to do with the fact that, with the independent presence of $\langle E_\parallel B \rangle$ and $\langle E_\parallel / B \rangle$, there appear to be more forces than fluxes.

Analytic evaluation of the transport coefficients is possible if the Spitzer problem described by Eq. (13) is solved by expanding $D(x)$ in the Laguerre polynomials $L_n^{3/2}(x)$. Keeping two terms in the expansion yields the solutions

$$D(x) = \tau_{ee}(d_1 + d_2 x^2), \quad d_1 = \frac{2\sqrt{2} - Z_i}{2Z_i(Z_i + \sqrt{2})},$$

$$d_2 = \frac{3}{2(Z_i + \sqrt{2})},$$

from which the Spitzer conductivity is calculated to be

$$\sigma_S = \frac{13 Z_i + 4\sqrt{2}}{4(Z_i + \sqrt{2})}\frac{n_e e^2 \tau_{ee}}{m_e}.$$

For the dimensionless transport coefficients, we need the following evaluations of integrals containing the function G':

$$\int_0^\infty dx\, e^{-x^2} x G' = \frac{\sqrt{2} - \ell n(1 + \sqrt{2})}{2},$$

$$\int_0^\infty dx\, e^{-x^2} x^3 G' = \frac{\sqrt{2}}{4}, \quad \int_0^\infty dx\, e^{-x^2} x^5 G' = \frac{9\sqrt{2}}{16},$$

to obtain

$$L_{11} = \tfrac{3}{2} Z_i + \tfrac{3}{2}[\sqrt{2} - \ell n(1 + \sqrt{2})], \quad L_{12} = \frac{3}{2} Z_i + \frac{3\sqrt{2}}{4},$$

$$L_{22} = 3 Z_i + \frac{27\sqrt{2}}{16},$$

$$L_{13} = L_{11} d_1 + L_{12} d_2, \quad L_{23} = L_{21} d_1 + L_{22} d_2,$$

$$L_{33} = L_{11} d_1^2 + 2 L_{12} d_1 d_2 + L_{22} d_2^2,$$

$$y = \frac{1}{2-\sqrt{2}\ell n(1+\sqrt{2})} = 1.33,$$

$$L_i = \frac{27\sqrt{2}}{16} - \frac{3\sqrt{2}}{4(2-\sqrt{2}\ell n\overline{1+\sqrt{2}})} = 0.98.$$

For $Z_i = 1$, we find $(d_1, d_2) = (0.38, 0.62)$, and $(L_{11}, L_{12}, L_{13}, L_{22}, L_{23}, L_{33}) = (2.30, 2.56, 2.46, 5.39, 4.32, 3.61)$. These give $\sigma_S = 1.93 ne^2 \tau_{ee}/m_e$ for Spitzer's conductivity, in which the numerical coefficient differs but slightly from the more exact value 1.96 from numerical integration.

The transport coefficients for $Z_i = 1$ are in excellent agreements with the results in Refs. [4], [7], [10] when specialized to circular flux surfaces. (In comparison with Ref. [4], the coefficients (1.12,0.43,0.19,2.44) in Eq. (168) in Ref. [4] are calculated to be (1.12,0.43,0.19,2.40); (1.53,1.81,0.27,1.75) in Eq. (170) are (1.56,1.84,0.26,1.76); (0.51,1.95,2.44, 0.69,0.42) in Eq. (175) are (0.52,1.83,2.40,0.61,0.41); 0.48 in Eq. (173) is exactly reproduced. In comparison with Ref. [7] which gives fitted coefficients $K^{(0)} = (1.04, 1.20, 2.55, 2.30, 4.19, 1.83)$ in Table III, we calculated (1.12,1.25,2.63,2.40, 4.22,1.80). In comparison with Ref. [10], the coefficients (1.53,0.59,0.26,1.67) in Eq. (11.30) are calculated to be (1.53,0.59,0.26,1.64); (2.12,2.51,0.37,1.19) in Eq. (11.33) are (2.13,2.53,0.36,1.22); (1.66,0.47,0.29,1.31) in the equation on p. 190 are (1.64,0.42,0.28,1.23); 0.92 in Eq. (11.23) is exactly reproduced.) However, we have not been able to reproduce the results from the published works of Galeev and Sagdeev.

E. Transport of fields and plasma

The field and moment equations derived in Sec. II assume much simpler forms in the cylinder limit, which can be easily obtained by keeping the lowest-order terms in these equations. In the following, they will be presented in such order and forms that make it easy to discuss how they are to be solved.

With the "poloidal flux" ψ defined from Eq. (43), and the stream function for the poloidal inductive electric field modified from Eq. (39) by the replacement $\psi_E/R_0 \rightarrow \psi_E$, Eq. (37) becomes

$$\frac{\partial}{\partial \psi}\left(p + \frac{1}{8\pi}B_z^2\right) + J_\parallel = 0, \tag{95}$$

while the toroidal component of Faraday's law, Eq. (41), becomes

$$\nabla^2 \psi_E = -\frac{\partial B_z}{\partial t} - E_\parallel \frac{\partial B_z}{\partial \psi}. \tag{96}$$

The moment equations assume the form

$$\left(\frac{\partial n_e}{\partial t}\right)_\psi + \langle E_\parallel \rangle \frac{\partial n_e}{\partial \psi} + \frac{1}{A'}\frac{\partial}{\partial \psi}A'\Gamma_e = 0, \tag{97}$$

$$\frac{3}{2}\left(\frac{\partial n_e T_e}{\partial t}\right)_\psi + \langle E_\parallel \rangle \frac{3}{2}\frac{\partial n_e T_e}{\partial \psi} + \frac{1}{A'}\frac{\partial}{\partial \psi}A'Q_e = -Q_\Delta + J_\parallel \langle E_\parallel \rangle - \Gamma_e \frac{T_i}{Z_i}\left(\frac{1}{n_i}\frac{\partial n_i}{\partial \psi} + \overline{y - \frac{3}{2}}\,\frac{1}{T_i}\frac{\partial T_i}{\partial \psi}\right), \tag{98}$$

$$\frac{3}{2}\left(\frac{\partial n_i T_i}{\partial t}\right)_\psi + \langle E_\parallel \rangle \frac{3}{2}\frac{\partial n_i T_i}{\partial \psi} + \frac{1}{A'}\frac{\partial}{\partial \psi}A'Q_i = Q_\Delta, \tag{99}$$

where the flux average operation $\langle\ \rangle$ and area derivative A' are defined by Eqs. (47) and (46). In deriving Eq. (98) from Eq. (30), we have dropped the last term on the right side and used the approximations $R'_{\parallel e} = e\Gamma_e$ and $u'_{\parallel i} = U'$, justified when δ is small. The poloidal component of Faraday's law [Eq. (40)] and the parallel component of Ampere's law [Eq. (36)] simplify to

$$\frac{\partial \psi}{\partial t} = E_\parallel, \tag{100}$$

and

$$\nabla^2 \psi = 4\pi J_\parallel. \tag{101}$$

Finally, the system is completed by the inclusion of the flux-force relations given by Eqs. (90) and (32). It proves advantageous to replace Eqs. (90c) and (32) by

$$\langle E_\parallel \rangle = \left(\sigma_S - \sqrt{\delta} a L_{33}\frac{ne^2\tau_{ee}}{m_e}\right)^{-1}[J_\parallel + \sqrt{\delta} a n_e T_e (L_{31}A_1 + L_{32}A_2)], \tag{102}$$

in which $\langle E_\parallel \rangle$ is taken to be a flux and $J_\parallel$ a force.

Just as in the case of arbitrary aspect ratio discussed in Sec. II B, the quantities B_z and ψ_E can be eliminated using Eqs. (95) and (96). This leaves the variables $\psi, E_\parallel, n_e, T_e, T_i, J_\parallel$ to be determined from the remaining equations [Eqs. (97)–(102)], which can be considered to constitute a poloidal system. We consider the state of the plasma to be characterized by the flux functions n_e, T_e, T_i, and $J_\parallel$, and proceed to investigate how to evolve the state in time, determining along the way all other associated parameters. We observe that the time evolution of n_e, T_e, T_i can be obtained from the moment equations. The geometry factors a and A' required in these equations can be obtained by solving the parallel component of Ampere's law [Eq. (101)]. It remains to find means to advance $J_\parallel$ in time, for which we have at hands the last three equations of the set [Eqs. (100–102)]. The peculiarity of these three equations is that while Faraday's law involves the full poloidal dependence of $E_\parallel$ to describe the change of flux surface shapes, only $\langle E_\parallel \rangle$ appears in Ohm's law. Had a local form of Ohm's law been obtained, such as would be the case if $\langle E_\parallel \rangle$ in Eq. (102) were replaced by $E_\parallel$, it would be possible to eliminate $E_\parallel$ and $J_\parallel$ from the system, obtaining thereby a standard two-dimensional diffusion equation for the poloidal flux ψ. As it is, the system presents a nonstandard mathematical problem.

The difficulty of solving the neoclassical transport problem with shape changes of flux surfaces is well known and methods have been proposed for its solution [6,18,19]. In our discussion, we have traced the source of the difficulty to the nonlocal nature of Ohm's law. But we refrain from a discussion of the possible methods of solution. Difficulty also occurs when resistive magnetohydrodynamics theory is applied to describe the combined evolution of plasma and magnetic field [20]. However, the mathematical nature of the problem is not the same as the neoclassical transport problem.

IV. SUMMARY

In this work, we have rederived and generalized some existing results in neoclassical transport theory using a method that is mathematically rigorous and that avoids the use of the variational principle. The transport equations for large aspect ratio flux surfaces are also rewritten in a more transparent form, leading to a better understanding of the mathematical nature of the transport problem.

Using the method of matched-asymptotic expansions, we are able to analytically calculate the transport coefficients, providing justification for the use of a simplified collision operator. In this method, separate treatments are accorded to the freely circulating particles, which represent the majority of particles affected but slightly by the magnetic mirror, and the slow particles that are greatly affected. For the latter, we have reproduced the existing distribution functions by means of a consistent approximation to the collision operator. For the former, we have derived equations and boundary conditions that are not previously known, but are nevertheless essential in justifying the calculation of transport fluxes using only the distribution functions of the slow particles. The fluxes share a common geometry factor that has been obtained in a form accurate to leading order in an inverse aspect-ratio expansion.

In addition, we have calculated the corrections to the transport coefficients due to departure from the asymptotic banana regime, using an extension, to general geometry, of an existing treatment of the boundary layer in the region of the slow particles that separates the trapped and the circulating particles.

Finally, we have presented a complete set of field and moment equations in the cylinder limit that describes the joint evolution of the plasma and the electromagnetic field in the transport time scale. We have traced the origin of the nonstandard mathematical nature of the problem to the special form of Ohm's law in neoclassical theory, which relates a current density that is constant on a flux surface to the average inductive electric field on that surface.

With the exception of angular momentum transport and the associated dynamical evolution of the radial electrostatic field, the results in this paper are comprehensive, but are clearly very restrictive. The generalization to realistic aspect ratios and wider ranges of collisionality while maintaining the same degree of mathematical rigor is probably too hard an undertaking to be attempted by analytical means. In this regard, our results are best used as limits for checking numerical works. An important part of the purpose of this work would be served if it has directed attention to the many aspects of neoclassical transport theory, which require careful considerations if accurate results are desired.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy under Contract No. DE-FG03-95ER54309. The authors thank F. L. Hinton for numerous discussions on neoclassical transport theory.

APPENDIX: BOUNDARY LAYER

The analytic solution [Eq. (72)] for slow electrons and a similar one for slow ions in the banana regime suffer from discontinuities in the derivative with respect to κ^2 at the boundary $\kappa^2=1$ separating the trapped and circulating particles. This can be remedied when it is realized that the perturbation treatment of collisions for Eq. (61) and its analog for ions break down in a thin layer across this boundary because terms with the highest derivative in κ^2 have been neglected in a first approximation. In Ref. [5], a boundary layer analysis is described for circular flux surfaces, showing how the discontinuity can be removed and obtaining a correction to the diffusion coefficient. In the following, we shall adapt the analysis to the flux surfaces of arbitrary shape, and obtain corrections to all of the transport coefficients in the main text.

Consider first Eq. (61) for slow electrons. We can apply the transformation [Eq. (63)] without requiring g_S to be independent of the poloidal coordinate as implied by the perturbation treatment in the main text. The function g_S then satisfies the equation

$$\pm b\cdot\vec{\nabla} g_S^{\pm} - \frac{\nu_{*e}}{2\nu}\frac{\partial}{\partial\kappa^2}\xi_*\frac{\partial g_S^{\pm}}{\partial\kappa^2}=0, \qquad \text{(A1)}$$

where the dependence on sgn(ξ) is explicitly displayed. In the terminology of matched-asymptotic expansions, the solution [Eq. (65)] obtained by treating the collision term in Eq. (A1) as small is called the outer solution. The inner solution applies in the range of variables for which the two terms in Eq. (A1) are comparable, which turns out to be a layer with a thickness of the order of $\sqrt{\nu_{*e}}$ in κ^2 across the boundary $\kappa^2=1$ (see Fig. 1). In this layer, we can use the approximation $\xi_*=\sqrt{2(1-\hat{B})}$ in Eq. (A1). Using $\hat{b}\cdot\vec{\nabla} g_S^{\pm}=(|\vec{\nabla}\psi\times\vec{\nabla}\theta|/B_z)\partial g_S^{\pm}/\partial\theta$, Eq. (A1) can be further simplified by replacing θ by the variable φ for which

$$\frac{d\varphi}{d\theta}=\frac{B_z}{qR_0}\frac{1}{\langle\sqrt{1-\hat{B}}\rangle}\frac{\sqrt{1-\hat{B}}}{|\vec{\nabla}\psi\times\vec{\nabla}\theta|}, \qquad \text{(A2)}$$

where the safety factor q is given by

$$q=\frac{B_z}{2\pi R_0}\oint\frac{d\theta}{|\vec{\nabla}\psi\times\vec{\nabla}\theta|}, \qquad \text{(A3)}$$

and which has the range $[-\pi,+\pi]$. We also introduce the inner variable

$$\eta=\frac{\kappa^2-1}{\left(\frac{\nu_{*e}qR_0}{\sqrt{2}\nu}\langle\sqrt{1-\hat{B}}\rangle\right)^{1/2}}, \tag{A4}$$

which is allowed to vary between $\pm\infty$. With these transformations, Eq. (A1) in the boundary layer becomes

$$\pm\frac{\partial g_S^\pm}{\partial\varphi}=\frac{\partial^2 g_S^\pm}{\partial\eta^2}. \tag{A5}$$

The boundary conditions at $\varphi=\pm\pi$ are $g_S^\pm(\pi)=g_S^\pm(-\pi)$ for $\eta>0$, which pertains to the circulating electrons, and $g_S^+(\pm\pi)=g_S^-(\pm\pi)$ in the trapped electron region $\eta<0$. The boundary conditions for $\eta=\pm\infty$ are dictated by the requirement of matching to the outer solution. The vanishing of the outer solution for trapped electrons according to Eq. (72) requires $g_S^\pm\to 0$ as $\eta\to-\infty$.

The solution of Eq. (A5) with the aforementioned boundary conditions has been obtained in Ref. [5] using the Wiener-Hopf method. The most important aspect of the solution is that it has the asymptotic behavior

$$g_S^\pm=\pm A(\eta+1.21), \quad \eta\to+\infty, \tag{A6}$$

with an arbitrary constant A, where the numerical constant comes from the evaluation $1.21=2(1-1/\sqrt{2}+1/\sqrt{3}-\cdots)$.

For completeness, we paraphrase the solution given in Ref. [5]. We first replace $g_S^\pm$ by $S=g_S^+ + g_S^-$ and $D=g_S^+ - g_S^-$, and seek solutions with the symmetry properties $S(-\varphi)=-S(\varphi)$, $D(-\varphi)=D(\varphi)$. [The symmetries $S(-\varphi)=S(\varphi)$, $D(-\varphi)=-D(\varphi)$ lead to the trivial solution.] In the infinite strip between $\varphi=0$ and $\varphi=\pi$, the boundary conditions are now as follows:

$$S=0, \quad \partial D/\partial\varphi=0 \quad \text{for} \quad \varphi=0, \tag{A7}$$

$$S=0 \quad \text{for} \quad \varphi=\pi, \quad \eta>0, \tag{A8}$$

$$D=0 \quad \text{for} \quad \varphi=\pi, \quad \eta<0. \tag{A9}$$

Introducing the Fourier transforms

$$\tilde{S}(k,\varphi)=\int_{-\infty}^{+\infty}\frac{d\eta}{2\pi}e^{-ik\eta}S(\eta,\varphi),$$

$$\tilde{D}(k,\varphi)=\int_{-\infty}^{+\infty}\frac{d\eta}{2\pi}e^{-ik\eta}D(\eta,\varphi), \tag{A10}$$

where $\mathrm{Im}\,k>0$ for the first integral and $\mathrm{Im}\,k<0$ for the second to guarantee convergence in the presence of the nonvanishing values of S for $\eta\to-\infty$ and D for $\eta\to+\infty$, the differential equations and the boundary conditions in Eq. (A7) are satisfied by

$$\tilde{S}(k,\varphi)=-\phi(k)\sinh k^2\varphi, \quad \tilde{D}(k,\varphi)=\phi(k)\cosh k^2\varphi, \tag{A11}$$

where $\phi(k)$ is an arbitrary function. Imposing the boundary conditions in Eqs. (A8) and (A9),

$$\tilde{S}(k,\pi)=\int_{-\infty}^{0}\frac{d\eta}{2\pi}e^{-ik\eta}S(\eta,\pi),$$

$$\tilde{D}(k,\pi)=\int_{0}^{+\infty}\frac{d\eta}{2\pi}e^{-ik\eta}D(\eta,\pi), \tag{A12}$$

which imply that $\tilde{S}(k,\pi)$ and $\tilde{D}(k,\pi)$ are analytic in the upper and lower halves of the k planes, respectively. We now write $\tilde{S}(k,\pi)=-\tilde{D}(k,\pi)\tanh\pi k^2$ and, following Ref. [5], perform the factorization $\tanh\pi k^2=U(k)L(k)$, where

$$U(k)=\left(1+\frac{k}{\hat{k}_0}\right)^{-1}\prod_{n\neq 0}\left(1+\frac{k}{k_n}\right)\left(1+\frac{k}{\hat{k}_n}\right)^{-1}, \tag{A13a}$$

$$L(k)=\pi k^2\left(1-\frac{k}{\hat{k}_0}\right)^{-1}\prod_{n\neq 0}\left(1-\frac{k}{k_n}\right)\left(1-\frac{k}{\hat{k}_n}\right)^{-1}, \tag{A13b}$$

with $k_n=\sqrt{|n|}e^{i\pi/4}$, $\hat{k}_n=\sqrt{|n+\frac{1}{2}|}e^{i\pi/4}$ for $n<0$ and $k_n=\sqrt{n}e^{i3\pi/4}$, $\hat{k}_n=\sqrt{n+\frac{1}{2}}e^{i3\pi/4}$ for $n\geqslant 0$. As a result,

$$\frac{\tilde{S}(k,\pi)}{U(k)}=-\tilde{D}(k,\pi)L(k). \tag{A14}$$

Since U has no zeros in the upper half plane and L has no poles in the lower half plane, the left and right sides of Eq. (A14) are analytic in the upper and lower half planes, respectively. They also remain bounded as $|k|\to\infty$ because of the asymptotic behaviors $U\sim 1/k$, $L\sim k$, and $\tilde{S}(k,\pi)\sim\tilde{D}(k,\pi)\sim 1/k$. From Liouvilles' theorem, both sides of Eq. (A14) are equal to the same constant, which will be denoted by A. Therefore the following solution is obtained:

$$S=-A\int_{-\infty}^{+\infty}dke^{ik\eta}U\frac{\sinh k^2\varphi}{\sinh k^2\pi},$$

$$D=-A\int_{-\infty}^{+\infty}dke^{ik\eta}\frac{1}{L}\frac{\cosh k^2\varphi}{\cosh k^2\pi}, \tag{A15}$$

where the contours of integration are taken to be just below the real line. The asymptotic behaviors as $\eta\to-\infty$ can be obtained by closing the contours on the lower half plane, showing both S and D to vanish. For $\eta\to+\infty$, the contours are closed on the upper half plane. While S still vanishes, D does not because of the pole at $k=0$. The residue at this pole for the integrand is $(i/\pi)(\eta+2-2/\sqrt{2}+2/\sqrt{3}-\cdots)$, which leads to the asymptotic behavior [Eq. (A6)].

The asymptotic behavior [Eq. (A6)] is to be matched with the following expansion of the outer solution [Eq. (65)]:

$$g_S^{\pm}=\pm\left(K_2+\frac{K_1}{\sqrt{2}}\frac{\kappa^2-1}{\langle\sqrt{1-\hat{B}}\rangle}\right),\quad \kappa^2\to 1+. \tag{A16}$$

The matching is made after transforming η in Eq. (A6) back into κ^2 using Eq. (A4), and leads to

$$A=\left(\frac{\nu_{*e}qR_0}{2\sqrt{2}\nu}\frac{1}{\langle\sqrt{1-\hat{B}}\rangle}\right)^{1/2}K_1,\quad K_2=1.21A. \tag{A17}$$

As a result, the asymptotic behavior [Eq. (72)] is modified to become

$$f_S\to\pm\left[a-1.21\left(\frac{\nu_{*e}qR_0}{2\sqrt{2}\nu}\frac{1}{\langle\sqrt{1-\hat{B}}\rangle}\right)^{1/2}\right]A_e f_{e0},$$
$$\xi_*\to\pm\infty. \tag{A18}$$

When this is used to evaluate fluxes in the same manner as described in the main text, the transport coefficients L_{ij} in Eq. (90) are replaced by $L_{ij}-\gamma_e L'_{ij}$, where

$$\gamma_e=\frac{1.21(3\sqrt{\pi})^{1/2}}{4a\langle\sqrt{1-\hat{B}}\rangle^{1/2}}\sqrt{\overline{\nu}_{*e}},\quad \overline{\nu}_{*e}=\frac{qR_0}{\delta^{3/2}\tau_{ee}}\sqrt{\frac{m_e}{T_e}}, \tag{A19}$$

$$L'_{ij}=3\int_0^\infty dx\, e^{-x^2}\frac{(Z_i+G')^{3/2}}{x}\alpha_i\alpha_j. \tag{A20}$$

Replacing $A_e f_{e0}$ by $A_i f_{i0}$ and ν_{*e} by ν_{*i} in Eq. (A18) gives the corresponding asymptotic behavior for the distribution function of the slow ions. This leads to a modification of Eq. (84) for U'. Keeping linear terms in the correction, the parameter y in Eq. (84) is replaced by $y-\gamma_i y'$, where γ_i is given by a similar equation as Eq. (A19), and

$$\frac{y'}{y}=\int_0^\infty dx\, e^{-x^2}xG'^{3/2}\Big/\int_0^\infty dx\, e^{-x^2}x^3G'$$
$$-\int_0^\infty dx\, e^{-x^2}x^{-1}G'^{3/2}\Big/\int_0^\infty dx\, e^{-x^2}xG'. \tag{A21}$$

For the ion heat flux, the coefficient L_i is replaced by $L_i-\gamma_i L'_i$, where

$$L'_i=3\left(\int_0^\infty dx\, e^{-x^2}x^3G'^{3/2}-y\int_0^\infty dx\, e^{-x^2}xG'^{3/2}\right.$$
$$\left.-y'\int_0^\infty dx\, e^{-x^2}x^3G'\right). \tag{A22}$$

Using the numerical results

$$\int_0^\infty dx\, e^{-x^2}(x^{-1},x,x^3)G'^{3/2}=(0.34,0.21,0.30),$$

it is shown that $y'=-0.90$ and $L'_i=1.04$.

For the electron fluxes, the integral for L'_{11} is logarithmically divergent. Since this can be traced to the expansion in the electron-ion mass ratio, we approximate the integral with a cutoff at $\sqrt{m_e/m_i}$. With the numerical results

$$\int_0^\infty dx\, e^{-x^2}(x^{-1},x,x^3)(1+G')^{3/2}$$
$$=(0.70-\ln\sqrt{m_e/m_i},\ 0.96,\ 1.12),$$

the modification in the transport coefficients for a hydrogen plasma is found to be given by $(L'_{11},L'_{12},L'_{13},L'_{22},L'_{23},L'_{33})=(9.91,2.87,5.53,3.36,3.17,4.06)$.

For circular cross-section flux surfaces, using $\langle\sqrt{1-\hat{B}}\rangle=2/\pi$, we find that $\gamma_e=0.89\sqrt{\overline{\nu}_{*e}}$, $\gamma_i=0.89\sqrt{\overline{\nu}_{*i}}$. It is then possible to give the corrected transport coefficients as follows: [the coefficients (0.61, 0.95, 0.39, 1.00, 2.01, 0.56, 0.66, 1.00) below the correction terms can be compared with those given in Ref. [7], which are obtained by fitting numerical solutions. The corresponding values are (0.88, 1.03, 2.01, 0.76, 1.02, 0.45, 0.57, 0.68), as deduced from Eq. (6.133), Eq. (6.135), and Table III of Ref. [7]. The agreements are far less favorable than those for the leading terms as noted before]

$$y=1.33(1+0.61\sqrt{\overline{\nu}_{*i}}),\quad L_i=0.98(1-0.95\sqrt{\overline{\nu}_{*i}}),$$
$$L_{11}=2.30(1-0.39\sqrt{\overline{\nu}_{*e}}),\quad L_{12}=2.56(1-1.00\sqrt{\overline{\nu}_{*e}}),$$
$$L_{13}=2.46(1-2.01\sqrt{\overline{\nu}_{*e}}),$$
$$L_{22}=5.39(1-0.56\sqrt{\overline{\nu}_{*e}}),$$
$$L_{23}=4.32(1-0.66\sqrt{\overline{\nu}_{*e}}),\quad L_{33}=3.61(1-1.00\sqrt{\overline{\nu}_{*e}}).$$

[1] A. A. Galeev and R. Z. Sagdeev, Zh. Eksp. Teor. Fiz. **53**, 348 (1967) [Sov. Phys. JETP **26**, 233 (1968)].
[2] A. A. Galeev, Zh. Eksp. Teor. Fiz. **59**, 1378 (1970) [Sov. Phys. JETP **32**, 752 (1971)].
[3] A. A. Galeev and R. Z. Sagdeev, *Advances in Plasma Physics*, edited by A. Simon and W. B. Thompson (Wiley, New York, 1976), Vol. 6, p. 311.
[4] M. N. Rosenbluth, R. D. Hazeltine, and F. L. Hinton, Phys. Fluids **13**, 116 (1972).
[5] F. L. Hinton and M. N. Rosenbluth, Phys. Fluids **16**, 836 (1973).
[6] R. D. Hazeltine, F. L. Hinton, and M. N. Rosenbluth, Phys. Fluids **16**, 1645 (1973).
[7] F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976).
[8] S. P. Hirshman and D. J. Sigmar, Nucl. Fusion **21**, 1079 (1981).
[9] R. Balescu, *Transport Processes in Plasmas* (North-Holland,

Amsterdam, 1988), Vols. I and II.
[10] P. Helandar and D. J. Sigmar, *Collisional Transport in Magnetized Plasmas* (Cambridge University Press, Cambridge, 2002).
[11] K. H. Burrell *et al.*, Phys. Plasmas **8**, 2153 (2001).
[12] J. Kevorkian and J. D. Cole, *Perturbation Methods in Applied Mathematics* (Springer-Verlag, New York, 1981).
[13] P. H. Rutherford, Phys. Fluids **13**, 482 (1970).
[14] A. H. Glasser and W. B. Thompson, Phys. Fluids **16**, 95 (1973).
[15] J. Blum and J. Le Foll, Comput. Phys. Rep. **1**, 465 (1984).
[16] M. N. Rosenbluth, P. H. Rutherford, J. P. Taylor, E. A. Frieman, and L. M. Kovrizhnykh, *Plasma Physics and Controlled Nuclear Fusion Research* (IAEA, Vienna, 1971), Vol. 1, p. 495.
[17] F. L. Hinton and S. K. Wong, Phys. Fluids **28**, 3082 (1985).
[18] F. J. Helton, R. L. Miller, and J. M. Rawls, J. Comput. Phys. **24**, 117 (1977).
[19] S. P. Hirshman and S. C. Jardin, Phys. Fluids **22**, 731 (1979).
[20] H. Grad, P. N. Hu, and D. Stevens, Proc. Natl. Acad. Sci. U.S.A. **72**, 3789 (1975).