Stability analysis of a delayed Hopfield neural network

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In this paper, we study a class of neural networks, which includes bidirectional associative memory networks and cellular neural networks as its special cases. By Brouwer's fixed point theorem, a continuation theorem based on Gains and Mawhin's coincidence degree, matrix theory, and inequality analysis, we not only obtain some different sufficient conditions ensuring the existence, uniqueness, and global exponential stability of the equilibrium but also estimate the exponentially convergent rate. Our results are less restrictive than previously known criteria and can be applied to neural networks with a broad range of activation functions assuming neither differentiability nor strict monotonicity.

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I. INTRODUCTION

Recently, theoretical and applied researches of the artificial neural networks have been the new world-wide focus (see Refs. [1–23]). Some of the reasons why Hopfield neural networks have received a great deal of attention are because it can be used in applications to signal and image processing [14], quadratic optimization [2], and fixed-point computation [3]. The number of equilibria of the neural network relates to its storage capacity. Some neural networks may have infinite associative memories. The networks described by differential equations include examples in which the neural network can have nondenumerably many equilibria. Hence, if the neural network is viewed as an associative memory, the more equilibria the neural network has, the greater the storage capacity. But when we use the neural network to solve optimization problems, we want to design a neural network with fewer equilibria. For example, in Refs. [13,23], if an equilibrium is unique, it will be the global minimum point of the related energy function. In such cases, it is almost necessary to have a unique equilibrium, which is global asymptotic stability, ensuring the convergence to an optimal solution starting from any initial guess. Therefore, in both applications, the stability of the neural networks is a prerequisite. On the other hand, Hopfield neural networks have the potential of performing parallel computation, and some electronic implementations of Hopfield neural networks in very large scale integrated technology have already been realized. However, in the implementation of artificial neural networks, time delays are unavoidably encountered. In fact, in models of electronic networks, time delays are likely to be present, due to the finite switching speed of amplifiers. It is known that time delays in the response of neurons can influence the stability of a network creating oscillatory and unstable characteristics. See, for example, Refs. [6,9,20], and the references cited therein. Therefore, it is crucial to take time delays into consideration and to investigate the global asymptotic stability of the Hopfield neural networks with delays. In the following, we assume that the outputs of neurons reach other receiving neurons with certain time delays. If neuron j has fired, a solitonlike pulse propagates along the axon of neuron j to a synapse of neuron i; the signal transport through the axon takes a finite amount of time known as the transmission delay τ_{ij}. In this paper, we consider such delayed Hopfield neural networks:

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} T_{ij} f_j(x_j(t-\tau_{ij})) + I_i, \qquad (1)$$

in which $i=1,2,\dots,n$, a_i represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, T_{ij} denotes the strength of the jth unit on the ith unit, τ_{ij} corresponds to the transmission delay of the ith unit along the axon of the jth unit, I_i denotes the external bias or clamped input from outside the network to the ith unit, x_i corresponds to the membrane potential of the ith unit at time t, $f_j(x_j)$ denotes the conversion of the membrane potential of the jth unit into its firing rate. Throughout this paper, we assume that $a_i > 0$, $\tau_{ij} \geqslant 0$, $T_{ij} \in R$, and $I_i \in R$ are constants.

Model (1) is the most popular and typical neural network model. Some other models, such as continuous BAM (bidirectional associative memory) networks and CNNs (cellular neural networks), are special cases of the network model (1). For instance, the following BAM networks (see, for example, Ref. [9]):

$$\dot{u}_i(t) = -\alpha_i u_i(t) + \sum_{j=1}^{p} a_{ij} g_j(v_j(t-\tau_{ij})) + I_i,$$

$$i=1,2,\dots,k;$$

$$\dot{v}_j(t) = -\beta_j v_j(t) + \sum_{i=1}^{k} b_{ji} h_i(u_i(t-\sigma_{ji})) + J_j,$$

$$j=1,2,\dots,p, \qquad (2)$$

can be deduced from the model of the form (1) with $n=k+p$, $x_i(t)=u_i(t)$, $f_i(u)=h_i(u)$, and $a_i=\alpha_i$ for $i=1,2,\dots,k$; and $x_i(t)=v_{i-k}(t)$, $f_i(u)=g_{i-k}(u)$, and $a_i=\beta_{i-k}$ for $i=k+1,k+2,\dots,k+p$. The connection strengths T_{ij} are specified by

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$$T_{ij}=\begin{cases}0 & \text{if } 1\leqslant i, \quad j\leqslant k \quad \text{or} \quad k+1\leqslant i, \quad j\leqslant n,\\ a_{i,j-k} & \text{if } 1\leqslant i\leqslant k \quad \text{and} \quad k+1\leqslant j\leqslant n,\\ b_{i-k,j} & \text{if } k+1\leqslant i\leqslant n \quad \text{and} \quad 1\leqslant j\leqslant k.\end{cases} \tag{3}$$

By means of Brouwer's fixed point theorem, a continuation theorem based on Gains and Mawhin's coincidence degree, matrix theory, and inequality analysis, we not only obtain some new sufficient conditions ensuring the existence, uniqueness, and global exponential stability of the equilibrium but also estimate the exponentially convergent rate. These conclusions are presented in terms of system parameters, and have an important leading significance in the design and applications of neural circuits for neural networks with delays. We not only unify and improve some previous results but also give some different criteria expressed in terms of matrix norms.

II. STABILITY ANALYSIS

In this section, we first introduce some elementary notations and lemmas. Let $\tau=\max_{1\leqslant i,j\leqslant n}\{\tau_{ij}\}$. As usual, we introduce the phase space $C([-\tau,0];R^n)$ as the Banach space of continuous mappings from $[-\tau,0]$ to R^n equipped with the supremum [24] norm defined by

$$\|\varphi\|=\max_{1\leqslant i\leqslant n}\ \sup_{-\tau\leqslant t\leqslant 0}|\varphi_i(t)|$$

for all $\varphi\in C([-\tau,0];R^n)$. Note that for each given initial value $\varphi\in C([-\tau,0];R^n)$, one can solve system (1) by method of steps to obtain a unique mapping $x:[-\tau,\infty)\to R^n$ such that $x|_{[-\tau,0]}=\varphi$, x is continuous for all $t\geqslant 0$, almost differentiable and satisfies Eq. (1) for $t>0$. This gives a unique solution of Eq. (1) defined for all $t\geqslant -\tau$.

For any given matrix $A=(a_{ij})_{n\times n}$, let $\rho(A)$ denote the spectral radius of A. A matrix or a vector $A\geqslant 0$ means that all the entries of A are greater than or equal to zero, similarly define $A>0$. The following lemma is needed in the proof of our main results.

Lemma 1 [25]. If $\rho(A)<1$ for $A\geqslant 0$, then $(E-A)^{-1}\geqslant 0$, where E denotes the identity matrix of size n.

A point $x^*=(x_1^*,x_2^*,\ldots,x_n^*)^T$ in R^n is called to be an equilibrium of system (1) if this point x^* satisfies the following equation:

$$a_i x_i^*=\sum_{j=1}^{n} T_{ij}f_j(x_j^*)+I_i, \quad i=1,2,\ldots,n. \tag{4}$$

Generally, Eq. (4) may have more than one solution x^*. In fact, we have the following theorem.

Theorem 1. Assume that each activation function satisfies

$$|f_i(u)|\leqslant p_i|u|+q_i \quad \text{for all} \quad u\in R, \quad i=1,2,\ldots,n, \tag{5}$$

where p_i,q_i, $i=1,2,\ldots,n$, are non-negative constants. Moreover, $\rho(A)<1$, where $A=(a_{ij})_{n\times n}$ and $a_{ij}=a_i^{-1}|T_{ij}|p_j$. Then, system (1) has at least one equilibrium.

Thus, there exist some patterns or memories (or equilibria) associated with each set of the external inputs when the connection weights T_{ij} are fixed.

It should be noted that the assumptions of the activation functions in Theorem 1 are very general, assuming neither differentiability nor strict monotonicity. In particular [6,21], if the activation functions in system (1) are all bounded on R, i.e., there exist positive constants q_i, $i=1,2,\ldots,n$, such that $|f_i(u)|<q_i$ for all $u\in R$ and $i=1,2,\ldots,n$, then system must have at least one equilibrium. For example, in CNN model, the activation function takes the form $f(u)=0.5(|u+1|-|u-1|)$, which is bounded. Of course, the computation of $\rho(A)$ could be expensive for a large network. Recall that for a given matrix M, its spectral radius $\rho(M)$ is equal to the minimum of its all matrix norms of M, i.e., for any matrix norm $\|\cdot\|$, $\rho(M)\leqslant\|M\|$. Therefore, we have the following two corollaries. Especially, Corollary 1 puts the constraints directly on the elements of the connection matrix and decay rate, and can be used more conveniently.

Corollary 1. In addition to assumption (5), suppose that there exist positive real numbers d_i $(i=1,2,\ldots,n)$ such that one of the following inequalities is satisfied: (i) $-d_i a_i+\sum_{j=1}^{n}|T_{ij}|p_j d_j<0$ for all $i=1,2,\ldots,n$; (ii) $-d_i a_i+\sum_{j=1}^{n}|T_{ji}|p_i d_j<0$ for all $i=1,2,\ldots,n$; (iii) $\sum_{i=1}^{n}|T_{ij}|d_j p_j/(a_i d_i)<1$ for all $j=1,2,\ldots,n$; (iv) $\sum_{i=1}^{n}\sum_{j=1}^{n}(|T_{ij}|d_j p_j/(a_i d_i))^2<1$. Then there exists at least one equilibrium of system (1).

Corollary 2. In addition to assumption (5), suppose that $\rho(A^TA)<1$ where A is defined as in Theorem 1. Then, there exists at least one equilibrium of system (1).

The equilibrium or pattern $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ of Eq. (1) is said to be globally asymptotically stable independent of the delays if every solution of Eq. (1) corresponding to an arbitrary given set of initial values satisfies $\lim_{t\to\infty}x_i(t)=x_i^*$, $i=1,2,\ldots,n$. Moreover, if there exist constants $M\geqslant 1$ and $\lambda>0$ such that for every solution $x(t)$ of Eq. (1) with any initial value $\varphi\in C([-\tau,0];R)$,

$$|x_i(t)-x_i^*|\leqslant M\|\varphi-x^*\|e^{-\lambda t}, \quad i=1,2,\ldots,n,$$

then x^* is called to be globally exponentially stable and λ is called to be globally exponentially convergent rate.

If we further assume that $f_i, i=1,2,\ldots,n$, are globally Lipschitz, then we shall obtain the uniqueness and global exponential stability of the equilibrium. Namely, we have the following theorem.

Theorem 2. Assume that there exist non-negative constants p_j, $j=1,2,\ldots,n$ such that $|f_j(x)-f_j(y)|\leqslant p_j|x-y|$ for any $x,y\in R$ and $j=1,2,\ldots,n$, and $\rho(A)<1$, where $A=(a_{ij})_{n\times n}$ and $a_{ij}=a_i^{-1}|T_{ij}|p_j$. Then system (1) has exactly one equilibrium x^*. Moreover, x^* is globally exponentially stable, and the globally exponentially convergent rate $\lambda<\lambda^*$, where λ^* is the minimal positive real root of the following equation:

$$\lambda+a_0e^{\lambda\tau}\rho(A)-a_0=0, \tag{6}$$

and $a_0=\min_{1\leqslant i\leqslant n}\{a_i\}$.

Theorem 2 implies that the patterns associated with external inputs are recalled by the convergence or global attractivity of system dynamics; global exponential stability means that the recall is ''perfect'' in the sense no hints or guesses are needed as in the case of local stability analysis; that is, when the external inputs are provided to the system, irrespective of the initial values, system (1) converges to the equilibrium associated with the inputs. Recall with the help of hints and guesses corresponds to local stability of equilibrium; since the initial values have to be in suitable neighborhood of the corresponding equilibrium. Moreover, in order to improve network performance, we can increase the exponentially convergent rate to reduce the time that is required for the system to recall.

Ye, Michel, and Wang [21] also obtained some sufficient conditions ensuring the global stablity of Hopfield neural networks with delays, i.e., $\lim_{t\to\infty}x(t)$ exists for any solution $x(t)$. However, we consider the existence, uniqueness and globally exponential stability of the equilibrium. These are two different notations. Moreover, in Ref. [21], there exist some more restrictive conditions about the connection weight matrix and the activation functions, i.e., T is symmetric, $\lim_{x\to\infty}f_j(x)=1$, $\lim_{x\to-\infty}f_j(x)=-1$, $f_j'(x)>0$, and $\lim_{|x|\to\infty}f_j'(x)=0$ for $i=1,2,\ldots,n$. Again, since matrix's spectral radius is equal to the minimum of its all matrix norms, we have the following corollary.

Corollary 3. Assume that there exist non-negative constants p_j, $j=1,2,\ldots,n$ such that $|f_j(x)-f_j(y)|\leq p_j|x-y|$ for any $x,y\in R$ and $i=1,2,\ldots,n$, suppose further that either $\rho(A^TA)<1$, where $A=(a_{ij})_{n\times n}$ and $a_{ij}=a_i^{-1}|T_{ij}|p_j$, or there exist positive real numbers d_i ($i=1,2,\ldots,n$) such that one of the inequalities (i)–(iv) in Corollary 1 is satisfied. Then, there exists exactly one equilibrium of system (1). Moreover, all other solutions of system (1) converge exponentially to it as $t\to\infty$.

Because continuous BAM network (2) is a special case of the network model (1). Thus, applying Theorems 1 and 2 and Corollaries 1–3, we have the following corollary.

Corollary 4. For system (2), assume that there exist $p_j>0$, $q_i>0$, and γ_j and δ_i such that $|g_j(u)|\leq p_j|u|+\gamma_j$ and $|h_i(u)|\leq q_i|u|+\delta_i$ for any $u\in R$ and $i=1,2,\ldots,k$ and $j=1,2,\ldots,p$, then each one of the following inequalities ensures the existence of equilibrium of system (2): (i) $\rho(A^TA)<1$ and $\rho(B^TB)<1$, where matrices $A=(|a_i^{-1}a_{ij}|)_{k\times p}$ and $B=(|b_i^{-1}b_{ij}|)_{p\times k}$; (ii) there exist positive real numbers d_i ($i=1,2,\ldots,n$) such that $-d_ia_i+\sum_{j=1}^{p}|a_{ij}|p_jd_{k+j}<0$ for $i=1,2,\ldots,k$ and $-d_{k+j}b_j+\sum_{i=1}^{k}|b_{ji}|q_id_i<0$ for $j=1,2,\ldots,p$; (iii) there exist positive real numbers d_i ($i=1,2,\ldots,n$) such that $-d_ia_i+\sum_{j=1}^{p}|b_{ji}|q_id_{k+j}<0$ for $i=1,2,\ldots,k$ and $-d_{k+j}b_j+\sum_{i=1}^{n}|a_{ij}|p_jd_i<0$ for $j=1,2,\ldots,p$.

In particular, if g_j and h_i are globally Lipschitz, i.e., there exists $p_i>0$ and $q_i>0$ such that $|g_j(x)-g_j(y)|\leq p_j|x-y|$ and $|h_i(x)-h_i(y)|\leq q_i|x-y|$ for any $x,y\in R$ and $i=1,2,\ldots,k$ and $j=1,2,\ldots,p$; and one of the above inequalities (i)–(iii) holds then the equilibrium of system (2) is unique and all other solutions of system (2) converge exponentially to it as $t\to\infty$.

III. EXISTENCE OF PERIODIC SOLUTIONS

In this section, we investigate the periodic solutions of the model of the form

$$\dot{x}_i(t)=-a_ix_i(t)+\sum_{j=1}^{n}T_{ij}f_j(x_j(t-\tau_{ij}))+I_i(t), \tag{7}$$

where $i=1,2,\ldots,n$ and $I_i:R^+\to R$ is a continuously periodic function with period ω, i.e., $I_i(t+\omega)=I_i(t)$. First, applying the continuation theorem of Gaines and Mawhin [27], we can obtain the following results.

Theorem 3. Assume that all the conditions in Theorem 1 hold, then there exists at least one ω-periodic solution of system (7).

Thus, all conditions in Corollary 1 or 2 can also ensure the existence of periodic solutions of system (7). If we further assume that $f_i, i=1,2,\ldots,n$, are globally Lipschitz, then we shall obtain the uniqueness and global exponential stability of the periodic solution of system (7). Namely, we have the following theorem.

Theorem 4. Assume that all the conditions in Theorem 2 hold, then system (7) has exactly one ω-periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable and the globally exponentially convergent rate $\lambda<\lambda^*$, where λ^* is the minimal positive real root of Eq. (6) and matrix A is defined as that in Theorem 2.

Corollary 5. Under the assumptions of Corollary 3, there exists exactly one ω-periodic solution of system (7) and all other solutions of Eq. (7) converge exponentially to it as $t\to\infty$.

IV. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following Hopfield neural networks with delays:

$$x_1'(t)=-3x_1(t)+af_1(x_1(t-1))-2f_2(x_2(t-0.4))+4\pi,$$

$$x_2'(t)=-2x_2(t)+2f_1(x_1(t-0.5))+af_2(x_2(t-1))+\frac{2-a}{2}\pi, \tag{8}$$

where the signal transmission functions $f_1(x)=\sin x$ and $f_2(x)=x$, then $p_1=p_2=1$, $a_1=3$, $a_2=2$, and

$$A=\begin{pmatrix} a/3 & 2/3 \\ 1 & a/2 \end{pmatrix}.$$

By a simple calculation, if $|a|<(5-\sqrt{17})/2$ then $\rho(A)=(5|a|+\sqrt{a^2+96})/12<1$. By Theorem 2, it is easy to see that system (8) has a globally asymptotically stable equilibrium $(\pi,\pi/2)$. Moreover, the globally exponentially convergent rate λ can be estimated by the inequality: $6\lambda+e^{\lambda}(5|a|+\sqrt{a^2+96})-12<0$. For example, if $a=0$, then the equilibrium $(\pi,\pi/2)$ of system (8) is globally asymptotically stable and the globally exponentially convergent rate $\lambda<0.1336$.

Example 2. Consider the following Hopfield neural networks with delays:

$$x_1'(t)=-3x_1(t)+0.2f_1(x_1(t-\pi))+f_2(x_2(t-\pi))+I_1(t),$$

$$x_2'(t)=-2x_2(t)+0.4f_1(x_1(t-0.5\pi))+f_2(x_2(t-0.5\pi))+I_2(t), \quad (9)$$

where the signal transmission functions $f_1(x)=2x$ and $f_2(x)=-x$, $I_1(t)=3.4\sin t-\cos t$, and $I_2(t)=2.8\cos t+\sin t$ then $p_1=2$, $p_2=1$, $a_1=3$, $a_2=2$, and

$$A=\begin{pmatrix} 2/15 & 1/3 \\ 2/5 & 1/2 \end{pmatrix}.$$

By easy computation, $\rho(A)\approx 0.7253<1$. Then by Theorem 4, there exists exactly one 2π-periodic solution of system (9) and all other solutions of Eq. (9) converge exponentially to it as $t\to\infty$. It is easy to verify that $(\sin t,\cos t)$ is the 2π-periodic solution of the model (9). Moreover, the globally exponentially convergent rate $\lambda<0.0879$.

V. CONCLUSION

Some sufficient conditions for global exponential stability for a kind of neural networks with delays have been obtained. For delayed neural networks (1) or its special forms, Cao and Zhou [5], Gopalsamy and He [9,10], Lu [18], Lu and He [19], and Zhou and Cao [22] derived some stability criteria. It is easy to see that Corollary 3 is a little generation and improvement of their theorems. Therefore, our results not only unify and improve some previous results about stability analysis but also give some different criteria expressed in terms of matrix norms. In addition, the results in this paper also allow nonsymmetric weight matrices and a broad range of activation functions assuming neither differentiability nor strict monotonicity. As we noted above, all results in this paper are presented in terms of system parameters, and have an important leading significance in the design and applications of neural circuits for neural networks with delays.

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APPENDIX A: PROOF OF THEOREM 1

Let $F_i(u)=a_i^{-1}[\sum_{j=1}^n T_{ij}f_j(u_j)+I_i]$, $i=1,2,\ldots,n$, and $F(u)=(F_1(u),F_2(u),\ldots,F_n(u))^T$. Then Eq. (4) can be rewritten as $x^*=F(x^*)$, where x^* is an equilibrium or a fixed point of the system (1). Let $D_i=a_i^{-1}[\sum_{j=1}^n|T_{ij}|q_j+|I_i|]$, $i=1,2,\ldots,n$, and $D=(D_1,D_2,\ldots,D_n)^T$. In view of $\rho(A)<1$ and Lemma 1, we have $(E-A)^T\geqslant 0$ and $h=(E-A)^{-1}D\geqslant 0$, i.e., $h_i=\sum_{j=1}^n a_{ij}h_j+D_i$, $i=1,2,\ldots,n$, where h_i is the ith component of vector h. Let

$$\Omega=\{(x_1,x_2,\ldots,x_n)^T\in R^n;|x_i|\leqslant h_i,\ i=1,2,\ldots,n\}.$$

Then, it follows from Eq. (5) that for all $u\in\Omega$,

$$|F_i(u)|\leqslant a_i^{-1}\left[\sum_{j=1}^n|T_{ij}||f_j(u_j)|+|I_i|\right]\leqslant\sum_{j=1}^n a_{ij}|u_j| + a_i^{-1}\sum_{j=1}^n|T_{ij}|q_j+a_i^{-1}|I_i|\leqslant\sum_{j=1}^n a_{ij}h_j+D_i=h_i.$$

Hence, $F:\Omega\to\Omega$ is a continuous mapping. By Brouwer's fixed point theorem, F has at least one fixed point or equilibrium. The proof is complete.

APPENDIX B: PROOF OF COROLLARY 1

For any matrix norm $\|\cdot\|$ and any nonsingular matrix S, $\|A\|_S=\|S^{-1}AS\|$ also defines a matrix norm. Let D and Q be positive diagonal matrices $D=\text{diag}\{d_1,d_2,\ldots,d_n\}$ and $Q=\text{diag}\{a_1,a_2,\ldots,a_n\}$, respectively. Then, Conditions (i)–(iv) in Corollary 1 correspond to the row norms, column norms, and Frobenius norm (or Euclidean norm) of matrix DAD^{-1} [or $DQA(DQ)^{-1}$], respectively.

APPENDIX C: PROOF OF THEOREM 2

Since $A=D^{-1}(DAD^{-1})D$ where $D=\text{diag}(a_1,a_2,\ldots,a_n)$, $\rho(DAD^{-1})=\rho A<1$. Thus, $E-D^{-1}A^TD$ is an M matrix [26], where E denotes an identity matrix of size n. Therefore, there exists a diagonal matrix $Q=\text{diag}(d_1,d_2,\ldots,d_n)$ with positive diagonal elements such that the product matrix $(E-D^{-1}A^TD)Q$ is strictly diagonally dominant with positive diagonal entries. Namely,

$$-a_id_i+\sum_{j=1}^n d_j|T_{ji}|p_i<0,\quad i=1,2,\ldots,n.$$

Hence, there must exist sufficiently small constant $\lambda\in(0,a_0)$ such that

$$(\lambda-a_i)d_i+e^{\lambda\tau}\sum_{j=1}^n d_j|T_{ji}|p_i<0,\quad i=1,2,\ldots,n. \quad \text{(C1)}$$

It follows from the existence of globally Lipschitz constants of f_j that $|f_j(x)|\leqslant p_j|x|+|f_j(0)|$, $j=1,2,\ldots,n$. Hence, all the hypotheses in Theorem 1 hold with $q_j=|f_j(0)|$ $(j=1,2,\ldots,n)$. Thus, from Theorem 1, system (1) has at least one equilibrium x^*. Let $x(t)$ be an arbitrary solution of system (1) and define $y(t)=x(t)-x^*$, then we have

$$\dot{y}_i(t)=-a_iy_i(t)+\sum_{j=1}^n T_{ij}g_j(y_j(t-\tau_{ij})),\quad i=1,2,\ldots,n, \quad \text{(C2)}$$

where $g_j(y_j(t-\tau_{ij}))=f_j(x_j(t-\tau_{ij}))-f_j(x_j^*)$. Thus, it is sufficient to prove that $(0,0,\ldots,0)^T$ is globally exponentially stable for system (C2).

We consider the Lyapunov functional

$$V(y)(t)=\sum_{i=1}^{n} d_i\left(|y_i(t)|e^{\lambda t}+\sum_{j=1}^{n}|T_{ij}|p_j\int_{t-\tau_{ij}}^{t}|y_j(s)|e^{\lambda(s+\tau_{ij})}ds\right). \tag{C3}$$

Obviously, for any $y(t)$ except 0, $V(y)(t)>0$. Calculating the upper right derivative of V along the solution $y(t)=(y_1(t),y_2(t),\ldots,y_n(t))^T$ of Eq. (1) with any initial value $\varphi\in C([-\tau,0],R^n)$, we have

$$\begin{aligned}D^+V\leqslant&\sum_{i=1}^{n} d_i\left(-a_i|y_i(t)|e^{\lambda t}+\sum_{j=1}^{n}|T_{ij}|p_j|y_j(t-\tau_{ij})|e^{\lambda t}+\lambda|y_i(t)|e^{\lambda t}\right)+\sum_{i=1}^{n} d_i\left(\sum_{j=1}^{n}|T_{ij}|p_j[|y_j(t)|e^{\lambda(t+\tau_{ij})}-|y_j(t-\tau_{ij})|e^{\lambda t}]\right)\\ \leqslant&\sum_{i=1}^{n} d_i\left((\lambda-a_i)|y_i(t)|e^{\lambda t}+e^{\lambda\tau}\sum_{j=1}^{n}|T_{ij}|p_j|y_j(t)|e^{\lambda t}\right)\\ =&\sum_{j=1}^{n}\left((\lambda-a_j)d_j+e^{\lambda\tau}\sum_{i=1}^{n}d_i|T_{ij}|p_j\right)|y_j(t)|e^{\lambda t}\leqslant 0,\end{aligned}$$

and so,

$$V(y)(t)\leqslant V(y)(0).$$

Therefore, we obtain

$$\begin{aligned}\sum_{i=1}^{n} d_i|y_i(t)|e^{\lambda t}\leqslant&\sum_{i=1}^{n} d_i\left(|y_i(0)|+\sum_{j=1}^{n}|T_{ij}|p_j\times\int_{-\tau_{ij}}^{0}|y_j(s)|e^{\lambda(s+\tau_{ij})}ds\right)\\ \leqslant&\sum_{i=1}^{n} d_i\left(1+\lambda^{-1}(e^{\lambda\tau}-1)\sum_{j=1}^{n}|T_{ij}|p_j\right)\|\varphi\|.\end{aligned}$$

Thus,

$$|y_i(t)|\leqslant Me^{-\lambda t},$$

for all $t\geqslant 0$ and $i=1,2,\ldots,n$, where $M=d_i^{-1}\Sigma_{i=1}^{n}d_i[1+\lambda^{-1}(e^{\lambda\tau}-1)\Sigma_{j=1}^{n}|T_{ij}|p_j]$. This implies that the periodic solution $x^*(t)$ is globally exponentially stable. Moreover, the exponentially convergent rate is λ, which is a positive number satisfying inequality (C1).

APPENDIX D: PROOF OF THEOREM 3

In order to use the continuation theorem for Eq. (7), we denote Z (resp X) as the set of all continuously (resp. differentiably) ω-periodic functions $u(t)=(x_1(t),x_2(t)\ldots,x_n(t))^T$ definded on R, and denote

$$|x_i|_0=\max_{t\in[0,w]}|x_i(t)|,\quad i=1,2,\ldots,n,$$

$$|u|_0=\max_{1\leqslant i\leqslant n}\{|x_i|_0\},\quad |u|_1=\max\{|u|_0,|\dot{u}|_0\}.$$

Then, X and Z are Banach spaces when they are endowed with the norms $|\cdot|_1$ and $|\cdot|_0$, respectively. Set

$$(Nu)_i(t)=-a_ix_i(t)+\sum_{j=1}^{n}T_{ij}f_j(x_j(t-\tau_{ij}))+I_i(t),$$

$$(Lu)(t)=\dot{u}(t),\quad Pu=\frac{1}{w}\int_0^w u(t)dt,\quad u\in X,$$

$$Qz=\frac{1}{w}\int_0^w z(t)dt,\quad z\in Z.$$

It is not difficult to show that

$$\ker L=R^n,\quad \mathrm{Im}L=\left\{z\in Z:\int_0^w z(t)dt=0\right\}\ \text{is closed in}\ Z,$$

$$\dim \mathrm{Ker}L=n=\mathrm{codim}\ \mathrm{Im}L,$$

and P,Q are continuous projectors such that

$$\mathrm{Im}P=\ker L,\quad \ker Q=\mathrm{Im}L=\mathrm{Im}(I-Q).$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (of L) $K_P:\mathrm{Im}L\to\ker P\cap\mathrm{Dom}L$ reads as

$$(K_Pu)_i(t)=\int_0^t x_i(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^t x_i(s)dsdt$$

for $i=1,2,\ldots,n$, $u\in Z$. Thus, we have

$$(QNu)_i=\frac{1}{\omega}\int_0^\omega\left[-a_ix_i(t)+\sum_{j=1}^{n}T_{ij}f_j(x_j(t-\tau_{ij}))+I_i(t)\right]dt,$$

$$\begin{aligned}(K_P(I-Q)Nu)_i(t)=&\int_0^t\left[-a_ix_i(s)+\sum_{j=1}^{n}T_{ij}f_j(x_j(s-\tau_{ij}))+I_i(s)\right]ds-\frac{1}{\omega}\int_0^\omega\int_0^t\left[-a_ix_i(s)+\sum_{j=1}^{n}T_{ij}f_j(x_j(s-\tau_{ij}))+I_i(s)\right]dsdt\end{aligned}$$

$$-\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[-a_ix_i(s)+\sum_{j=1}^{n}T_{ij}f_j(x_j(s-\tau_{ij}))+I_i(s)\right]ds$$

for $i=1,2,\ldots,n$. Clearly, QN and $K_p(I-Q)N$ are continuous. An application of the Arzela-Ascoli theorem to $K_p(I-Q)N$ results in the fact that $K_p(I-Q)N(\bar{\Omega})$ is compact for any open bounded $\Omega\subset X$. Moreover, $QN(\bar{\Omega})$ is clearly bounded. Thus, N is L compact on $\bar{\Omega}$ with any open bounded set $\Omega\subset X$.

Now we reach the position to search for an appropriate open bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lx=\lambda Nx,\lambda\in(0,1)$, we have

$$\dot{x}_i(t)=-\lambda a_ix_i(t)+\lambda\sum_{j=1}^{n}T_{ij}f_j(x_j(t-\tau_{ij}))+\lambda I_i(t) \tag{D1}$$

for $i=1,2,\ldots,n$. Assume that $u=u(t)\in X$ is a solution of Eq. (D1) for a certain $\lambda\in(0,1)$. Then, for any $i=1,2,\ldots,n$, $x_i(t)$, as the components of $u(t)$, are all continuously differentiable. Thus, there exist $t_i\in[0,\omega]$ such that $|x_i(t_i)|=\max_{t\in[0,\omega]}|x_i(t)|$. Hence, $\dot{x}_i(t_i)=0$. This implies that

$$a_ix_i(t_i)=\sum_{j=1}^{n}T_{ij}f_j(x_j(t_i-\tau_{ij}))+I_i(t_i) \tag{D2}$$

for $i=1,2,\ldots,n$. Thus,

$$\begin{aligned}|x_i(t_i)|&=\left|\sum_{j=1}^{n}\frac{T_{ij}}{a_i}f_j(x_j(t_i-\tau_{ij}))+\frac{I_i(t_i)}{a_i}\right|\\&\leqslant\sum_{j=1}^{n}\frac{|T_{ij}|p_j}{a_i}|x_j(t_i-\tau_{ij})|+\sum_{j=1}^{n}\frac{|T_{ij}|q_j}{a_i}+\frac{|I_i(t)|}{a_i}\\&\leqslant\sum_{j=1}^{n}\frac{|T_{ij}|p_j}{a_i}|x_j(t_j)|+\sum_{j=1}^{n}\frac{|T_{ij}|q_j}{a_i}+\frac{|I_i(t)|}{a_i}\\&\leqslant\sum_{j=1}^{n}a_{ij}|x_j(t_j)|+D_i,\end{aligned}$$

where $D_i=a_i^{-1}[\Sigma_{j=1}^{p}|T_{ij}|q_j+\max_{t\in[0,\omega]}\{|I_i(t)|\}]$. In view of $\rho(A)<1$ and Lemma 1, we have $(E-A)^T\geqslant0$ and $h=(E-A)^{-1}D\geqslant0$, where $D=(D_1,D_2,\ldots,D_n)^T$. Therefore,

$$|x_i(t_i)|\leqslant h_i\quad\text{for}\quad i=1,2,\ldots,n,$$

where h_i is the ith component of vector h. Thus, we have

$$|x_i|_0\leqslant h_i\quad\text{for}\quad i=1,2,\ldots,n. \tag{D3}$$

Clearly, $h_i,i=1,2,\ldots,n$ are independent of λ. Moreover, it follows from Eq. (D1) that

$$\begin{aligned}|\dot{x}_i(t)|&\leqslant\lambda a_i|x_i(t)|+\lambda\sum_{j=1}^{n}|T_{ij}||f_j(x_j(t-\tau_{ij}))|+\lambda|I_i(t)|\\&\leqslant a_i\left[h_i+\sum_{j=1}^{n}a_{ij}h_j+D_i\right]=2h_ia_i.\end{aligned}$$

Namely, we have

$$|\dot{x}_i|_0\leqslant2h_ia_i,\quad i=1,2,\ldots,n. \tag{D4}$$

Let $A=\max_{1\leqslant i\leqslant n}\{h_i(1+2a_i)\}$. Then, there exists some $d>1$ such that $dh_i>A$ for all $i=1,2,\ldots,n$. We take

$$\Omega=\{u\in X;-dh<u(t)<dh\quad\text{for all}\quad t\}.$$

If $u=(x_1,x_2,\ldots,x_n)^T\in\partial\Omega\cap\ker L=\partial\Omega\cap R^n$, then u is a constant vector in R^n with $|x_i|=dh_i$ for $i=1,2,\ldots,n$. It follows that

$$(QNu)_i=-a_ix_i+\sum_{j=1}^{n}T_{ij}f_j(x_j)+\frac{1}{\omega}\int_0^{\omega}I_i(t)dt,$$

for $i=1,2,\ldots,n$. We claim that

$$|(QNu)_i|>0\quad\text{for}\quad i=1,2,\ldots,n. \tag{D5}$$

Contrarily, suppose that there exists some $i\in\{1,2,\ldots,n\}$ such that $|(QNu)_i|=0$, i.e., $-a_ix_i+\Sigma_{j=1}^{n}T_{ij}f_j(x_j)+\frac{1}{\omega}\int_0^{\omega}I_i(t)dt=0$. Then, there exists some $t^*\in[0,\omega]$ such that

$$-a_ix_i+\sum_{j=1}^{n}T_{ij}f_j(x_j)+I_i(t^*)=0.$$

Thus,

$$dh_i=|x_i|\leqslant\sum_{j=1}^{n}\frac{|T_{ij}|}{a_i}|f_j(x_j)|+\frac{|I_i(t^*)|}{a_i}\leqslant\sum_{j=1}^{n}a_{ij}dh_j+D_i.$$

In view of $d>1$ and $dh=d(Ah+D)>Adh+D$, we have $dh_i>\Sigma_{j=1}^{n}a_{ij}dh_j+D_i$ for $i=1,2,\ldots,n$. It follows from the above inequality that $dh_i<dh_i$, which is a contradiction. Therefore, Eq. (D5) holds, and hence,

$$QNu\neq0,\quad\text{for}\quad u\in\partial\Omega\cap\ker L.$$

Define $\psi:\Omega\cap\ker L\times[0,1]\to X/\mathrm{Im}L=X^c$ by

$$\psi(u,\mu)=\mu\,\mathrm{diag}(-a_1,-a_2,\ldots,-a_n)u+(1-\mu)QNu$$

for all $u=(x_1,x_2,\ldots,x_n)^T\in\Omega\cap\ker L=\Omega\cap R^n$ and $\mu\in[0,1]$.

When $u\in\partial\Omega\cap\ker L$ and $\mu\in[0,1]$, $u=(x_1,\ldots,x_n)^T$ is a constant vector in R^n with $|x_i|=dh_i(i=1,2,\ldots,n)$. Thus,

$$|\psi(u,\mu)|_0 = \max_{1\leq i\leq n}\left\{\left|-a_i x_i + (1-\mu)\left[\sum_{j=1}^{n} T_{ij} f_j(x_j) + \frac{1}{\omega}\int_0^{\omega} I_i(t)dt\right]\right|\right\}.$$

We claim that

$$|\psi(u,\mu)|_0 > 0. \tag{D6}$$

Contrarily, suppose that $|\psi(u,\mu)|_0 = 0$, then we have

$$-a_i x_i + (1-\mu)\left[\sum_{j=1}^{n} T_{ij} f_j(x_j) + \frac{1}{\omega}\int_0^{\omega} I_i(t)dt\right] = 0, \tag{D7}$$

for all $i=1,2,\ldots,n$. It follows that there exists some $t^* \in [0,\omega]$ such that

$$-a_i x_i + (1-\mu)\left[\sum_{j=1}^{p} T_{ij} f_j(x_j) + I_i(t^*)\right] = 0.$$

Thus,

$$dh_i = |x_i| \leq (1-\mu)\left[\sum_{j=1}^{n} \frac{|T_{ij}|}{a_i}|f_j(x_j)| + \frac{|I_i(t^*)|}{a_i}\right] \leq \sum_{j=1}^{n} a_{ij} dh_j + D_i,$$

which contradicts that $dh_i > \Sigma_{j=1}^{n} a_{ij} dh_j + D_i$ for $i = 1,2,\ldots,n$. Thus, Eq. (D6) holds. Therefore,

$$\psi(u,\mu) \neq 0 \quad \text{for any} \quad u \in \partial\Omega \cap \ker L.$$

Using the property of topological degree and taking $J:\mathrm{Im}Q \to \ker L$, $(x_1,\ldots,x_n)^T \to (x_1,\ldots,x_n)^T$, we have

$$\begin{aligned}\deg(JQN,\Omega\cap\ker L,0) &= \deg(\psi(\cdot,0),\Omega\cap\ker L,0)\\ &= \deg(\psi(\cdot,1),\Omega\cap\ker L,0)\\ &= \deg(\mathrm{diag}(-a_1,\ldots,\\ &\quad -a_n),\Omega\cap\ker L,0) = 1.\end{aligned}$$

Therefore, according to the continuation theorem of Gaines and Mawhin [27], system (7) has at least one ω-periodic solution.

APPENDIX E: PROOF OF THEOREM 4

Using the similar arguments to the proof of Theorem 2, system (7) has at least one ω-periodic solution $x^*(t)$. Let $x(t)$ be an arbitrary solution of system (7) and define $y(t) = x(t) - x^*(t)$ then we have Eq. (C2) with $g_j(y_j(t-\tau_{ij})) = f_j(x_j(t-\tau_{ij})) - f_j(x_j^*(t-\tau_{ij}))$. Thus, it is sufficient to prove that 0 is globally exponentially stable for system (C2). We can apply the same arguments as that in the proof of Theorem 2 to complete the proof of this theorem.

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