PHYSICAL REVIEW E **67**, 065601(R) (2003)

Nonlinearity of a metamaterial arising from diode insertions into resonant conductive elements

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(Received 23 October 2002; published 13 June 2003)

We consider a possibility to create a metamaterial with nonlinear magnetic response in the microwave frequency range. Such a metamaterial is a regular periodic three-dimensional-array of resonant conductive elements with diodes as nonlinear insertions. We calculate the arising quadratic nonlinear susceptibility and show how it is controlled by the properties and arrangement of the structure elements as well as by the type and characteristics of the diode. We discuss the requirements for the diode necessary to optimize the nonlinear response of the proposed metamaterial.

DOI: 10.1103/PhysRevE.67.065601 PACS number(s): 41.20.Jb, 42.65.An, 78.20.Bh

Increasing attention paid to metamaterials during the last several years is to a great extent motivated by growing interest to the microwave range in electromagnetics. Advances in the microwave technology imply the development of the media suitable for manipulations with microwaves. Metamaterials, being artificial structures, arranged as regular arrays of identical conductive elements or sets of elements, are likely to be particularly useful in this area.

One of the possible applications of metamaterials is concerned with the realization of media with negative refraction index, the idea of which first appeared in the early paper by Veselago [1]. Whereas such media are not available in nature, metamaterials were appropriate for the demonstration of the negative refraction phenomena [2,3]. In practice, metamaterials with negative refraction index could provide numerous unusual applications, such as high precision focusing [4,5], mirrorless phase compensation, cavity resonators with arbitrary size [6]. Apart from negative refraction, metamaterials were suggested for magnetoinductive waves guidance [7] or microwave phase conjugation [8].

In most cases the size of the structural element is of the order of few millimeters, much smaller than the wavelength of microwaves (several centimeters and more). Then, the response of metamaterials to microwaves can be described by macroscopic permittivity and permeability. We have demonstrated this with a linear theory of the magnetic response, for an example, of a metastructure assembled as a regular lattice of circular resonant conductive elements [9]. The developed theory has much in common with the standard methods of analysis of condensed matter optical properties [10–12]. The drastically distinct scale, however, shifts the operating range for metamaterials to microwaves. An important advantage is that the properties of the whole metastructure can be directly controlled by adjusting the characteristics of individual elements—an option, not available in optical crystals. We have shown that surroundings influence leads to a significant shift of the resonance frequency of the linear permeability from the resonance of a single element.

In this paper, we would like to generalize the approach and to consider a metamaterial organized as a three-dimensional (3D) array of arbitrary flat resonant conductive elements (RCE), which can be represented by linear contours. This representation is appropriate not only for the simple resonant elements we discussed [7,9], but also, e.g., for prominent [13–15] split ring resonators, as these can be represented by an effective contour, provided that the width of the stripes is small enough compared to the element size [16,17]. If the dimensions of the contour are smaller than the wavelength, then the linear current is the same along the contour line, and the element can be characterized with effective resistance R, inductance L, and capacitance C. These parameters can be either estimated theoretically like in Ref. [9] or evaluated experimentally [14]. Then, the linear response of the element to the external periodic electromotive force (emf) at frequency ω is described by the self-impedance $Z(\omega)=R-i\omega L+i(\omega C)^{-1}$.

Following the analogy between metamaterials and crystals, we expect that numerous nonlinear phenomena known in optics can be observed in metamaterials if the response of the structure elements is nonlinear. In Ref. [8], it was suggested to construct an artificial nonlinear medium from dipoles with nonlinear inclusions. We propose to create a magnetic nonlinear metamaterial by inserting a diode into each RCE. We show how the nonlinear voltage-current characteristic of the diode provides nonlinearity to the response of the metamaterial. The arising multiwave interactions allow to affect the wave propagation directly in a convenient "all-optical" manner, i.e., without conversion into electronic signals. We find that the resulting macroscopic susceptibility depends not only on the single element's susceptibility but also contains renormalization factors determined by the linear response of the material at the frequencies of the interacting waves.

At the first step, we analyze an RCE with a diode. The dc current-voltage characteristic of a diode in the case of weak nonlinearity is often approximated by

$$I=\frac{1}{R_D}(U+\gamma U^2), \tag{1}$$

where I is the current through the diode to which the voltage U is applied, R_D is the diode's ohmic resistance, and γ is a

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parameter standing for the nonlinearity. Here, the voltage should be much smaller than $\tilde{U}\sim 1/\gamma$.

In the ac case, it is necessary to account for frequency dispersion, which is easier to do if one deals with Fourier components. Accordingly, we substitute all the time dependencies by sums of monochromatic components, so that, e.g., the time dependence of the current is $I(t)=\Sigma_\nu I(\omega_\nu)e^{-i\omega_\nu t}$, $\nu=\pm 1,\pm 2,\ldots,$ where we use the notation with $\omega_{-\nu}\equiv -\omega_\nu$, $I(\omega_{-\nu})\equiv I^*(\omega_\nu)$, which includes automatically the complex conjugates.

Due to the nonlinear term in Eq. (1), Fourier components at combinational frequencies will be coupled. Keeping up to bilinear in I terms, we can represent the voltage-current characteristic as the following:

$$U(\omega_\nu)=\mathcal{Z}(\omega_\nu)I(\omega_\nu)+\frac{1}{2}\sum_\eta \gamma(\omega_\nu;\omega_\eta,\omega_\nu-\omega_\eta)\times\mathcal{Z}(\omega_\nu-\omega_\eta)\mathcal{Z}(\omega_\eta)I(\omega_\nu-\omega_\eta)I(\omega_\eta), \tag{2}$$

where $\gamma(\omega_\nu;\omega_\eta,\omega_\nu-\omega_\eta)$ is generally complex and $\mathcal{Z}(\omega)$ is the linear impedance of the diode, i.e., $\mathcal{Z}\rightarrow R_D$ at $\omega\rightarrow 0$.

When the diode is inserted into the RCE, the response of the whole element depends on the particular position and connections. However, for the reasonable arrangement, the element with the diode can be still described by linear contour. If the nonlinearity is weak, the current in the RCE with diode under the action of external emf is determined by

$$Z(\omega)I(\omega)+U^{(2)}(\omega)=\mathcal{E}(\omega), \tag{3}$$

$$U^{(2)}(\omega)=\frac{1}{2}\sum_\eta \zeta(\omega_\nu;\omega_\eta,\omega_\nu-\omega_\eta)I(\omega_\nu-\omega_\eta)I(\omega_\eta), \tag{4}$$

where $U^{(2)}$ is the nonlinear part of the response. Here, for the ease of notation, we include the linear characteristics of the diode (resistance and capacitance) into the linear impedance of the RCE $Z(\omega)$, whereas nonlinear properties are described by the parameter ζ, which generally depends on the diode characteristics (γ and $\mathcal{Z}$) as well as on the way of insertion. For example, if the diode is inserted in series into a split conductive ring,

$$\zeta(\omega_\nu;\omega_\eta,\omega_v)=\gamma(\omega_\nu;\omega_\eta,\omega_v)\mathcal{Z}(\omega_v)\mathcal{Z}(\omega_\eta). \tag{5}$$

Now we are ready with the single RCE's response.

Considering the electromagnetic wave interaction with the metastructure, we assume that the response is formed at a distances much smaller than the wavelength. The corresponding quasistatic limit allows us to separate magnetic and electric effects, so that only the magnetic field affects the magnetization of the metamaterial, defining the permeability. At this scale, we can also neglect the inhomogeneity of magnetization and averaged fields, and the problem is reduced to the behavior of a RCE array in an external homogeneous oscillating magnetic field [9].

We suppose that all the RCE's lie in parallel planes normal to z axis, forming an infinite regular spatial lattice such that each RCE has the same surroundings. For this arrangement, only z components of magnetic fields and zz and zzz components of the magnetic susceptibility tensors are important. Further, we omit all appearing z, zz, and zzz indices.

We assume that the mutual interaction of the structure elements k and k' (the running indices of the RCE's) is described by mutual impedances $Z_{kk'}(\omega)$. Then, with the help of the multi-impedance matrix and using Eq. (3), one can write the following system of equations:

$$Z(\omega)I_k(\omega)+\sum_{k'\neq k} Z_{kk'}(\omega)I_{k'}(\omega)+U_k^{(2)}(\omega)=\mathcal{E}_k(\omega), \tag{6}$$

where $\mathcal{E}_k$ is the external emf induced in the kth element.

Obviously, in the homogeneous infinite medium under the action of the homogeneous varying field H_0, all the RCEs are in the same situation. Therefore, the same emf

$$\mathcal{E}(\omega)=i\mu_0 S\omega H_0(\omega) \tag{7}$$

(where S is the area surrounded by the flat contour equivalent to the RCE) is induced and all the currents are equal, $I_k(\omega)=I(\omega)$. Then, the system (6) is reduced to the equation

$$\left[Z(\omega)+\sum_{k'\neq k} Z_{kk'}(\omega)\right]I(\omega)+U^{(2)}(\omega)=\mathcal{E}(\omega). \tag{8}$$

The mutual impedance of the RCE's is represented by the mutual inductance. In the limit of an infinitely thin wire, it is given by the double integral along the contours:

$$Z_{kk'}(\omega)=-i\omega\frac{\mu_0}{4\pi}\int\int\frac{(d\mathbf{l}_k\cdot d\mathbf{l}_{k'})}{|\mathbf{s}_k-\mathbf{s}_{k'}|}, \tag{9}$$

where $\mathbf{s}$ is the radius vector of that point of the contour where $d\mathbf{l}$ is taken. From the form of Eq. (9), it is clear that the sum of the matrix elements in Eq. (8) can be represented as $\Sigma_{k'\neq k}Z_{kk'}(\omega)=-i\omega\mu_0 r\Sigma$, where r is some characteristic dimension of the RCE and Σ is the dimensionless parameter which depends only on the geometry of the RCE arrangement in the metastructure and can be calculated numerically. For example, for simple split rings, r is the radius of a ring and the value of Σ is of the order of 1–10 for typical lattices [9]. For the rings, we have shown that although the infinite summation is implied in Σ, in fact, the summation over a finite and relatively small number of rings is sufficient, as further increase in the number of rings taken into account does not influence the result. This fact is in accordance with the local response principle, and we can safely assume that the response is formed within the distance of several lattice constants for more complicated RCE's as well.

Now we can denote all the impedances involved as $Z_\Sigma(\omega)=Z(\omega)+\Sigma_{k'\neq k}Z_{kk'}(\omega)$. Note that it can be further written as $Z_\Sigma(\omega)=R-i\omega L_\Sigma+i(\omega C)^{-1}$, if we combine the mutual and self-inductances into $L_\Sigma=L+\mu_0 r\Sigma$. Writing now Eq. (8) for multiple frequencies, with the help of Eq. (3), we obtain

$$Z_\Sigma(\omega_\nu)I(\omega_\nu)=\mathcal{E}(\omega_\nu)+\frac{1}{2}\sum_\eta \zeta(\omega_\nu;\omega_\eta,\omega_\nu-\omega_\eta)\times I(\omega_\nu-\omega_\eta)I(\omega_\eta). \tag{10}$$

The system (10) shows that the current component at each frequency is determined not only by the emf at that frequency but also by the current components induced at two other frequencies, so that a three-wave interaction occurs. Accordingly, we consider below ω_1, ω_2, and ω_3, such that $\omega_1+\omega_2=\omega_3$. Then, from Eq. (10), we get for the current induced at ω_3

$$Z_\Sigma(\omega_3)I(\omega_3)=\mathcal{E}(\omega_3)+\zeta(\omega_3;\omega_1,\omega_2)I(\omega_1)I(\omega_2). \tag{11}$$

For the linear in ζ approximation, we can substitute $I(\omega_2)$ and $I(\omega_1)$ in the right-hand side of Eq. (11) for the expressions, obtained from Eq. (10) written for ω_1, ω_2, neglecting the terms with ζ. Then, we can express $I(\omega_3)$ via $\mathcal{E}(\omega_1)$, $\mathcal{E}(\omega_2)$, and $\mathcal{E}(\omega_3)$. This allows us to calculate the magnetization $M(\omega_3)$ of the metamaterial as the magnetic moment density $M(\omega)=nSI(\omega)$, where n is the volume density of the RCE's. Taking Eq. (7) into account, then we obtain

$$M(\omega_3)=\frac{in\mu_0S^2\omega_3}{Z_\Sigma(\omega_3)}H_0(\omega_3)-\frac{\zeta(\omega_3;\omega_1,\omega_2)n\mu_0^2S^3\omega_1\omega_2}{Z_\Sigma(\omega_3)Z_\Sigma(\omega_2)Z_\Sigma(\omega_1)}H_0(\omega_1)H_0(\omega_2). \tag{12}$$

The macroscopic induction is proportional to the spatially averaged microscopic magnetic field $B=\mu_0\langle H_{\rm mic}\rangle$. Corresponding volume integration gives generally $B=\mu_0(H_0+\frac{2}{3}M)$, notwithstanding the structure element's peculiarities [9]. With the general definition $B=\mu_0(H+M)$, we can express $H=H_0-\frac{1}{3}M$. Using this relation, we can solve Eq. (12) for $M(\omega_3)$. In the linear approximation, we obtain $M(\omega_3)=\chi_{\rm M}(\omega_3)H(\omega_3)$, where the factor

$$\chi_{\rm M}(\omega)=\mu_0nS^2\omega\left(\frac{1}{\omega C}-\omega L_\Sigma-\frac{1}{3}\mu_0nS^2\omega-iR\right)^{-1} \tag{13}$$

stands for the linear part of the magnetic susceptibility. This is in agreement with the results of Ref. [9], where $\chi_{\rm M}(\omega)$ was shown to have a general resonance form with the resonance frequency

$$\omega_r=\omega_0\left(\frac{L_\Sigma}{L}+\frac{1}{3}\frac{\mu_0nS^2}{L}\right)^{-1/2}, \tag{14}$$

which depends via Σ in L_Σ on the lattice parameters. Here $\omega_0=(LC)^{-1/2}$ is the resonance frequency of a single RCE.

Returning to the first order nonlinear consideration and keeping the terms linear in ζ, we express finally $M(\omega_3)$ in a form that is analogous to the polarization of a medium with quadratic dielectric nonlinearity:

$$M(\omega_3)=\chi_{\rm M}(\omega_3)H(\omega_3)+\chi_{\rm M}^{(2)}(\omega_3;\omega_1,\omega_2)H(\omega_1)H(\omega_2) \tag{15}$$

with the quadratic nonlinear susceptibility

$$\chi_{\rm M}^{(2)}(\omega_3;\omega_1,\omega_2)=\frac{\zeta(\omega_3;\omega_1,\omega_2)}{i\mu_0S^3\omega_3}n^{-2}\chi_{\rm M}(\omega_1)\chi_{\rm M}(\omega_2)\chi_{\rm M}(\omega_3). \tag{16}$$

Therefore, in spite of the completely different physical background, one can deal with the nonlinear interaction of electromagnetic waves in the proposed metamaterial using the well-developed apparatus of the nonlinear optics. The general symmetry of Maxwell equations with respect to the magnetic field—electric field transposition allows us to expect that the whole variety of known nonlinear optical processes can have corresponding analoges in metamaterials.

Analyzing the structure of Eq. (16), it is easy to notie that the first factor is completely determined by the single element properties. Its multiplication with the linear susceptibilities taken at the frequencies of interacting waves performs a kind of renormalization and can be treated as the result of the influence of the surroundings. This kind of renormalization appears, for instance, in the derivation of nonlinear dielectric susceptibility in optical materials [18,19]. Similar to the optical nonlinearity, the nonlinearity of the magnetic metamaterial increases resonantly as one of the frequencies involved approaches the resonance of the linear susceptibility.

To estimate the macroscopic characteristics of the nonlinear metamaterial of the presented type, we consider an example of metastructure based on split rings with radius $r_0=2$ mm and wire diameter 0.1 mm, arranged with the density $n\sim r_0^{-3}$. Among numerous diode types, backward diodes were reported to possess the best sensitivity and the highest nonlinearity [20,21]. Such a diode insertion having the same cross section as the wire is characterized [21] by $\gamma\approx30\ V^{-1}$ and $\mathcal{Z}\approx R_D\approx1\ \Omega$. Although diodes might allow for higher nonlinearity, Eq. (2) is valid only under the assumption that the nonlinear contribution is much smaller than the linear one. These become comparable when the current reaches characteristic value $\tilde{I}(\omega)\sim(|\gamma\mathcal{Z}(\omega)|)^{-1}$. Therefore, the magnetization should not exceed $\tilde{M}(\omega)=nS\tilde{I}(\omega)$, and, accordingly, the magnetic field in the metamaterial must be lower than $\tilde{H}(\omega)=\tilde{M}(\omega)/\chi_{\rm M}(\omega)$. Assuming that the pumping frequency is ω_1 and $H(\omega_1)\sim\tilde{H}(\omega_1)$, and using Eq. (5), we can estimate the maximal amplitude of the nonlinear modulation of the magnetic susceptibility as

$$\chi_{\rm M}^{(2)}(\omega_3;\omega_1,\omega_2)H(\omega_1)\sim\frac{\mathcal{Z}(\omega_2)}{\mu_0nS^2\omega_3}\chi_{\rm M}(\omega_2)\chi_{\rm M}(\omega_3). \tag{17}$$

For frequencies not close to the resonance, we can estimate $\chi_{\rm M}(\omega_2)\chi_{\rm M}(\omega_3)\sim 0.1$, which provides a noticeable nonlinear contribution of the order of $\chi_{\rm M}^{(2)}(\omega_3;\omega_1,\omega_2)H(\omega_1)\sim 0.001$, with the pumping field limited by about 10 A/m.

The nonlinearity can be further enhanced by either decreasing the diode cross section (that raises $\mathcal{Z}$) or choosing the frequencies closer to the resonance. However, both of these ways are accompanied by the increase of dissipation losses in the media. For practical purposes, one has to ensure that the losses do not exceed the nonlinear contribution. To remain in the transparency region $\mathrm{Re}(\chi_{\rm M}(\omega))\gg \mathrm{Im}(\chi_{\rm M}(\omega))$, the condition $|1-\omega_r^2/\omega^2|\gg R_D/\omega L$ must be fulfilled (the overall ohmic resistance R of the whole element is build mostly by the diode resistance R_D). Then, the figure of merit for this metastructure takes a simple form

$$\frac{\chi_{\rm M}^{(2)}(\omega_3;\omega_1,\omega_2)H(\omega_1)}{\mathrm{Im}(\chi_{\rm M}(\omega_3))}\sim\frac{|\mathcal{Z}(\omega_2)|}{R_D}\;\frac{1-\omega_r^2/\omega_3^2}{1-\omega_r^2/\omega_2^2}, \tag{18}$$

and its larger values are favorable. This relation shows that choosing one of the frequencies closer to the resonance, one can win in the parameters at one frequency, but inevitably loose at the other. The ratio (18) appears to be independent of the diode cross section. Nonlinearity and losses grow equally as the diode gets smaller.

Therefore, the only way to improve the ratio (18) significantly is to increase the ratio $|\mathcal{Z}(\omega)|/R_D$. As in backward diodes, the linear impedance is mostly concerned with the ohmic losses, $|\mathcal{Z}(\omega)|\sim R_D$, their usage can be limited by a significant damping. Certainly, applying higher pumping fields will provide higher nonlinear contribution. However, high nonlinearity would make the susceptibility expansion in the form (15) inapplicable, and the corresponding analysis is beyond the approach followed in this paper.

A promising opportunity is offered by varactors [22]. For varactors, the capacitive impedance can be much higher than the resistance and it is only necessary to ensure that this condition is fulfilled in the desired frequency range.

In summary, we have demonstrated that with the help of simple electronic components, it is possible to construct a metamaterial which possesses nonlinear response. We have shown how the macroscopic properties of this material can be controlled by the characteristics of the structure elements, their arrangement, and the properties of nonlinear insertions. This encourages us to expect that various processes, known in nonlinear optics, can be performed in the microwave range with the help of the proposed metamaterial.

ACKNOWLEDGMENTS

The authors are grateful to Professor L. Solymar for useful ideas and to Dr. B. Sturman and Professor K. Betzler for fruitful discussions. Financial support from the Deutsche Forschungsgemeinschaft is acknowledged by M.L. (Graduate College 695).

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