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Linear and nonlinear response in the aging regime of the one-dimensional trap model

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We investigate the behavior of the response function in the one-dimensional trap model using scaling arguments that we confirm by numerical simulations. We study the average position of the random walk at time t_w+t, given that a small bias h is applied at time t_w. Several scaling regimes are found, depending on the relative values of t, t_w, and h. Comparison with the diffusive motion in the absence of bias allows us to show that the fluctuation-dissipation relation is valid even in the aging regime, at least for times such that linear response is obeyed. However, for sufficiently long times, the response always becomes nonlinear in h.

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The one-dimensional trap model (hereafter denoted as 1DTM) has been the focus of renewed attention, both in the mathematical community [1,2], and also using a physics approach [3,4]. This model was proposed in 1970s to describe the properties of one-dimensional disordered conductors [5,6]. Very recently, the direct relevance of this model for the dynamics of DNA "denaturation bubbles" under torsion was emphasized [7]. Although the "annealed" version of the model, in which a new trapping time is randomly chosen from an *a priori* distribution at each step, is now well documented [8–10], the full analysis of the quenched model considered here remains challenging. It was shown in Refs. [3,4] that, besides interesting dynamical localization properties, the 1DTM exhibits different time scalings, depending on which correlation function is considered. To be more specific, the probability of not moving between t_w and t_w+t, called $\Pi(t_w+t,t_w)$ in Ref. [3], and the probability of occupying the same site at t_w and t_w+t, called $C(t_w+t,t_w)$, exhibit different scalings: Π scales as t/t_w^ν with $\nu<1$, whereas C behaves as t/t_w. Then a natural question arises in this context: what is the relevant time scale that governs the response function of the particle to an external bias? This question is interesting in the context of the physical applications mentioned above, and also in the context of the out-of-equilibrium fluctuation-dissipation theorem that was much discussed recently [11].

Let us first briefly recall the definition of the 1DTM. Consider a one-dimensional lattice, and associate to each site i a quenched random variable $E_i>0$, the energy barrier, chosen from an exponential distribution $\rho(E)=T_g^{-1}e^{-E/T_g}$. One follows the evolution of a particle driven by a thermal noise at temperature T on the lattice. The particle has to overcome the energy barrier E_i in order to leave site i and reach one neighbor. This naturally leads to a mean-trapping time τ_i on site i given by an Arrhenius law $\tau_i=e^{E_i/T}$, and distributed according to $p(\tau)=\mu/\tau^{1+\mu}$, with $\mu=T/T_g$; we focus in the following on the aging phase $\mu<1$. Note that the trapping times are a coarse grained description of the underlying Langevin dynamics, which is not explicitly described in this model. The quantity Γ_0 is a microscopic frequency scale that will be set to unity in the following. Once that particle has escaped the trap, it chooses one of the two nearest neighboring sites, with probability q_- for the left one and $q_+=1-q_-$ for the right one. Two particular cases have already been studied in detail in the literature, namely, the unbiased case $q_+=1/2$ [1,2,3] and the fully directed case $q_+=1$ [6,12,13].

In order to compute a response function, one has to first define an external field to which the response will be associated. The simplest choice is to consider small deviations from the unbiased case, i.e., to introduce a small bias h, independent of the site, such that $q_\pm=(1\pm h)/2$. Note that this is equivalent, as long as $h\ll 1$, to the introduction of a uniform small force field F, which transforms the transition probabilities $q_\pm$ into $q_\pm=q_\pm^0\exp(\pm Fa/2kT)$. Here a is the lattice spacing, which is set to unity in the following; the correspondence is then $h=F/2kT$. Physically, this force is due to the external electric field in the case of conductors [5], or to an asymmetry of the DNA composition along the chain in the model considered in Ref. [7].

As far as the response is concerned, the two natural choices are the average position of the walk after the bias is applied, or the average probability current. Interestingly, a formal relation exists between the former and the latter, namely, that the current is the time derivative of the average position. Switching on a small bias h at time t_w and measuring quantities at a subsequent time t_w+t, we define for a *given* sample of the disorder the average position $x_h(t,t_w)$ and the total probability current $J_h(t,t_w)$ as

$$x_h(t,t_w)=\sum_n nP_n(t,t_w), \tag{1}$$

$$J_h(t,t_w)=\sum_n \phi_{n-1,n}(t,t_w), \tag{2}$$

with $\phi_{n-1,n}\equiv W_{n-1\to n}^h P_{n-1}-W_{n\to n-1}^h P_n$ being the local current. Taking the derivative of $x_h(t,t_w)$ with respect to t, one has

$$\frac{\partial x_h}{\partial t}=\sum_n n\frac{\partial P_n}{\partial t}=\sum_n n[\phi_{n-1,n}-\phi_{n,n+1}]. \tag{3}$$

Assuming periodic boundary conditions on a lattice of size L, and letting eventually $L\to\infty$, one can show that the sum reduces to $\Sigma_n\phi_{n-1,n}$, finally leading to

$$\frac{\partial x_h}{\partial t}(t,t_w)=J_h(t,t_w). \qquad (4)$$

Note that in the 1DTM, transition rates $W^h_{n\to n\pm 1}$ depend only on n, so that $J_h(t,t_w)=0$ for $h=0$, leading to a value of $x(t,t_w)=x(0,0)$ in the absence of bias, for any given sample. Averaging over the disorder, it is clear that Eq. (4) is also valid for the averaged quantities $\langle x\rangle_h(t,t_w)$ and $\langle J\rangle_h(t,t_w)$. Therefore, in the following we shall focus only on the average position after a bias is applied.

In this section, we shall give some simple scaling arguments in order to predict the behavior of $\langle x\rangle_h(t,t_w)$ as a function of the three variables t, t_w, and h. Interestingly, nontrivial regimes appear due to the fact that the limits $h\to 0$ and $t,t_w\to\infty$ cannot be inverted. Note that the case $t_w=0$ was studied in Ref. [6], where it was shown that a nontrivial crossover line appears in the plane $h,1/t$. Let us recall briefly these scaling arguments, since this will be useful in the following. It is convenient to introduce the typical number N of steps of the walk after time t, and to express both $\langle x\rangle_h$ and t as a function of N. It is clear that $\langle x\rangle_h\simeq Nh$; now considering the typical number $\mathcal{N}_s$ of sites visited by the walk, $\mathcal{N}_s$ can be approximately written as the sum of a drift contribution and of a diffusive one:

$$\mathcal{N}_s\sim Nh+\sqrt{N}. \qquad (5)$$

Consider first the case $Nh\gg\sqrt{N}$, corresponding to $\mathcal{N}_s\sim Nh$. Given that the trapping times τ_i are distributed according to $p(\tau)=\mu/\tau^{1+\mu}$, the sum of M independent variables τ_k behaves as $\Sigma_{k=1}^{M}\tau_k\sim M^{1/\mu}$. Since each site is visited of the order of $N/\mathcal{N}_s$ times, t can be expressed as

$$t\sim\frac{N}{\mathcal{N}_s}\sum_{i=-\mathcal{N}_s/2}^{\mathcal{N}_s/2}\tau_i\sim N\mathcal{N}_s^{(1-\mu)/\mu}, \qquad (6)$$

which can be rewritten as $N\sim t^{\mu}h^{\mu-1}$. Finally, $\langle x\rangle_h$ reads

$$\langle x\rangle_h\sim h^{\mu}t^{\mu}. \qquad (7)$$

The criterion $Nh\gg\sqrt{N}$ translates into $h^{\mu}t^{\mu}\gg h^{(\mu-1)/2}t^{\mu/2}$, or equivalently $t\gg t_h\sim h^{-(1+\mu)/(\mu)}$. On the contrary, if $Nh\ll\sqrt{N}$ then $\mathcal{N}_s\sim\sqrt{N}$, and N and t are related through $N\sim t^{2\mu\nu}$, so that

$$\langle x\rangle_h\sim ht^{2\mu\nu} \qquad (8)$$

with $\nu\equiv 1/(1+\mu)$. Therefore, the response is linear in h but nonlinear in t for $t\ll t_h$, and nonlinear both in h and t for $t\gg t_h$.

Let us now turn to the response in the aging regime $t_w\gg 1$. Then if t is small enough (to be specified later), the walk will essentially evolve within the space region of size $\xi(t_w)$ already visited (the diffusion correlation length). The average trapping time $\overline{\tau}(t_w)$ within this region is defined by

$$\overline{\tau}(t_w)\sim\int_1^{t_w^{\nu}}\frac{\mu d\tau}{\tau^{1+\mu}}\sim t_w^{\nu(1-\mu)}, \qquad (9)$$

which is the average value of τ computed from the distribution $p(\tau)$, taking into account the natural cutoff induced by the dynamics t_w^{ν} [3]. Therefore, as long as the particle does not escape from the initial region, the average position should drift at a constant velocity $h/\overline{\tau}(t_w)$ (the lattice spacing a is taken as the unit of length), leading to

$$\langle x\rangle_h\sim ht/t_w^{\nu(1-\mu)}. \qquad (10)$$

This short time regime is limited by two conditions: first, $|\xi(t_w+t)-\xi(t_w)|\ll\xi(t_w)$, which implies $t\ll t_w$, and also $|\langle x\rangle_h|\ll\xi(t_w)$, which requires $t\ll t^*$, where t^* is a new time scale defined by $t^*\equiv t_w^{\nu}/h$. Note, however, that this time scale is only relevant if $t^*<t_w$. If we are in the opposite limit $t\gg t_w$, then t_w no longer plays any role and one recovers the results found above in the particular case $t_w=0$. So one has to distinguish between several regimes, depending on the relative values of t, t_w, and t^*.

(1) $t^*\gg t_w$ (or $h\ll h^*\sim t_w^{-\mu\nu}$). In this case, three different regimes appear:

$$\langle x\rangle_h\sim ht/t_w^{\nu(1-\mu)},\quad t\ll t_w, \qquad (11)$$

$$\langle x\rangle_h\sim ht^{2\mu\nu},\qquad t_w\ll t\ll t_h, \qquad (12)$$

$$\langle x\rangle_h\sim h^{\mu}t^{\mu},\qquad t\gg t_h, \qquad (13)$$

where $t_h\sim h^{-(1+\mu)/\mu}$ as above.

(2) $t^*\ll t_w$ (or $h\gg h^*$). The behavior of $\langle x\rangle_h$ can be summarized as follows:

$$\langle x\rangle_h\sim h\,t/t_w^{\nu(1-\mu)},\quad t\ll t^*, \qquad (14)$$

$$\langle x\rangle_h\sim h^{\mu}t^{\mu},\qquad t\gg t^*, \qquad (15)$$

It is interesting to reformulate the above equations in terms of scaling functions,

$$\langle x\rangle_h=ht_w^{2\mu\nu}f_1(t/t_w),\quad h\ll h^*, \qquad (16)$$

$$\langle x\rangle_h=t_w^{\mu\nu}f_2(ht/t_w^{\nu}),\quad h\gg h^*, \qquad (17)$$

with the following asymptotic behavior for the functions $f_1(z)$ and $f_2(z)$:

$$f_1(z)\sim z\ (z\ll 1),\quad f_1(z)\sim z^{2\mu\nu}\ (z\gg 1), \qquad (18)$$

$$f_2(z)\sim z\ (z\ll 1),\quad f_2(z)\sim z^{\mu}\ (z\gg 1). \qquad (19)$$

The scaling function $f_1(z)$ only accounts for the time regimes described by Eqs. (11) and (12), i.e., for times smaller than t_h. However, since we are in the small bias case $h\ll t_w^{-\mu\nu}$, t_h is much larger than t_w, so that this crossover scale is difficult to exhibit numerically. But one should bear in mind that the response always becomes nonlinear at sufficiently long times, even when $h\to 0$.

We have not found the way to include all the t, t_w, and h regimes in a single scaling function. We now report on the numerical results on the response function, and compare them to the scaling predictions of the preceding section. In

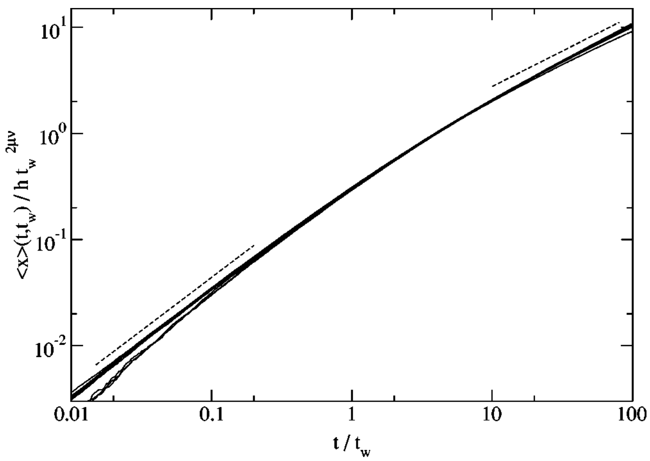

FIG. 1. Rescaled response in the small field regime [$h \ll h^*(t_w)$], using Eq. (16), for $h=2\times10^{-3}$, 5×10^{-3}, 8×10^{-3}, and $t_w=10^3$, 10^4, 10^5 ($\mu=1/2$). The resulting collapse is good, even though some deviations appear both at short and large times. The predicted asymptotic scaling behavior is well obeyed (dashed lines).

the small field regime $h \ll h^*(t_w)$, the scaling predicted by Eq. (16) is well satisfied. Figure 1 shows the resulting collapse of the data for three different values of $h (h=2\times10^{-3}$, 5×10^{-3}, and $8\times10^{-3})$ and three different values of the waiting time $t_w (t_w=10^3$, 10^4, and $10^5)$ at temperature $\mu=\frac{1}{2}$.

In the opposite regime, $h \gg h^*(t_w)$, Eq. (17) is also well obeyed, as shown in Fig. 2 for bias $h=0.1$, 0.2, 0.3, waiting times $t_w=10^4$, 10^5, 10^6, 10^7, and $\mu=\frac{1}{2}$. Note that for clarity, data corresponding to $h=0.2$ and 0.3 are presented for $t_w=10^5$ only. We have checked for several values of μ that the short time and large time behaviors of the scaling functions $f_1(z)$ and $f_2(z)$ given by Eqs. (18) and (19) are also correctly predicted, see Figs. 1 and 2 for the case $\mu=\frac{1}{2}$.

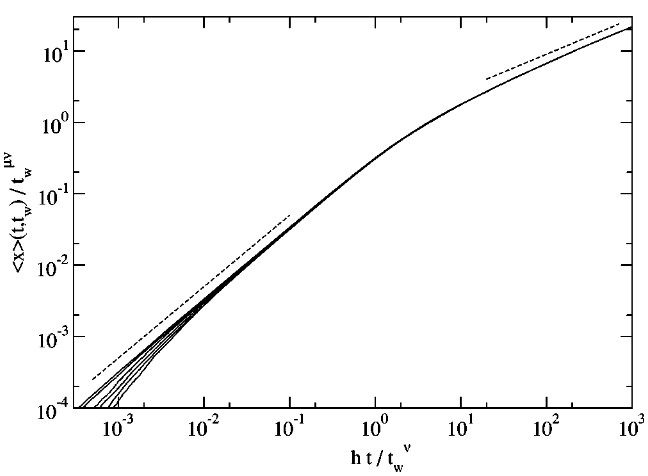

FIG. 2. Rescaled response in the large bias regime [$h \gg h^*(t_w)$], using Eq. (17), for $h=0.1$, 0.2, 0.3, $t_w=10^4$, 10^5, 10^6, 10^7 (see text), and $\mu=1/2$, showing a very good collapse of the data, except for small t where finite time effects become noticeable. The asymptotic behavior of the scaling function is also well predicted (dashed lines).

If one wishes to draw a link with the correlation functions $C(t_w+t,t_w)$ and $\Pi(t_w+t,t_w)$ defined above (see also Ref. [3]), it might appear that $\langle x\rangle_h(t,t_w)$ should be associated with $C(t_w+t,t_w)$, due to the t/t_w scaling in Eq. (16). However, this is limited to the very small bias case. Interestingly, in the opposite regime $h \gg h^*(t_w)$, $\langle x\rangle_h(t,t_w)$ is a function of t/t^* $(t^* \sim t_w^\nu/h)$, up to a prefactor dependent on t_w. So at fixed h, $\langle x\rangle_h(t,t_w)$ scales as t/t_w^ν, as $\Pi(t_w+t,t_w)$ does. In fact, this relation is not purely formal, but indeed corresponds to the underlying physics. During the unbiased time interval $[0,t_w]$, the walk typically visits deep traps with characteristic time $\sim t_w^\nu$.

Once the bias is applied, the evolution is dominated by these deepest traps, before eventually reaching a long time regime independent of t_w. If the bias is very small, the walk will visit these traps a large number of times (of the order of $t_w^{\mu\nu}$ [3]), and the aging dynamics thus resembles closely that of $C(t_w+t,t_w)$ in the absence of bias. On the contrary, if h is large enough, the walk will visit only a few times (of the order of $1/h$) the deepest traps occupied at time t_w, so that the aging dynamics is dominated by the time needed to leave the deep traps for the first time, in close analogy with $\Pi(t_w+t,t_w)$. Moreover, it is worth mentioning that if one applies the bias from $t=0$ instead of t_w, and compute the resulting average displacement between times t_w and t_w+t, different scalings are obtained. For small fields, the displacement $\langle x\rangle(t,t_w)$ behaves as $\langle x\rangle(t,t_w)=ht_w^{2\mu\nu}f_1(t/t_w)$ as in the previous case, but for larger h, one finds a scaling of the form $\langle x\rangle(t,t_w)=t_w^\mu f_3(ht/t_w)$. So in this case, a t/t_w scaling also appears, but for a different reason: because of the bias, the trapping times reached after a time t_w are now of the order of t_w itself instead of t_w^ν, but the walk visits these traps a finite number of times $(\sim 1/h)$ after time t_w.

Besides the correlation C and Π considered hereabove, the natural self-correlation associated with $\langle x\rangle_h$ is the mean square displacement restricted to the time interval $[t_w,t_w+t]$, in the absence of bias field:

$$\langle \Delta x^2\rangle_0(t,t_w) \equiv \langle [x(t_w+t)-x(t_w)]^2\rangle_0, \tag{20}$$

where the brackets $\langle \cdots \rangle$ denote both a thermal average *and* an average over the disorder. Using the effective trapping time $\overline{\tau}(t_w)$, one can also estimate $\langle \Delta x^2\rangle_0(t,t_w)$ in the short time diffusive regime to be $\langle \Delta x^2\rangle_0(t,t_w) \sim t/\overline{\tau}(t_w)$. This expression can only be valid if $\langle \Delta x^2\rangle_0(t,t_w) \ll \xi(t_w)^2$, which yields $t \ll t_w$. In the opposite regime $t \gg t_w$, one recovers the $t_w=0$ result $\langle \Delta x^2\rangle_0(t,t_w) \sim \xi(t_w+t)^2 \sim \xi(t)^2$. In summary

$$\langle \Delta x^2\rangle_0(t,t_w) \sim t/t_w^{\nu(1-\mu)}, \quad t \ll t_w, \tag{21}$$

$$\langle \Delta x^2\rangle_0(t,t_w) \sim t^{2\mu\nu}, \qquad t \gg t_w. \tag{22}$$

This means that $\langle \Delta x^2\rangle_0(t,t_w)$ can be written in the scaling form:

$$\langle \Delta x^2\rangle_0(t,t_w) = t_w^{2\mu\nu} g(t/t_w). \tag{23}$$

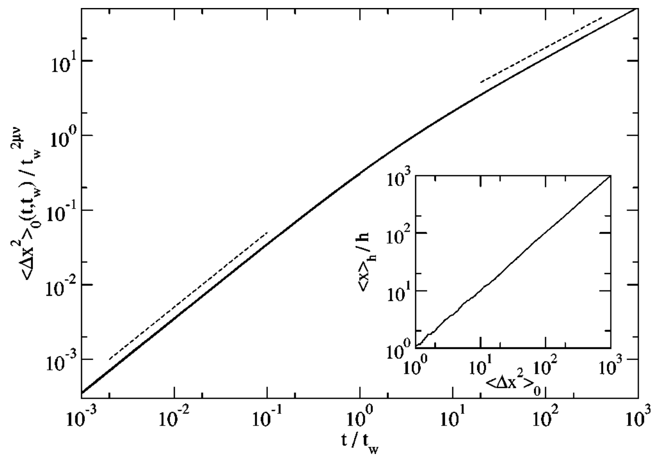

FIG. 3. Rescaled diffusion $\langle \Delta x^2\rangle_0(t,t_w)$ in the absence of bias, using Eq. (23), for $t_w = 10^3$, 10^4, 10^5, 10^6, and 10^7, and $\mu = 1/2$. The different curves are indistinguishable to the eye. The dashed lines indicate the slopes 1 and 2/3, respectively. Note, however, that for computational reasons, time t is limited to 10^7. Inset: FDR plot $\langle x\rangle_h(t,t_w)/h$ versus $\langle \Delta x^2\rangle_0(t,t_w)$ for $t_w = 10^3$, $h = 5\times 10^{-3}$, and $\mu = 1/2$. FDR is clearly satisfied over the whole range of time where the response is linear, i.e., when $t \ll t_h \approx 10^7$.

Figure 3 displays the numerical data obtained for $\langle \Delta x^2\rangle_0(t,t_w)$, for waiting times t_w ranging from 10^3 to 10^7, using Eq. (23). One can see that the collapse is perfect.

Given the quantities computed above, it is natural to test the fluctuation-dissipation (or Einstein) relation (FDR). This is done in the inset of Fig. 3, which displays $\langle x\rangle_h(t,t_w)/h$ versus $\langle \Delta x^2\rangle_0(t,t_w)$ in log-log scale, for $h = 5\times 10^{-3}$, $t_w = 10^3$, and $\mu = 1/2$. This relation appears to be very well satisfied over the whole range of time where the response is linear, although the system is strongly out of equilibrium. This contrasts with many disordered models where the FDR is modified even in the linear regime [11].

We now give a general argument in order to demonstrate the validity of the FDR for the trap model in the aging regime. It was shown in Ref. [6] that for a given configuration of the disorder and a given initial position $x(t_w)$, the following fluctuation-dissipation relation holds, in the limit $h \to 0$:

$$\langle \Delta x\rangle_h = \langle \Delta x\rangle_0 + h[\langle \Delta x^2\rangle_0 - \langle \Delta x\rangle_0^2], \tag{24}$$

with $\Delta x \equiv x(t_w+t) - x(t_w)$ and t is finite. For the purpose of clarity, we distinguish here between average on thermal histories after t_w ($\langle \cdots \rangle$) and average over thermal histories before t_w—thus over $x(t_w)$—*and* over the disorder ($\overline{\cdots}$). Now applying this second type of average to the previous equation, we get

$$\overline{\langle x\rangle_h}(t,t_w) = \frac{F}{2kT}\overline{\langle \Delta x^2\rangle_0}(t,t_w), \tag{25}$$

where we have used the fact that for the trap model $\langle \Delta x\rangle_0 = 0$, see Eq. (4), and h has been replaced by its "physical" expression $F/2kT$. So we conclude that the fluctuation-dissipation relation is indeed valid in this out-of-equilibrium and disordered system, with a temperature equal to the bath temperature T. In particular, this implies that the scaling functions $f_1(\cdot)$ and $g(\cdot)$ are identical. Note that such an "aging Einstein relation" has already been found in the annealed version of the model [9].

As a conclusion, we note that the influence of an external bias in disordered system can be highly nontrivial. In the simple one-dimensional trap model discussed here, we already find several regimes in the t, t_w, h "cube." One of the most interesting result is that the response becomes nonlinear at long times, even in the limit where the external bias tends to zero. We expect this effect to be rather generic, and was actually already noticed in both theoretical and experimental works on the role of magnetic field in spin glass dynamics [14,15].

We have also extended the above arguments to the Sinai model with an external bias (for a review, see Ref. [16]), which for reasons of space we cannot detail here. Already for $t_w = 0$ and small external force F, one finds in general four different time regimes for the average displacement $\langle x\rangle$. There is, in particular, a regime where $\langle x\rangle$ grows as $\ln^2 t$, but with an F independent prefactor, before the asymptotic regime where $\langle x\rangle \sim t^{\alpha F}$ sets in. In other examples, such as a walker on a percolation network, the response can even be nonmonotonous with F [6,17].

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