Exact finite-size corrections of the free energy for the square lattice dimer model under different boundary conditions

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We express the partition functions of the dimer model on finite square lattices under five different boundary conditions (free, cylindrical, toroidal, Möbius strip, and Klein bottle) obtained by others (Kasteleyn, Temperley and Fisher, McCoy and Wu, Brankov and Priezzhev, and Lu and Wu) in terms of the partition functions with twisted boundary conditions $Z_{\alpha,\beta}$ with $(\alpha,\beta)=(1/2,0)$, $(0,1/2)$ and $(1/2,1/2)$. Based on such expressions, we then extend the algorithm of Ivashkevich, Izmailian, and Hu [J. Phys. A **35**, 5543 (2002)] to derive the exact asymptotic expansion of the logarithm of the partition function for all boundary conditions mentioned above. We find that the aspect-ratio dependence of finite-size corrections is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis. We have also established several groups of identities relating dimer partition functions for the different boundary conditions.

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I. INTRODUCTION

The dimer model was originally introduced to represent physical adsorption of diatomic molecules on crystal surfaces [1]. The surface may be considered as a regular lattice which attracts the diatomic molecules (dimers) in such a way that each dimer fills two neighboring lattice sites and with crucial constraint that no lattice site is covered by two dimers. The exact calculation of partition functions of the dimer model on the $\mathcal{M}\times\mathcal{N}$ square lattice under different boundary conditions (Figs. 1 and 2) has attracted the attention of researchers for more than 40 years. In 1961, Kasteleyn [2] obtained exact partition functions for the dimer model on the square lattice with both free and toroidal boundary conditions. Fisher [3], Temperley and Fisher [4] also solved the case of free boundary case independently. Ferdinand [5] calculated finite-size corrections up to the first order for the free energy of the dimer model on $\mathcal{M}\times\mathcal{N}$ square lattices with both free and toroidal boundary conditions for different parities of $\mathcal{M}$ and $\mathcal{N}$. In 1973, McCoy and Wu [6] calculated exact partition functions for cylindrical boundary conditions. In 1985, Bhattacharjee and Nagle [7] studied the finite-size effect of an anisotropic dimer model of domain walls on the brick lattice. In 1993, Brankov and Priezzhev [8] obtained the exact partition function for a Möbius strip. In 1999 and 2002, Lu and Wu obtained exact partition functions for a Möbius strip and a Klein bottle [9,10] and calculated finite-size corrections up to the first order for $\mathcal{M}\times\mathcal{N}$ lattices when both $\mathcal{M}$ and $\mathcal{N}$ are even. Very recently Wu [11] obtained exact partition function for the dimer model on the $2M\times(2N-1)$ square lattice with cylindrical boundary conditions. The interest in dimer model was renewed with the discovery of high-temperature superconductivity and also with recent work on domino tilings (which are equivalent to dimers on a square lattice) of an Aztec diamond, demonstrating a strong effect of the boundary on a typical domino configuration [12].

Finite-size scaling and finite-size corrections in finite critical systems and their boundary effects have attracted much attention in recent decades [13–28], especially in the Ising [17–28] and percolation models [15,16]. Many of such studies have been based on Monte Carlo simulations [29] and a few of them are based on analytic calculations [19,21–28]. Very recently, Ivashkevich, Izmailian, and Hu (IIH) [23] proposed a systematic method to compute finite-size corrections to the partition functions and their derivatives of free models on torus, including Ising model, dimer model, and Gaussian model. Their approach is based on relations between the terms of the asymptotic expansion and the so-called Kronecker's double series [23] which are directly related to elliptic θ functions. Expressing the final result in terms of θ functions avoids messy sums (as in some earlier

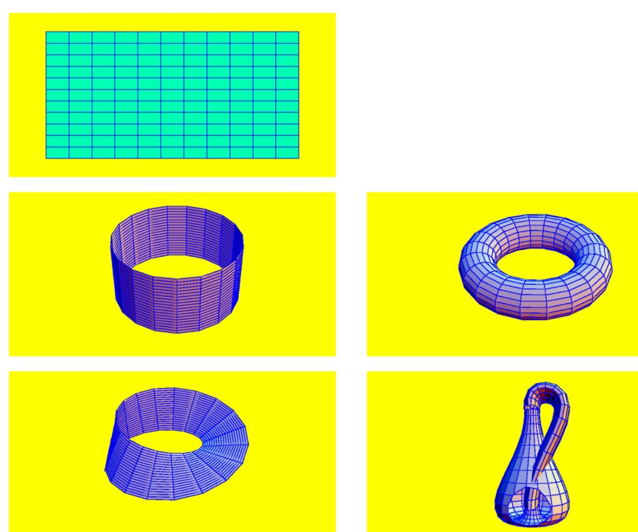

FIG. 1. Illustration of the rectangular lattice with different boundary conditions: free, cylinder, torus, Möbius strip, and Klein bottle.

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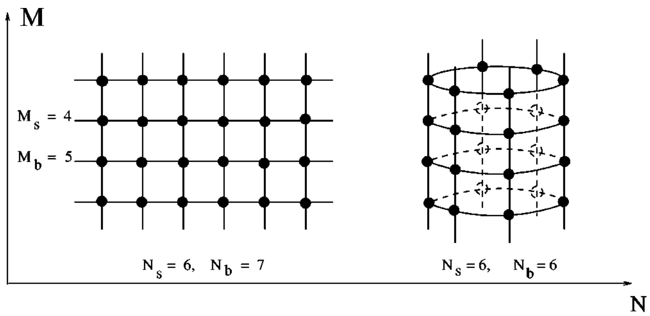

FIG. 2. Example of lattices with the free and cylindrical boundary conditions; relations between the number of the edges ($\mathcal{M}=M_b,\mathcal{N}=N_b$) and the number of the sites (M_s,N_s) of the lattice are given in the first paragraph of Sec. II.

works) and greatly simplifies the task of verifying the behavior of the different terms in the asymptotic expansion under duality transformation $M\leftrightarrow N$. Using this approach, Salas [24] computed the finite-size corrections to the free energy, internal energy, and specific heat of the critical Ising model on triangular and honeycomb lattices wrapped on a torus and quite recently Izmailian, Oganesyan, and Hu [27] obtained similar finite-size corrections of the Ising model on a square lattice with Braskamp-Kunz boundary conditions. Using exact partition functions [30] and finite-size corrections of the critical Ising model on the square, plane triangular, and honeycomb lattices with periodic-aperiodic boundary conditions, Wu, Hu, and Izmailian [28] obtained universal finite-size scaling functions for the free energy, internal energy, and specific heat of the Ising model with exact nonuniversal metric factors.

In the present paper, we relate the exact partition functions of the dimer model on the square lattice under free, cylindrical, toroidal, Möbius strip, and Klein bottle boundary conditions obtained by Kasteleyn [2], Temperley and Fisher [3,4], McCoy and Wu [6], Brankov and Priezzhev [8], and Lu and Wu [9–11] to the partition functions with twisted boundary conditions $Z_{\alpha,\beta}$ with $(\alpha,\beta)=(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$ (Sec. II). Based on such expressions, we derive several groups of identities relating dimer partition functions for the different boundary conditions (Sec. III). We then extend IIH's algorithm [23] to derive the exact asymptotic expansions of the logarithm of the partition functions for all boundary conditions and write down the expansion coefficients up to the second order (Sec. IV). We find that the aspect-ratio dependence of finite-size corrections is sensitive to boundary conditions and the parity of the number of lattice sites along the lattice axis (Fig. 3). We also discuss our results and problems for further studies (Sec. V).

II. DIMER MODEL UNDER VARIOUS BOUNDARY CONDITIONS

Consider a dimer model on an $M_s\times N_s$ square lattice of M_sN_s sites with M_s rows and N_s columns. The lattice forms a cylinder if there are periodic boundary conditions in the horizontal directions and free boundary conditions in the vertical direction, a torus if there are periodic boundary conditions in both directions, a Möbius strip if there are twisted boundary conditions in the horizontal direction and free boundary conditions in the vertical direction, and a Klein bottle if, in addition to the twisted boundary conditions in the horizontal directions, there are periodic boundary conditions in the vertical direction (Fig. 1). We have the following correspondence between the number of the edges ($\mathcal{M},\mathcal{N}$) and the number of the sites (M_s,N_s) under different boundary conditions (Fig. 2): $\mathcal{M}=M_s$ and $\mathcal{N}=N_s$ for torus and Klein bottle, $\mathcal{M}=M_s+1$ and $\mathcal{N}=N_s$ for cylinder and Möbius strip, $\mathcal{M}=M_s+1$ and $\mathcal{N}=N_s+1$ for free boundary conditions.

The partition function of the dimer model on an $\mathcal{M}\times\mathcal{N}$ lattice is given by

$$Z_{\mathcal{M},\mathcal{N}}(z_v,z_h)=\sum z_v^{n_v}z_h^{n_h}, \tag{1}$$

where summation is taken over all dimer covering configurations, z_v and z_h are, respectively, dimer weight in the horizontal and vertical directions, n_v and n_h are, respectively, the number of vertical and horizontal dimers. In what follows, we will show that the exact asymptotic expansion of the logarithm of the partition function can be written as

$$\ln Z_{\mathcal{M},\mathcal{N}}(z)=f_{bulk}S+\mathcal{N}f_{1s}(z_h,z_v)+\mathcal{M}f_{2s}(z_h,z_v)+f_0(z\rho)+\sum_{p=1}^{\infty}f_p(z\rho)S^{-p}, \tag{2}$$

with $S=\mathcal{M}\mathcal{N}$, $z=z_h/z_v$, and $\rho=\mathcal{M}/\mathcal{N}$, which is the aspect ratio and will be denoted by ρ_1 for the Möbius strip and cylindrical boundary conditions, and by ρ_2 for the free boundary conditions. The explicit expression of the partition function depends crucially on whether M_s and N_s are even or odd, and since the total number of sites must be even if the lattice is to be completely covered by dimers, we will consider three cases: $M_s=2M$, $N_s=2N$, $M_s=2M-1$, $N_s=2N$; $M_s=2M$, $N_s=2N-1$.

Dimers on 2M×2N lattices. The partition function of the dimer model on $2M\times2N$ torus has been obtained by Kasteleyn [2] and can be written as

$$Z_{2M,2N}^{torus}(z)=\frac{z_v^{2MN}}{2}[Z_{1/2,1/2}^2(z,M,N)+Z_{0,1/2}^2(z,M,N)+Z_{1/2,0}^2(z,M,N)]. \tag{3}$$

Here we have introduced the partition function with twisted boundary conditions $Z_{\alpha,\beta}(z,M,N)$,

$$Z_{\alpha,\beta}^2(z,M,N)=\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}4\left[z^2\sin^2\left(\frac{\pi(n+\alpha)}{N}\right)+\sin^2\left(\frac{\pi(m+\beta)}{M}\right)\right]. \tag{4}$$

With the help of the identity [31]

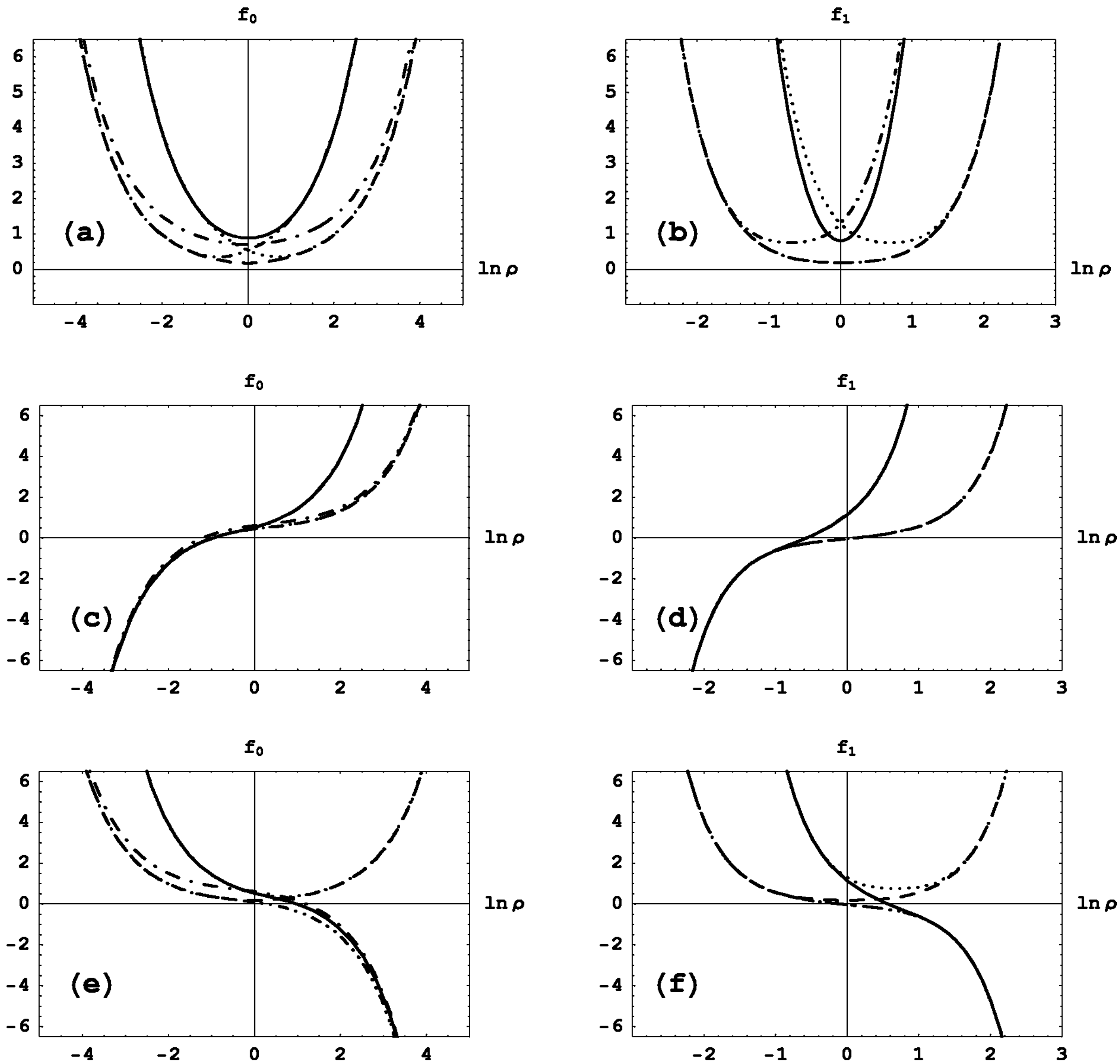

FIG. 3. Aspect-ratio (ρ) dependence of finite-size correction terms f_0 and f_1 for the free energy of the square lattice dimer model with the toroidal (solid lines), free (dot-dashed lines), cylindrical (dot-dot–dashed lines), Möbius strip (dashed lines), and Klein bottle (dotted lines) boundary conditions (BC): (a) and (b) for the $2M\times 2N$ lattice, (c) and (d) for the $(2M-1)\times 2N$ lattice, and (e) and (f) for the $2M\times(2N-1)$ lattice. Aspect ratio ρ is defined in the second paragraph of Sec. II. In (a) the lowest symmetry curve is for the Möbius strip (dashed line), which collapses with the curve for the Klein bottle for $\ln\rho>1$ and with the curve for the cylinder for $\ln\rho<-1$. In (b) the lowest symmetry curve is for the Möbius strip and plane (free boundary conditions). In (c) and (d) near $\ln\rho=1$ the upper curves are for the torus and cylinder and lower curves are for the other three cases. Note that f_0 vanishes at $\rho\approx 0.303\,468$ for the free boundary conditions, at $\rho\approx 0.378\,978$ for the torus and cylinder, and at $\rho\approx 0.378\,408$ for the Möbius strip and Klein bottle; f_1 vanishes at $\rho\approx 0.567\,436$ for the torus and cylinder and at $\rho\approx 1.132\,912$ for the Möbius strip, Klein bottle, and free boundary conditions. In (e) and (f) near $\ln\rho=2$ the upper curves are for the Möbius strip and Klein bottle and lower curves are for the other three cases. Note that f_0 vanishes at $\rho\approx 3.263\,732$ for the free boundary conditions, at $\rho\approx 2.641\,441$ for the torus, and at $\rho\approx 1.313\,279$ for the cylinder; f_1 vanishes at $\rho\approx 1.761\,351$ for the torus and at $\rho\approx 0.881\,437$ for the cylindrical and free boundary conditions.

$$4|\sinh(M\,\omega+i\,\pi\beta)|^2=4[\sinh^2 M\,\omega+\sin^2\pi\beta]=\prod_{m=0}^{M-1}4\left[\sinh^2\omega+\sin^2\left(\frac{\pi(m+\beta)}{M}\right)\right], \tag{5}$$

$Z_{\alpha,\beta}(z,M,N)$ can be transformed into a simpler form:

$$Z_{\alpha,\beta}(z,M,N)=\prod_{n=0}^{N-1}2\left|\sinh\left[M\,\omega_z\left(\frac{\pi(n+\alpha)}{N}\right)+i\,\pi\beta\right]\right|, \tag{6}$$

where $\omega_z(k)=\mathrm{arcsinh}(z\sin k)$. Note that the general theory about the asymptotic expansion of $Z_{\alpha,\beta}(1,M,N)$ has been given in Ref. [23], which can be easily extended to the case with arbitrary z (see the Appendix).

Let us consider the symmetry properties of the partition function $Z_{\alpha,\beta}(z,M,N)$. From Eq. (4), one can easily verify that

$$Z_{\alpha,\beta}(z,M,N)=Z_{\alpha,-\beta}(z,M,N)=Z_{-\alpha,\beta}(z,M,N),$$

$$Z_{\alpha,\beta}(z,M,N)=Z_{1+\alpha,\beta}(z,M,N)=Z_{\alpha,1+\beta}(z,M,N).$$

These imply that we need only consider α and β in $[0,1]$. Other useful identities are

$$Z_{\alpha,\beta}(z,M,2N)=Z_{\alpha/2,\beta}(z,M,N)Z_{(1-\alpha)/2,\beta}(z,M,N), \quad (7)$$

$$Z_{\alpha,\beta}(z,2M,N)=Z_{\alpha,\beta/2}(z,M,N)Z_{\alpha,(1-\beta)/2}(z,M,N), \quad (8)$$

$$Z_{\alpha,\beta}\left(\frac{1}{z},M,N\right)=\frac{1}{z^{MN}}Z_{\beta,\alpha}(z,N,M). \quad (9)$$

In particular, from the identities of Eqs. (7) and (8) one can obtain that

$$Z_{1/2,0}(z,M,2N)=Z^2_{1/4,0}(z,M,N), \quad (10)$$

$$Z_{1/2,1/2}(z,M,2N)=Z^2_{1/4,1/2}(z,M,N), \quad (11)$$

$$Z_{1/4,0}(z,2M,N)=Z_{1/4,0}(z,M,N)Z_{1/4,1/2}(z,M,N), \quad (12)$$

$$Z_{1/2,0}(z,2M,N)=Z_{1/2,0}(z,M,N)Z_{1/2,1/2}(z,M,N). \quad (13)$$

Finally, Eq. (9) implies that we need only consider z in the interval $[0,1]$.

Thus the partition function for the dimer model on $2M\times 2N$ torus is expressed in terms of the only object $Z_{\alpha,\beta}(z,M,N)$ with $(\alpha,\beta)=(0,\frac{1}{2}),(\frac{1}{2},0),(\frac{1}{2},\frac{1}{2})$.

In what follows, we will show that the partition function of the dimer model under four different boundary conditions (free, cylindrical, Klein bottle, and Möbius strip) can be expressed in terms of $Z_{1/2,1/2}(z,K,L)$ only, namely,

$$Z^{free}_{2M,2N}(z)=z_v^{2MN}\left[\frac{(1+z^2)^{1/2}Z_{1/2,1/2}(z,2M+1,2N+1)}{2z^{2N+1}\cosh[(2M+1)\operatorname{arcsinh} z]\cosh\left[(2N+1)\operatorname{arcsinh}\frac{1}{z}\right]}\right]^{1/2}, \quad (14)$$

$$Z^{cyl}_{2M,2N}(z)=z_v^{2MN}\frac{Z_{1/2,1/2}(z,2M+1,N)}{2z^N\cosh\left(N\operatorname{arcsinh}\frac{1}{z}\right)}, \quad (15)$$

$$Z^{Klein}_{2M,2N}(z)=z_v^{2MN}Z_{1/2,1/2}(z,M,2N), \quad (16)$$

$$Z^{Mob}_{2M,2N}(z)=z_v^{2MN}\left[\frac{Z_{1/2,1/2}(z,2M+1,2N)}{2z^{2N}\cosh\left(2N\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}. \quad (17)$$

The partition function of $2M\times 2N$ Klein bottle is given by [9]

$$Z^{Klein}_{2M,2N}(z)=z_v^{2MN}\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}4\left[z^2\sin^2\left(\frac{\pi(n+1/4)}{N}\right)+\sin^2\left(\frac{\pi(m+1/2)}{M}\right)\right]. \quad (18)$$

It is easy to see from Eqs. (4) and (18) that

$$Z^{Klein}_{2M,2N}(z)=z_v^{2MN}Z^2_{1/4,1/2}(z,M,N). \quad (19)$$

Now using identity given by Eq. (11), we finally obtain Eq. (16).

In the case of the free boundary conditions, the exact partition function [2] is

$$Z^{free}_{2M,2N}(z)=z_v^{2MN}\prod_{n=1}^{N}\prod_{m=1}^{M}4\left[z^2\cos^2\frac{\pi n}{2N+1}+\cos^2\frac{\pi m}{2M+1}\right]. \quad (20)$$

Let us change variables n and m in the following way: $(n\to N-n$ and $m\to M-m)$. Then the partition function given by Eq. (20) can be transformed to the following form:

$$Z^{free}_{2M,2N}(z)=z_v^{2MN}\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}4\left[z^2\sin^2\frac{\pi(n+1/2)}{2N+1}+\sin^2\frac{\pi(m+1/2)}{2M+1}\right]. \quad (21)$$

Now we first express double products $\Pi_{n=0}^{2N}\Pi_{m=0}^{2M}f(n,m)$ in terms of $\Pi_{n=0}^{N-1}\Pi_{m=0}^{M-1}f(n,m)$, where

$$f(n,m)=4\left[z^2\sin^2\frac{\pi(n+1/2)}{2N+1}+\sin^2\frac{\pi(m+1/2)}{2M+1}\right]. \quad (22)$$

It it easy to show that $f(2N-n,m)=f(n,2M-m)=f(n,m)$ and thus

$$\prod_{n=0}^{2N}\prod_{m=0}^{2M} f(n,m) = \frac{\prod_{n=0}^{2N} f(n,M)\prod_{m=0}^{2M} f(N,m)}{f(N,M)}\left(\prod_{n=0}^{N-1}\prod_{m=0}^{M-1} f(n,m)\right)^4, \quad (23)$$

with $f(N,M)=4(1+z^2)$. With the help of the identity, Eq. (5) the products $\Pi_{m=0}^{2M} f(N,m)$ and $\Pi_{n=0}^{2N} f(n,M)$ can be written as

$$\prod_{n=0}^{2N} f(n,M)=4z^{2(2N+1)}\cosh^2\left[(2N+1)\operatorname{arcsinh}\frac{1}{z}\right], \quad (24)$$

$$\prod_{m=0}^{2M} f(N,m)=4\cosh^2[(2M+1)\operatorname{arcsinh} z]. \quad (25)$$

Now using Eqs. (21)–(25), the partition function of dimers with the free boundary conditions finally can be written as Eq. (14).

The partition functions of the dimer model for the cylindrical boundary condition [6] and the Möbius strip [9,8] are given by

$$Z^{cyl}_{2M,2N}(z)=z_v^{2MN}\prod_{n=1}^{N}\prod_{m=1}^{M}4\left[z^2\sin^2\frac{\pi(n-1/2)}{N}+\cos^2\frac{\pi m}{2M+1}\right]$$
$$=z_v^{2MN}\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}4\left[z^2\sin^2\frac{\pi(n+1/2)}{N}+\sin^2\frac{\pi(m+1/2)}{2M+1}\right],$$

$$Z^{Mob}_{2M,2N}(z)=z_v^{2MN}\prod_{n=1}^{N}\prod_{m=1}^{M}4\left[z^2\sin^2\frac{\pi(n-1/4)}{N}+\cos^2\frac{\pi m}{2M+1}\right]$$
$$=z_v^{2MN}\prod_{n=0}^{N-1}\prod_{m=0}^{M-1}4\left[z^2\sin^2\frac{\pi(n+1/4)}{N}+\sin^2\frac{\pi(m+1/2)}{2M+1}\right].$$

Following the same way as in the case of the free boundary condition, we can obtain Eq. (15) for the cylindrical boundary condition and

$$Z^{Mob}_{2M,2N}(z)=\frac{z_v^{2MN}}{\sqrt{2}z^N\cosh^{1/2}\left(2N\operatorname{arcsinh}\frac{1}{z}\right)}\times Z_{1/4,1/2}(z,2M+1,2N) \quad (26)$$

for the Möbius strip. Using the identity given by Eq. (11), we finally arrived at Eq. (17).

Dimers on (2M−1)×2N lattices. In what follows, we will show that the partition function of the dimer model under five different boundary conditions of Fig. 1 can be expressed in terms of $Z_{1/2,0}(z,K,L)$ only, namely,

$$Z^{torus}_{2M-1,2N}(z)=z_v^{N(2M-1)}Z_{1/2,0}(z,2M-1,N), \quad (27)$$

$$Z^{free}_{2M-1,2N}(z)=z_v^{N(2M-1)}\left[\frac{(1+z^2)^{1/2}Z_{1/2,0}(z,2M,2N+1)}{2z^{2N+1}\sinh(2M\operatorname{arcsinh} z)\cosh\left((2N+1)\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}, \quad (28)$$

$$Z^{cyl}_{2M-1,2N}(z)=\frac{z_v^{N(2M-1)}}{2z^N}\frac{Z_{1/2,0}(z,2M,N)}{\cosh\left(N\operatorname{arcsinh}\frac{1}{z}\right)}, \quad (29)$$

$$Z^{Klein}_{2M-1,2N}(z)=\sqrt{2}z_v^{N(2M-1)}Z^{1/2}_{1/2,0}(z,2M-1,2N), \quad (30)$$

$$Z^{Mob}_{2M-1,2N}(z)=\frac{z_v^{N(2M-1)}}{z^N}\left[\frac{Z_{1/2,0}(z,2M,2N)}{\cosh\left(2N\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}. \quad (31)$$

The partition function for torus [2] has a form

$$Z^{torus}_{2M-1,2N}(z)=\frac{1}{2}z_v^{N(2M-1)}\Bigg\{\left(\prod_{n=1}^{N}\prod_{m=1}^{2M-1}4\left[z^2\sin^2\frac{\pi n}{N}+\sin^2\frac{\pi(2m-1)}{2M-1}\right]\right)^{1/2}+\left(\prod_{n=1}^{N}\prod_{m=1}^{2M-1}4\left[z^2\sin^2\frac{\pi\left(n-\frac{1}{2}\right)}{N}+\sin^2\frac{2\pi m}{2M-1}\right]\right)^{1/2}+\left(\prod_{n=1}^{N}\prod_{m=1}^{2M-1}4\left[z^2\sin^2\frac{\pi\left(n-\frac{1}{2}\right)}{N}+\sin^2\frac{\pi(2m-1)}{2M-1}\right]\right)^{1/2}\Bigg\}. \quad (32)$$

The first term on the right-hand side of Eq. (32) is zero; the second and third terms are equal to each other according to the following relations:

$$\prod_{m=1}^{2M-1}\left(a+\sin^2\frac{\pi(2m-1)}{2M-1}\right)=\prod_{m=1}^{2M-1}\left(a+\sin^2\frac{\pi 2m}{2M-1}\right)=\prod_{m=1}^{2M-1}\left(a+\sin^2\frac{\pi m}{2M-1}\right). \quad (33)$$

Using Eq. (33) in Eq. (32), we obtain Eq. (27). It is interesting to note that in this case the partition function with the toroidal boundary condition [Eq. (27)] is very simple and expressed only in terms of $Z_{1/2,0}(z,2M-1,2N)$.

The partition function for the Klein bottle boundary condition [10] has the following form:

$$Z^{Klein}_{2M-1,2N}=\frac{z_v^{N(2M-1)}}{z^N}\prod_{n=1}^{N}\prod_{m=1}^{M}4\left[z^2\sin^2\frac{\pi\left(n-\frac{1}{4}\right)}{N}+\sin^2\frac{\pi(2m-1)}{2M-1}\right]. \quad (34)$$

Using the same transformations as in the case (M_s,N_s) =(even, even) and the relation

$$\prod_{m=1}^{M-1}\left(a+\sin^2\frac{\pi(2m-1)}{2M-1}\right)=\prod_{m=1}^{M-1}\left(a+\sin^2\frac{\pi 2m}{2M-1}\right)=\prod_{m=1}^{M-1}\left(a+\sin^2\frac{\pi m}{2M-1}\right), \quad (35)$$

we can obtain Eq. (30).

The partition functions of the dimer model for the free boundary condition [2], the cylindrical boundary condition [6], and the Möbius strip [8,9] are given by

$$Z^{free}_{2M-1,2N}(z)=z_v^{N(2M-1)}\left(\prod_{n=1}^{N}\prod_{m=1}^{2M-1}4\left[z^2\cos^2\frac{\pi n}{2N+1}+\cos^2\frac{\pi m}{2M}\right]\right)^{1/2}, \quad (36)$$

$$Z^{cyl}_{2M-1,2N}(z)=z_v^{N(2M-1)}\left(\prod_{n=1}^{N}\prod_{m=1}^{2M-1}4\left[z^2\sin^2\frac{\pi\left(n-\frac{1}{2}\right)}{N}+\cos^2\frac{\pi m}{2M}\right]\right)^{1/2}, \quad (37)$$

$$Z^{Mob}_{2M-1,2N}=\frac{z_v^{N(2M-1)}}{z^N}\prod_{n=1}^{N}\prod_{m=1}^{M}4\left[z^2\sin^2\frac{\pi\left(n-\frac{1}{4}\right)}{N}+\cos^2\frac{\pi m}{2M}\right]. \quad (38)$$

Following the same procedure as in the case (M_s,N_s) =(even, even), we can obtain Eqs. (28), (29), and (31).

Dimers on 2M×(2N−1) lattices. Here we will show that the partition functions can be expressed in terms of $Z_{0,1/2}(z,K,L)$ or $Z_{1/2,1/2}(z,K,L)$ as

$$Z^{torus}_{2M,2N-1}(z)=z_v^{M(2N-1)}Z_{0,1/2}(z,M,2N-1), \tag{39}$$

$$Z^{free}_{2M,2N-1}(z)=z_v^{M(2N-1)}\left[\frac{(1+z^2)^{1/2}Z_{0,1/2}(z,2M+1,2N)}{2z^{2N}\sinh\left(2N\operatorname{arcsinh}\frac{1}{z}\right)\cosh[(2M+1)\operatorname{arcsinh} z]}\right]^{1/2}, \tag{40}$$

$$Z^{cyl}_{2M,2N-1}(z)=z_v^{M(2N-1)}\left[\frac{Z_{0,1/2}(z,2M+1,2N-1)}{2z^{2N-1}\sinh\left((2N-1)\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}, \tag{41}$$

$$Z^{Klein}_{4M,2N-1}(z)=z_v^{2M(2N-1)}Z_{1/2,1/2}(z,2M,2N-1), \tag{42}$$

$$Z^{Mob}_{4M,2N-1}(z)=z_v^{2M(2N-1)}\left[\frac{Z_{1/2,1/2}(z,4M+1,2N-1)}{2z^{2N-1}\cosh\left((2N-1)\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}. \tag{43}$$

The partition functions of the dimer model for the toroidal boundary condition [2], the free boundary condition [3], and the cylindrical boundary condition [11] are given by

$$Z^{torus}_{2M,2N-1}(z)=\frac{1}{2}z_h^{M(2N-1)}\left\{\left(\prod_{n=1}^{(2N-1)}\prod_{m=1}^{M}4\left[\sin^2\frac{\pi m}{M}+z^2\sin^2\frac{\pi(2n-1)}{2N-1}\right]\right)^{1/2}+\left(\prod_{n=1}^{(2N-1)}\prod_{m=1}^{M}4\left[\sin^2\frac{\pi\left(m-\frac{1}{2}\right)}{M}+z^2\sin^2\frac{2\pi n}{2N-1}\right]\right)^{1/2}+\left(\prod_{n=1}^{(2N-1)}\prod_{m=1}^{M}4\left[\sin^2\frac{\pi\left(m-\frac{1}{2}\right)}{M}+z^2\sin^2\frac{\pi(2n-1)}{2N-1}\right]\right)^{1/2}\right\}, \tag{44}$$

$$Z^{free}_{2M,2N-1}(z)=z_v^{M(2N-1)}\left(\prod_{m=1}^{M}\prod_{n=1}^{2N-1}4\left[z^2\cos^2\frac{\pi n}{2N}+\cos^2\frac{\pi m}{2M+1}\right]\right)^{1/2}, \tag{45}$$

$$Z^{cyl}_{2M,2N-1}(z)=z_v^{M(2N-1)}\left(\prod_{m=1}^{M}\prod_{n=1}^{2N-1}4\left[z^2\sin^2\frac{\pi n}{2N-1}+\cos^2\frac{\pi m}{2M+1}\right]\right)^{1/2}. \tag{46}$$

Following along the same lines as in previous cases, we can obtain Eqs. (39)–(41).

The partition functions of the dimer model for the Möbius strip and Klein bottle boundary conditions are given by [10]

$$Z^{Mob}_{2M,2N-1}(z)=z_v^{M(2N-1)}\mathrm{Re}\left[(1-i)\prod_{m=1}^{M}\prod_{n=1}^{2N-1}2\left(i(-1)^{M+m+1}z\sin\frac{(4n-1)\pi}{2(2N-1)}+\cos\frac{m\pi}{2M+1}\right)\right], \tag{47}$$

$$Z^{Klein}_{2M,2N-1}(z)=z_v^{M(2N-1)}\mathrm{Re}\left[(1-i)\prod_{m=1}^{M}\prod_{n=1}^{2N-1}2\left(i(-1)^{M+m+1}z\sin\frac{(4n-1)\pi}{2(2N-1)}+\sin\frac{(2m-1)\pi}{2M}\right)\right]. \tag{48}$$

For $M_s=4M$ using the same method as in the case (M_s,N_s) = (even, even) from Eqs. (47) and (48), we obtain the partition functions for the Klein bottle [Eq. (42)] and Möbius strip [Eq. (43)] boundary conditions. We cannot find such simplification for the case $M_s=4M-2$. It is interesting to note that partition functions of the Klein bottle and Möbius strip for the case $M_s=4M$ can be written in the common form for both (even, even) case [Eqs. (16) and (17)] and (even, odd) case [Eqs. (42) and (43)],

$$Z^{Klein}_{4M,N}(z)=z_v^{2MN}Z_{1/2,1/2}(z,2M,N), \tag{49}$$

$$Z^{Mob}_{4M,N}(z)=z_v^{2MN}\left[\frac{Z_{1/2,1/2}(z,4M+1,N)}{2z^{N}\cosh\left(N\operatorname{arcsinh}\frac{1}{z}\right)}\right]^{1/2}. \tag{50}$$

III. SYMMETRY AND IDENTITIES OF THE DIMER MODEL

In the case of the periodic and the free boundary conditions in both horizontal and vertical directions, we expect symmetry under the interchanges $z_h \leftrightarrow z_v$ (or $z \leftrightarrow 1/z$) and $M \leftrightarrow N$. To verify this, we use the identity given by Eq. (9) and the expressions for the partition functions [see Eqs. (3), (14), (27), (28), (39), and (40)], from which the results

$$Z_{M,N}^{torus}(1/z) = Z_{N,M}^{torus}(z), \tag{51}$$

$$Z_{M,N}^{free}(1/z) = Z_{N,M}^{free}(z) \tag{52}$$

are evident. It is also evident that we cannot expect such symmetry for the Klein bottle, Möbius strip, and cylindrical boundary conditions. Instead the partition functions for such boundary conditions under the interchange $z_h \leftrightarrow z_v$ (or $z \leftrightarrow 1/z$) transform in the following way:

$$Z_{2M,2N}^{Klein}(1/z) = z_v^{2MN} Z_{1/2,1/2}(z,2N,M), \tag{53}$$

$$Z_{2M-1,2N}^{Klein}(1/z) = \sqrt{2} z_v^{N(2M-1)} \sqrt{Z_{0,1/2}(z,2N,2M-1)}, \tag{54}$$

$$Z_{4M,2N-1}^{Klein}(1/z) = z_v^{2M(2N-1)} Z_{1/2,1/2}(z,2N-1,2M), \tag{55}$$

$$Z_{2M,2N}^{Mob}(1/z) = z_v^{2MN} \left[\frac{Z_{1/2,1/2}(z,2N,2M+1)}{2\cosh(2N \operatorname{arcsinh} z)} \right]^{1/2}, \tag{56}$$

$$Z_{2M-1,2N}^{Mob}(1/z) = z_v^{N(2M-1)} \sqrt{\frac{Z_{0,1/2}(z,2N,2M)}{\cosh(2N \operatorname{arcsinh} z)}}, \tag{57}$$

$$Z_{4M,2N-1}^{Mob}(1/z) = z_v^{2M(2N-1)} \left[\frac{Z_{1/2,1/2}(z,2N-1,4M+1)}{2\cosh[(2N-1)\operatorname{arcsinh}\ z]} \right]^{1/2}, \tag{58}$$

$$Z_{2M,2N}^{cyl}(1/z) = z_v^{2MN} \frac{Z_{1/2,1/2}(z,N,2M+1)}{2\cosh(N \operatorname{arcsinh} z)}, \tag{59}$$

$$Z_{2M-1,2N}^{cyl}(1/z) = \frac{1}{2} z_v^{N(2M-1)} \frac{Z_{0,1/2}(z,N,2M)}{\cosh(N \operatorname{arcsinh} z)}, \tag{60}$$

$$Z_{2M,2N-1}^{cyl}(1/z) = z_v^{M(2N-1)} \left[\frac{Z_{1/2,0}(z,2N-1,2M+1)}{2\sinh[(2N-1)\operatorname{arcsinh}\ z]} \right]^{1/2}. \tag{61}$$

Equation (53) implies that the partition function of the dimer model on the $4M \times 2N$ lattice with the Klein bottle boundary conditions obeys the unexpected symmetry under the interchanges $z_h \leftrightarrow z_v$ and $M \leftrightarrow N$, namely,

$$Z_{4M,2N}^{Klein}(1/z) = Z_{4N,2M}^{Klein}(z). \tag{62}$$

It is easy to see from Eqs. (14)–(17) that the partition functions of the dimer model on $2M \times 2N$ lattice with different boundary conditions obey the following identities:

$$Z_{2M,2N}^{free}(z) = \frac{(1+z^2)^{1/4}}{z_v^M \sqrt{\cosh[(2M+1)\operatorname{arcsinh} z]}} \sqrt{Z_{2M,4N+2}^{cyl}(z)}, \tag{63}$$

$$Z_{4M+2,2N}^{Klein}(z) = 2 z_h^{2N} \cosh\left(2N \operatorname{arcsinh} \frac{1}{z} \right) Z_{2M,4N}^{cyl}(z), \tag{64}$$

$$Z_{2M,2N}^{Mob}(z) = \sqrt{Z_{2M,4N}^{cyl}(z)}. \tag{65}$$

The relation of Eq. (65) was first established in the large M and N limit by Brankov and Priezzhev [8] and then was rigorously established by Lu and Wu [9].

Using Eqs. (27)–(31), one can write the following identities between partition functions of the dimer model on $(2M-1) \times 2N$ lattice with different boundary conditions:

$$Z_{2M-1,4N}^{torus}(z) = \tfrac{1}{2} [Z_{2M-1,2N}^{Klein}(z)]^2, \tag{66}$$

$$Z_{2M-1,4N}^{cyl}(z) = \tfrac{1}{2} [Z_{2M-1,2N}^{Mob}(z)]^2, \tag{67}$$

$$Z_{2M-1,4N+2}^{cyl}(z) = z_v^{2M-1} \frac{\sinh(2M \operatorname{arcsinh} z)}{(1+z^2)^{1/2}} [Z_{2M-1,2N}^{free}(z)]^2. \tag{68}$$

The relation given by Eq. (67) was first established by Lu and Wu [9].

Using Eqs. (39) and (41), one can write the following identity between partition functions of the dimer model on $2M \times (2N-1)$ lattice with the toroidal and cylindrical boundary conditions:

$$Z_{4M+2,2N-1}^{torus}(z) = 2 z_h^{2N-1} \sinh\left((2N-1)\operatorname{arcsinh} \frac{1}{z} \right) \times [Z_{2M,2N-1}^{cyl}(z)]^2. \tag{69}$$

And finally, using identity given by Eq. (13) and the expressions for the partition functions $Z_{2M,2N}^{Klein}(z)$, $Z_{2M-1,2N}^{cyl}(z)$, and $Z_{2M-1,2N}^{Mob}(z)$ [see Eqs. (16), (29), and (31)], we can obtain the following identities:

$$Z_{4M-1,2N}^{cyl}(z) = Z_{2M-1,2N}^{cyl}(z) Z_{4M,2N}^{Klein}(z), \tag{70}$$

$$Z_{4M-1,2N}^{Mob}(z) = Z_{2M-1,2N}^{Mob}(z) \sqrt{Z_{4M,4N}^{Klein}(z)}. \tag{71}$$

IV. ASYMPTOTIC EXPANSION OF THE FREE ENERGY

In Sec. II, we have shown that the partition functions of the dimer model with various boundary conditions can be expressed in terms of the partition function with twisted boundary conditions $Z_{1/2,0}(z,K,L)$, $Z_{0,1/2}(z,K,L)$, and $Z_{1/2,1/2}(z,K,L)$ [see Eqs. (3), (14)–(17), (27)–(31), (39)–(43)]. Based on such results, one can easily write down all the terms of the exact asymptotic expansion of the logarithm of the partition functions for the dimer model using Eq. (A1). We have found that the exact asymptotic expansion of $\ln Z_{\mathcal{M},\mathcal{N}}(z)$ can be written as Eq. (2).

The bulk free energy f_{bulk} is the same for all boundary conditions and given by

$$f_{bulk}=\frac{1}{2}\ln z_v+\frac{1}{2\pi}\int_0^{\pi}\omega_z(x)dx$$
$$=\frac{1}{2}\ln z_v+\frac{1}{2\pi}\int_0^{\pi}\text{arcsinh}(z\sin x)dx$$
$$=\frac{1}{2}\ln z_v+\frac{\Phi\left(-z^2,2,\frac{1}{2}\right)}{4\pi}, \tag{72}$$

where $\Phi(z,s,\alpha)$ is Lerch's transcendent function defined as

$$\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}(\alpha+n)^{-s}z^n. \tag{73}$$

In particular, one has $\Phi(-1,2,1/2)=4G$, where G is the Catalan constant given by $G=\Sigma_{n=0}^{\infty}(-1)^n/(2n+1)^2 =0.915\,965\,594\ldots$. The surface free energies $f_{1s}(z_h,z_v)$ and $f_{2s}(z_h,z_v)$ defined by Eq. (2) are

$$f_{1s}^{torus}(z_h,z_v)=f_{1s}^{Klein}(z_h,z_v)=0,$$

$$f_{1s}^{cyl}(z_h,z_v)=f_{1s}^{free}(z_h,z_v)=f_{1s}^{Mob}(z_h,z_v)$$
$$=-\tfrac{1}{2}\ln\left(z_v+\sqrt{z_h^2+z_v^2}\right),$$

$$f_{2s}^{torus}(z_h,z_v)=f_{2s}^{Klein}(z_h,z_v)=f_{2s}^{cyl}(z_h,z_v)=f_{2s}^{Mob}(z_h,z_v)=0,$$

$$f_{2s}^{free}(z_h,z_v)=f_{1s}^{free}(z_v,z_h)=-\tfrac{1}{2}\ln\left(z_h+\sqrt{z_h^2+z_v^2}\right). \tag{74}$$

Note that $f_{1s}(z_h,z_v)$ and $f_{2s}(z_h,z_v)$ depend on the type of boundary conditions but independent on the parities (even or odd) of M_s and N_s. This is not the case for the other coefficients $f_p(z\rho)$ $(p=0,1,2,\ldots)$ in the expansion of Eq. (2). In what follows, we will list expansion coefficients $f_p(z\rho)$ for $p=0$, 1, and 2 and show that they depend crucially on whether M_s and N_s are even or odd.

Dimers on 2M×2N lattices. For the periodic boundary conditions (torus), the coefficients in the expansion coefficients are

$$f_0^{torus}(z\rho)=\ln\frac{\theta_2^2+\theta_3^2+\theta_4^2}{2\eta^2},$$

$$f_1^{torus}(z\rho)=-\frac{\pi^3\rho^2 z_2}{15}\,\frac{\frac{7}{8}(\theta_2^{10}+\theta_3^{10}+\theta_4^{10})+\theta_2^2\theta_3^2\theta_4^2(\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2)}{\theta_2^2+\theta_3^2+\theta_4^2},$$

$$\begin{aligned}
f_2^{torus}(z\rho)=&-\frac{\pi^6\rho^4 z_2^2}{450}\left(\frac{\theta_2^2\theta_3^2\theta_4^2(\theta_2^2\theta_4^2-\theta_2^2\theta_3^2-\theta_3^2\theta_4^2)+\frac{7}{8}(\theta_2^{10}+\theta_3^{10}+\theta_4^{10})}{\theta_2^2+\theta_3^2+\theta_4^2}\right)^2\\
&+\frac{\pi^6\rho^4 z_2^2}{450}\,\frac{\theta_3^2\left(\frac{7}{8}\theta_3^8+\theta_2^4\theta_4^4\right)^2+\theta_2^2\left(\frac{7}{8}\theta_2^8-\theta_3^4\theta_4^4\right)^2+\theta_4^2\left(\frac{7}{8}\theta_4^8-\theta_2^4\theta_3^4\right)^2}{\theta_2^2+\theta_3^2+\theta_4^2}-\frac{\pi^6\rho^4 z_2^2}{21}\theta_3^4\theta_4^4\\
&\times\frac{\theta_3^8(\theta_2^2-\theta_4^2)+\theta_4^8(\theta_2^2-\theta_3^2)+\frac{5}{8}[\theta_3^2(\theta_2^8-\theta_4^8)+\theta_4^2(\theta_2^8-\theta_3^8)]+\frac{5}{16}(2\theta_2^{10}-\theta_3^{10}-\theta_4^{10})}{\theta_2^2+\theta_3^2+\theta_4^2}\\
&-\frac{31\pi^5\rho^3}{3024}\,\frac{\theta_3^{10}(\theta_2^4-\theta_4^4)+\theta_2^{10}(\theta_3^4+\theta_4^4)-\theta_4^{10}(\theta_3^4+\theta_2^4)}{\theta_2^2+\theta_3^2+\theta_4^2}\left(z_4+36\rho z_2^2\frac{\partial}{\partial(z\rho)}\ln\theta_2\right)\\
&+\frac{\pi^5\rho^3}{189}\,\frac{\theta_2^2\theta_3^2\theta_4^2[\theta_3^6(\theta_2^2-\theta_4^2)+\theta_2^6(\theta_3^2+\theta_4^2)-\theta_4^6(\theta_3^2+\theta_2^2)]}{\theta_2^2+\theta_3^2+\theta_4^2}\left(z_4+36\rho z_2^2\frac{\partial}{\partial(z\rho)}\ln\theta_2\right)\\
&\vdots
\end{aligned} \tag{75}$$

where $z_2 = -z(1+z^2)/3$, $z_4 = z(1+z^2)(1+9z^2)/5$, and $\theta_i = \theta_i(z\rho)$ with $i=2,3,4$.

For the Möbius strip boundary condition, the expansion coefficients are

$$f_0^{Mob}(z\rho_1) = \frac{1}{6}\ln\frac{2\,\theta_3^2}{\theta_2\theta_4},$$

$$f_1^{Mob}(z\rho_1) = -\frac{\pi^3\rho_1^2 z_2}{240}\left(\frac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right),$$

$$f_2^{Mob}(z\rho_1) = \frac{5\pi^6\rho_1^4 z_2^2}{21504}\theta_3^4\theta_4^4\left(\theta_3^8 - 2\,\theta_2^8 + \frac{26}{5}\theta_4^8\right) - \frac{\pi^5\rho_1^3}{12096}(\theta_2^4 - \theta_4^4)\left(\frac{31}{16}\theta_3^8 - \theta_2^4\theta_4^4\right) \times \left(z_4 + 36 z_2\rho_1\frac{\partial}{\partial(z\rho_1)}\ln\theta_2\right)$$

$$\vdots \tag{76}$$

where $\theta_i = \theta_i(z\rho_1)$ with $i=2,3,4$.

For the free, cylindrical, and Klein bottle boundary conditions, the expansion coefficients can be obtained from the following functional relations:

$$f_p^{free}(z\rho_2) = f_p^{Mob}(z\rho_2) + \frac{\delta_{p,0}}{4}\ln 4(z_v^2 + z_h^2),$$

$$f_p^{cyl}(z\rho_1) = 2^{p+1} f_p^{Mob}(2z\rho_1),$$

$$f_p^{Klein}(z\rho) = 2^{p+1} f_p^{Mob}(z\rho/2). \tag{77}$$

It is of interest to compare these findings with other results. In the case of the periodic boundary conditions, our results for $f_{bulk}(z\rho)$ and $f_0^{torus}(z\rho)$ reproduced those obtained by Fisher [3] and Ferdinand [5]. In the case $z_h = z_v = 1$ $(z=1)$, the expressions for $f_1^{torus}(z\rho)$ and $f_2^{torus}(z\rho)$ at $z=1$ reproduced results in Ref. [23]. Our results for $f_1^{torus}(z\rho)$ and $f_2^{torus}(z\rho)$ for arbitrary z have not been reported in papers cited in this paper.

For the case of the free, cylindrical, Möbius strip, and Klein bottle boundary conditions, the asymptotic expansion of the logarithm of the partition function $\ln Z_{(2M,2N)}(z)$ in the large M and N limit has the following form [3,9,5]:

$$\ln Z_{(2M,2N)} = 4MNf_{bulk} + 2Nc_1 + 2Mc_2 + c_3 + O(1/N). \tag{78}$$

There are following relations between coefficients in Eqs. (2) and (78):

$$c_1^{Klein} = f_{1s}^{Klein}(z_h, z_v), \quad c_2^{Klein} = f_{2s}^{Klein}(z_h, z_v),$$

$$c_3^{Klein} = f_0^{Klein}(z\rho),$$

$$c_1^{Mob} = f_{1s}^{Mob}(z_h, z_v) + f_{bulk}, \quad c_2^{Mob} = f_{2s}^{Mob}(z_h, z_v),$$

$$c_3^{Mob} = f_0^{Mob}(z\rho) + f_{2s}^{Mob}(z_h, z_v),$$

$$c_1^{cyl} = f_{1s}^{cyl}(z_h, z_v) + f_{bulk}, \quad c_2^{cyl} = f_{2s}^{cyl}(z_h, z_v),$$

$$c_3^{cyl} = f_0^{cyl}(z\rho) + f_{2s}^{cyl}(z_h, z_v),$$

$$c_1^{free} = f_{1s}^{free}(z_h, z_v) + f_{bulk}, \quad c_2^{free} = f_{2s}^{free}(z_h, z_v) + f_{bulk},$$

$$c_3^{free} = f_0^{free}(z\rho) + f_{1s}^{free}(z_h, z_v) + f_{2s}^{cyl}(z_h, z_v) + f_{bulk}, \tag{79}$$

which imply that our results for f_{1s}, f_{2s}, f_0 for the free, cylindrical, and Möbius strip boundary conditions are consistent with those obtained by Fisher [3], Ferdinand [5], and Lu and Wu [9]. For the Klein bottle, we have obtained different result for f_0 compared with Ref. [9]. Our results for f_1^{Klein}, f_1^{Mob}, f_1^{cyl}, f_1^{free}, f_2^{Klein}, f_2^{Mob}, f_2^{cyl}, and f_2^{free} have not been reported in papers cited in this paper.

Dimers on (2M−1)×2N lattices. For the Möbius strip boundary conditions, the expansion coefficients are

$$f_0^{Mob}(z\rho_1) = \frac{1}{2}\ln 2 + \frac{1}{6}\ln\frac{2\,\theta_4^2}{\theta_2\theta_3},$$

$$f_1^{Mob}(z\rho_1) = -\frac{\pi^3\rho_1^2 z_2}{240}\left(\frac{7}{8}\theta_4^8 - \theta_2^4\theta_3^4\right),$$

$$f_2^{Mob}(z\rho_1) = \frac{5\pi^6\rho_1^4 z_2^2}{21504}\theta_3^4\theta_4^4\left(\theta_4^8 - 2\,\theta_2^8 + \frac{26}{5}\theta_3^8\right) + \frac{\pi^5\rho_1^3}{12096}(\theta_2^4 + \theta_3^4)\left(\frac{31}{16}\theta_4^8 + \theta_2^4\theta_3^4\right) \times \left(z_4 + 36 z_2\rho_1\frac{\partial}{\partial(z\rho_1)}\ln\theta_2\right)$$

$$\vdots \tag{80}$$

where $\theta_i = \theta_i(z\rho_1)$ with $i=2,3,4$.

For the toroidal, free, cylindrical and Klein bottle boundary conditions, the expansion coefficients can be obtained from the following functional relations:

$$f_p^{torus}(z\rho) = 2^{p+1} f_p^{Mob}(2z\rho) - \frac{\delta_{p,0}}{2}\ln 2,$$

$$f_p^{free}(z\rho_2) = f_p^{Mob}(z\rho_2) + \frac{\delta_{p,0}}{4}\ln(z_v^2 + z_h^2),$$

$$f_p^{cyl}(z\rho_1) = 2^{p+1} f_p^{Mob}(2z\rho_1) - \frac{\delta_{p,0}}{2}\ln 2,$$

$$f_p^{Klein}(z\rho) = f_p^{Mob}(z\rho). \tag{81}$$

In the case of the periodic and free boundary conditions, our results for f_0^{torus} and f_0^{free} reproduced those obtained by Ferdinand [5]. Our results for f_0^{Klein}, f_0^{Mob}, f_0^{cyl}, f_1^{torus}, f_1^{Klein}, f_1^{Mob}, f_1^{cyl}, f_1^{free}, f_2^{torus}, f_2^{Klein}, f_2^{Mob}, f_2^{cyl}, and f_2^{free} have not been reported in papers cited in this paper.

Dimers on 2M×(2N−1) lattices. For the cylindrical and Möbius strip boundary conditions, the expansion coefficients are

$$f_0^{cyl}(z\rho_1)=\frac{1}{6}\ln\frac{2\,\theta_2^2}{\theta_4\theta_3},$$

$$f_0^{Mob}(z\rho_1)=\frac{1}{6}\ln\frac{2\,\theta_3^2}{\theta_2\theta_4},$$

$$f_1^{cyl}(z\rho_1)=-\frac{\pi^3\rho_1^2 z_2}{240}\left(\frac{7}{8}\theta_2^8-\theta_3^4\theta_4^4\right),$$

$$f_1^{Mob}(z\rho_1)=-\frac{\pi^3\rho_1^2 z_2}{240}\left(\frac{7}{8}\theta_3^8+\theta_2^4\theta_4^4\right),$$

$$f_2^{cyl}(z\rho_1)=-\frac{\pi^6\rho_1^4 z_2^2}{1344}\theta_3^4\theta_4^4\left(\frac{5}{8}\theta_2^8+\theta_3^8+\theta_4^8\right)-\frac{\pi^5\rho_1^3}{12096}(\theta_3^4+\theta_4^4)\left(\frac{31}{16}\theta_2^8+\theta_3^4\theta_4^4\right)\times\left(z_4+36z_2\rho_1\frac{\partial}{\partial(z\rho_1)}\ln\theta_2\right),$$

$$f_2^{Mob}(z\rho_1)=\frac{5\pi^6\rho_1^4 z_2^2}{21504}\theta_3^4\theta_4^4\left(\theta_3^8-2\,\theta_2^8+\frac{26}{5}\theta_4^8\right)-\frac{\pi^5\rho_1^3}{12\,096}(\theta_2^4-\theta_4^4)\left(\frac{31}{16}\theta_3^8-\theta_2^4\theta_4^4\right)\times\left(z_4+36z_2\rho_1\frac{\partial}{\partial(z\rho_1)}\ln\theta_2\right)$$

$$\vdots \qquad (82)$$

where $\theta_i=\theta_i(z\rho_1)$ with $i=2,3,4$.

Using the following functional relations:

$$f_p^{torus}(z\rho)=2^{p+1}f_p^{cyl}(z\rho/2),$$

$$f_p^{free}(z\rho_2)=f_p^{cyl}(z\rho_2)+\frac{\delta_{p,0}}{4}\ln 4(z_v^2+z_h^2),$$

$$f_p^{Klein}(z\rho)=2^{p+1}f_p^{Mob}(z\rho/2), \qquad (83)$$

one can obtain the expansion coefficients $f_p(z\rho)$ ($p=0,1,2,\dots$) for other boundary conditions: toroidal, free, and Klein bottle.

In the case of the periodic and free boundary conditions, our results for f_0^{torus} and f_0^{free} reproduced those obtained by Ferdinand [5]. Our results for f_0^{Klein}, f_0^{Mob}, f_0^{cyl}, f_1^{torus}, f_1^{Klein}, f_1^{Mob}, f_1^{cyl}, f_1^{free}, f_2^{torus}, f_2^{Klein}, f_2^{Mob}, f_2^{cyl}, and f_2^{free} have not been reported by papers cited in this paper.

Figures of expansion coefficients. We plot the aspect-ratio (ρ) dependence of f_0 and f_1 at $z=1$ for $2M\times 2N$, $(2M-1)\times 2N$, and $2M\times(2N-1)$ lattices under various boundary conditions in Fig. 3. We use the logarithmic scales for the horizontal axis. We have several interesting observations from Fig. 3.

For $2M\times 2N$ lattices [Figs. 3(a) and 3(b)], the plot of f_0 and f_1 as a function of ρ in logarithmic scale is symmetric for the torus, plane, and Möbius strip because of the symmetric property under the transformation under $z\leftrightarrow 1/z$ and $\rho\leftrightarrow 1/\rho$. f_0 and f_1 take the minimum at $\rho=1$. This is not the case for other geometries. For large enough ρ ($\geqslant 1$), the finite-size scaling (FSS) properties of the Klein bottle and those of the Möbius strip become the same because the boundaries along the shorter direction determine the FSS properties of the system; for both the Möbius strip and the Klein bottle, the boundary conditions along the horizontal direction are the twisted one. The FSS properties of the torus and the cylinder are the same for large enough ρ. In contrast, the systems with $\rho\ll 1$, the Klein bottle, and the torus show similar FSS behavior. For small enough ρ ($\ll 1$), the Möbius strip, the cylinder, and the plane show the same FSS properties because the boundaries along the shorter directions for these three are the same, that is, the free boundary condition. For large enough ρ ($\gg 1$), the FSS properties of the plane and those of the Klein bottle and the Möbius strip become the same. To summarize, we have found that in the limit of large enough ρ ($\gg 1$) the finite-size correction coefficients (f_0,f_1) can be classified into two groups: one group is torus and cylinder, and the other is Möbius strip, Klein bottle, and plane. For small enough ρ ($\ll 1$), the finite-size correction coefficients (f_0,f_1) are classified into another two groups; one group is torus and Klein bottle, and the other is Möbius strip, cylinder, and plane. Note that for $0\leqslant\rho\leqslant\infty$, the coefficients f_1 for free the and Möbius strip boundary conditions show similar behavior. Our results for the FSS behavior of the dimer model on $2M\times 2N$ lattice are consistent with the results obtained by Kaneda and Okabe [18] for the FSS behavior of the Binder parameter for square lattice Ising model.

For $(2M-1)\times 2N$ lattices [Figs. 3(c) and 3(d)] and $2M\times(2N-1)$ lattice [Figs. 3(e) and 3(f)], the FSS properties of the system are totally different. For $(2M-1)\times 2N$ lattice, the finite-size correction coefficients (f_0,f_1) are classified into two groups for large enough ρ ($\rho\gg 1$): one group is torus and cylinder, and the other is Möbius strip, Klein bottle, and plane. In contrast, for small enough ρ ($\rho\ll 1$), the finite-size correction coefficients (f_0,f_1) show similar behavior for all five boundary conditions.

For $2M\times(2N-1)$ lattices [Figs. 3(e) and 3(f)], the finite-size correction coefficients (f_0,f_1) for large enough ρ ($\rho\gg 1$) can again be classified into two groups: one group is torus, cylinder, and plane, and the other is Möbius strip and Klein bottle. For small enough ρ ($\ll 1$), the finite-size correction coefficients (f_0,f_1) are classified into another two groups; one group is torus and Klein bottle, and the other is Möbius strip, cylinder, and plane.

V. SUMMARY AND DISCUSSION

In this paper, we have used the method of Ref. [23] to derive exact finite-size corrections for the logarithm of the partition function $\ln Z_{M,N}$ of the dimer model on the $2M\times 2N$, $(2M-1)\times 2N$, and $2M\times(2N-1)$ square lattices with different boundary conditions (Fig. 1). We have found that the exact asymptotic expansion of $\ln Z_{(\mathcal{M},\mathcal{N})}(z)$ can be written in the form given by Eq. (2). We have shown that the coefficients $f_p(z\rho)$ for $p=0,1,2,\ldots$ in this expansion are sensitive to the boundary conditions and the parity of the number of lattice sites, M_s and N_s, along the axes. We have established several groups of identities relating dimer partition functions for different boundary conditions [see Eqs. (63)–(71)]. We have also established that the partition functions of the dimer model on the $4M\times 2N$ lattice with the Klein bottle boundary conditions obey the unexpected symmetry under the interchanges $z_h\leftrightarrow z_v$ and $M\leftrightarrow N$ [see Eq. (62)].

Previous studies [5,9] only obtained exact finite-size corrections up to the first order for certain boundary conditions or parities of M_s and N_s. The present paper is the one to calculate finite-size corrections up to the second order for the square lattice dimer model under five different boundary conditions shown in Fig. 1. Our results are a useful reference for following further studies on the dimer model:

(a) At present, the exact result is usually available only for closed-packed dimers on lattices, in which each lattice site is occupied by one dimer [32]. One can consider a more general dimer model consisting of mixtures of dimers and vacancies (monomer). There is no exact solution for such a general dimer model. To use numerical methods to study the general dimer model (which includes the closed-packed dimer model as a special case), one would like to know the convergent behavior of the calculated quantities as the system size increases. Our exact finite-size correction terms are useful for this purpose.

(b) Izmailian and Hu have found universal amplitude ratios for the Ising model on square (sq), plane triangular (pt), and honeycomb lattices (hc) [21]. It is of interest to extend the study of the present paper for the sq lattice dimer model to pt and hc lattices and try to find some universal amplitude ratios for such systems.

The results of this paper show that the method of Ref. [23] is quite useful for calculating exact finite-size corrections for critical systems. It is of interest to use this method to calculate higher-order terms.

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APPENDIX: ASYMPTOTIC EXPANSION OF $Z_{\alpha,\beta}(Z,M,N)$

$Z_{\alpha,\beta}(z,M,N)$ can be expanded in the similar way as in Ref. [23] and has a form

$$\ln Z_{\alpha,\beta}(z,M,N)=\frac{S}{\pi}\int_0^{\pi}\omega_z(x)dx+\ln\left|\frac{\theta_{\alpha,\beta}(iz\rho)}{\eta(iz\rho)}\right| -2\pi\rho\sum_{p=1}^{\infty}\left(\frac{\pi^2\rho}{S}\right)^p\frac{\Lambda_{2p}}{(2p)!}\frac{\mathrm{Re}K^{\alpha,\beta}_{2p+2}(iz\rho)}{2p+2}, \tag{A1}$$

where $S=MN$, $\rho=M/N$, $\eta(\tau)$ is the Dedekind–η function,

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{n=1}^{\infty}[1-e^{2\pi i\tau n}], \tag{A2}$$

$K^{\alpha,\beta}_{2p+2}(\tau)$ is Kronecker's double series [23] and functions $\theta_{\alpha,\beta}(\tau)$ are defined as

$$\begin{aligned}\theta_{\alpha,\beta}(\tau)&=\sum_{n\in Z}\exp\left[\pi i\tau\left(n+\frac{1}{2}-\alpha\right)^2+2\pi i\left(n+\frac{1}{2}-\alpha\right)\left(\frac{1}{2}-\beta\right)\right]\\&=\eta(\tau)\exp\left[\pi i\tau\left(\alpha^2-\alpha+\frac{1}{6}\right)+2\pi i\left(\frac{1}{2}-\alpha\right)\times\left(\frac{1}{2}-\beta\right)\right]\prod_{n=0}^{\infty}[1-e^{2\pi i\tau(n+\alpha)+2\pi i\beta}]\\&\quad\times[1-e^{2\pi i\tau(n+1-\alpha)-2\pi i\beta}].\end{aligned} \tag{A3}$$

Relations to standard notations are $\theta_{0,0}(i\tau)=\theta_1(\tau)$, $\theta_{0,1/2}(i\tau)=\theta_2(\tau)$, $\theta_{1/2,1/2}(i\tau)=\theta_3(\tau)$, $\theta_{1/2,0}(i\tau)=\theta_4(\tau)$, and $\eta(i\tau)=[\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)/2]^{1/3}$. The differential operators Λ_{2p} that have appeared in Eq. (A1) can be expressed via coefficients z_{2p} of Taylor expansion of the lattice dispersion relation $\omega_z(k)$,

$$\omega_z(k)=k\left(z+\sum_{p=1}^{\infty}\frac{z_{2p}}{(2p)!}k^{2p}\right) \tag{A4}$$

with $z_2=-z(1+z^2)/3$, $z_4=z(1+z^2)(1+9z^2)/5$, $z_6=-z(1+z^2)(1+90z^2+225z^4)/7$, etc.,

$$\Lambda_2=z_2,$$

$$\Lambda_4=z_4+3z_2^2\frac{\partial}{\partial z},$$

$$\Lambda_6=z_6+15z_4z_2\frac{\partial}{\partial z}+15z_2^3\frac{\partial^2}{\partial z^2},$$

$$\vdots$$

$$\Lambda_p=\sum_{r=1}^{p}\sum\left(\frac{z_{p_1}}{p_1!}\right)^{k_1}\cdots\left(\frac{z_{p_r}}{p_r!}\right)^{k_r}\frac{p!}{k_1!\cdots k_r!}\frac{\partial^k}{\partial z^k}. \tag{A5}$$

Here summation is over all positive numbers $\{k_1,\ldots,k_r\}$ and different positive numbers $\{p_1,\ldots,p_r\}$ such that $p_1k_1+\cdots+p_rk_r=p$ and $k=k_1+\cdots+k_r-1$.

We are interested in the asymptotic expansion of $Z_{\alpha,\beta}(z,M,N)$ with $\alpha=0,\frac{1}{2}$ and $\beta=0,\frac{1}{2}$. The function $K_{2p}^{\alpha,\beta}(\tau)$ can be expressed in terms of the elliptic θ functions [23], e.g.,

$$K_8^{0,0}(\tau)=-\frac{1}{30}[\theta_2^4\theta_3^4-\theta_2^4\theta_4^4+\theta_3^4\theta_4^4]^2,$$

$$K_8^{1/2,0}(\tau)=\frac{1}{30}\left[\frac{127}{128}\theta_4^{16}+\frac{7}{4}\theta_2^4\theta_4^{12}+\frac{3}{4}\theta_2^8\theta_4^8-2\theta_2^{12}\theta_4^4-\theta_2^{16}\right], \tag{A6}$$

$$K_8^{0,1/2}(\tau)=\frac{1}{30}\left[\frac{127}{128}\theta_2^{16}+\frac{7}{4}\theta_2^{12}\theta_4^4+\frac{3}{4}\theta_2^8\theta_4^8-2\theta_2^4\theta_4^{12}-\theta_4^{16}\right], \tag{A7}$$

$$K_8^{1/2,1/2}(\tau)=\frac{1}{30}\left[\frac{127}{128}\theta_3^{16}-\frac{7}{4}\theta_3^8\theta_2^4\theta_4^4-\theta_2^8\theta_4^8\right], \tag{A8}$$

$$K_{10}^{0,0}(\tau)=\frac{5}{132}[\theta_2^4+\theta_3^4][\theta_4^4+\theta_2^4][\theta_3^4+\theta_4^4][\theta_2^4\theta_3^4-\theta_2^4\theta_4^4+\theta_3^4\theta_4^4], \tag{A9}$$

$$K_{10}^{1/2,0}(\tau)=-\frac{5}{66}\left[\frac{511}{512}\theta_4^{20}+\frac{647}{256}\theta_2^4\theta_4^{16}+\frac{67}{32}\theta_2^8\theta_4^{12}+\frac{49}{16}\theta_2^{12}\theta_4^8+\theta_2^{20}\right], \tag{A10}$$

$$K_{10}^{0,1/2}(\tau)=\frac{5}{66}\left[\frac{511}{512}\theta_2^{20}+\frac{647}{256}\theta_2^{16}\theta_4^4+\frac{67}{32}\theta_2^{12}\theta_4^8+\frac{49}{16}\theta_2^8\theta_4^{12}+\theta_4^{20}\right], \tag{A11}$$

$$K_{10}^{1/2,1/2}(\tau)=\frac{5}{132}[\theta_2^4-\theta_4^4]\left[\frac{511}{256}\theta_3^{16}-\frac{17}{16}\theta_3^8\theta_2^4\theta_4^4+\theta_2^8\theta_4^8\right]. \tag{A12}$$

$$\vdots \tag{A13}$$

Equations for $K_{2p}^{\alpha,\beta}(\tau)$ with $p=2$, 3 and other useful relations for elliptic θ functions can be found in Ref. [23].

[1] R.H. Fowler and G.S. Rushbrooke, Trans. Faraday Soc. **33**, 1272 (1937).

[2] P.W. Kasteleyn, Physica (Amsterdam) **27**, 1209 (1961).

[3] M.E. Fisher, Phys. Rev. **124**, 1664 (1961).

[4] H.N.V. Temperley and M.E. Fisher, Philos. Mag. **6**, 1061 (1961).

[5] A.E. Ferdinand, J. Math. Phys. **8**, 2332 (1967).

[6] B.W. McCoy and T.T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, MA, 1973).

[7] S.M. Bhattacharjee and J.F. Nagle, Phys. Rev. A **31**, 3199 (1985).

[8] J.G. Brankov and V.B. Priezzhev, Nucl. Phys. B **400**, 633 (1993).

[9] W.T. Lu and F.Y. Wu, Phys. Lett. A **259**, 108 (1999).

[10] W.T. Lu and F.Y. Wu, Phys. Lett. A **293**, 235 (2002); **298**, 293(E) (2002); we believe that Eq. (66) reported here requires minor correction: $(-1)^{M/2+m+1}$ should be replaced by $(-1)^{[M/2]+m+1}$.

[11] F.Y. Wu (private communication).

[12] W. Jockusch, J. Propp and P. Shor, e-print math-CO/9801068; H. Cohn, N. Elkies, and J. Propp, Duke Math. J. **85**, 117 (1996); R. Kenyon, http://topo.math.u-psud.fr/∼kenyon/; V. Korepin and P. Zinn-Justin, J. Phys. A **33**, 7053 (2000); D.S. Rokhsar and S.A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988); P. Fendley, R. Moessner, and S.L. Sondhi, Phys. Rev. B **66**, 214513 (2002).

[13] M.E. Fisher, in *Critical Phenomena*, Proceedings of the 1970 International School of Physics ''Enrico Fermi,'' Course 51, edited by M.S. Green (Academic, New York, 1971); *Finite-size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990).
[14] V. Privman and M.E. Fisher, Phys. Rev. B **30**, 322 (1984).
[15] C.-K. Hu, J. Phys. A **27**, L813 (1994); C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Phys. Rev. Lett. **75**, 193 (1995); **75**, 2786(E) (1995); Physica A **221**, 80 (1995); C.-K. Hu and C.-Y. Lin, Phys. Rev. Lett. **77**, 8 (1996); C.-K. Hu and F.-G. Wang, J. Korean Phys. Soc. **31**, S271-277 (1997); C.-Y. Lin and C.-K. Hu, Phys. Rev. E **58**, 1521 (1998); H.P. Hsu, S.C. Lin, and C.-K. Hu, *ibid.* **64**, 016127 (2001); H. Watanabe *et al.*, J. Phys. Soc. Jpn. **70**, 1537 (2001).
[16] R.M. Ziff, S.R. Finch, and V.S. Adamchik, Phys. Rev. Lett. **79**, 3447 (1997); P. Kleban and R.M. Ziff, Phys. Rev. B **57**, R8075 (1998); C.D. Lorenz and R.M. Ziff, J. Phys. A **31**, 8147 (1998).
[17] K.-C. Lee, Phys. Rev. Lett. **69**, 9 (1992); Y. Okabe and M. Kikuchi, Int. J. Mod. Phys. C **7**, 287 (1996); F.-G. Wang and C.-K. Hu, Phys. Rev. E **56**, 2310 (1997); C.-K. Hu, J.-A. Chen, and C.-Y. Lin, Physica A **266**, 27 (1999); Y. Okabe *et al.*, Phys. Rev. E **59**, 1585 (1999); Y. Tomita, Y. Okabe, and C.-K. Hu, *ibid.* **60**, 2716 (1999).
[18] K. Kaneda and Y. Okabe, Phys. Rev. Lett. **86**, 2134 (2001).
[19] A.E. Ferdinand and M.E. Fisher, Phys. Rev. **185**, 832 (1969).
[20] C.-K. Hu *et al.*, Phys. Rev. E **60**, 6491 (1999); e-print cond-mat/9905203; this paper calculated finite-size corrections for the *q*-state Potts model (QPM) based on the connection between the QPM and a bond-correlated percolation model; see C.-K. Hu, Phys. Rev. B **29**, 5103 (1984); **29**, 5109 (1984).
[21] N.Sh. Izmailian and C.-K. Hu, Phys. Rev. Lett. **86**, 5160 (2001).
[22] N.Sh. Izmailian and C.-K. Hu, Phys. Rev. E **65**, 036103 (2002).
[23] E.V. Ivashkevich, N.Sh. Izmailian, and C.K. Hu, e-print cond-mat/0102470; J. Phys. A **35**, 5543 (2002).
[24] J. Salas, J. Phys. A **34**, 1311 (2001); **35**, 1833 (2002).
[25] W.T. Lu and F.Y. Wu, Phys. Rev. E **63**, 026107 (2001).
[26] W. Janke and R. Kenna, Phys. Rev. B **65**, 064110 (2002).
[27] N.Sh. Izmailian, K.B Oganesyan, and C.-K. Hu, Phys. Rev. E **65**, 056132 (2002).
[28] M.C. Wu, C.-K. Hu, and N.Sh. Izmailian, Phys. Rev. E **67**, 065103 (2003).
[29] *Monte Carlo Methods in Statistical Physics*, 2nd ed., edited by K. Binder, Topics in Current Physics Vol. 7 (Springer-Verlag, Heidelberg, 1986); C.-K. Hu, Phys. Rev. B **46**, 6592 (1992).
[30] M.C. Wu and C.-K. Hu, J. Phys. A **35**, 5189 (2002).
[31] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, New York, 1965).
[32] One exception is the dimer model on a simple-quartic net with a vacancy, see W.-J. Tzeng and F.Y. Wu, J. Stat. Phys. **110**, 671 (2003).