# Asymptotic distributions of periodically driven stochastic systems

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(Received 2 January 2003; published 23 June 2003; publisher error corrected 26 June 2003)

We study the large-time behavior of Brownian particles moving through a viscous medium in a confined potential, and which are further subjected to position-dependent driving forces that are periodic in time. We focus on the case where these driving forces are rapidly oscillating with an amplitude that is not necessarily small. We develop a perturbative method for the high-frequency regime to find the large-time behavior of periodically driven stochastic systems. The asymptotic distribution of Brownian particles is then determined to second order. To first order, these particles are found to execute small-amplitude oscillations around an effective static potential that can have interesting forms.

## I. INTRODUCTION

When a macroscopic system characterized by a Hamiltonian $H(C)$ is in contact with an environment with temperature $\beta^{-1}$, at long enough times it reaches an equilibrium state and is described by the Boltzmann-Gibbs distribution $P_{eq}(C) \sim \exp[-\beta H(C)]$ over the configurations of the system. If, however, the system is subjected to time-dependent forces of appreciable magnitude, there is no analogous general statement that can be made about the distribution $P(C,t)$ at large times. A case of particular interest arises when these forces vary periodically in time. While it is straightforward to see that the large-time distribution must be periodic as well, its full form is not known in general. It is thus of interest to seek explicit answers for particular physical systems subjected to periodic driving.

In this paper, we focus on a paradigmatic system: a Brownian particle which feels viscous forces and random impulses from the surrounding medium, and is confined by a potential well. We ask: What is the effect of a further oscillating potential on the state of the particle? We are primarily interested in the case where the fluctuating potential has a nontrivial spatial dependence. We analyze the problem mostly in the high-frequency limit and find the asymptotic state perturbatively. The resulting time-averaged asymptotic state is described effectively by a distribution of the Boltzmann form with an energy function that has three parts: a kinetic energy term that depends only on velocity; an effective potential that depends only on position coordinates; and a term with both velocity and position coordinates. However, to the leading order the result is particularly simple, and involves only the kinetic term and an effective frequency-dependent potential energy whose form can be specified exactly.

The effects of rapidly oscillating periodic forces on purely mechanical systems were demonstrated many years ago for a driven pendulum [1] and were also studied for more general cases [2]. Periodically driven stochastic systems too have been studied extensively over the past two decades in the context of stochastic resonance (see, e.g., Ref. [3], and references therein). However, most of these studies were restricted to weak periodic driving with position-independent forces [4–7], often only in the overdamped regime. Further,

it has sometimes been assumed that the driving frequency is larger than all typical frequencies, including that which is associated with the noise [5]. Our treatment generalizes that of Refs. [4] and [5] in the high-frequency regime, by allowing for arbitrary damping; by including position dependence of the driving force; and by taking frequencies to be higher than those set by the confining potential, yet not necessarily higher than those set by the noise.

Some applications of our results are possible. For instance, depending on the spatial variation of the fluctuating forces, the effective potential may have more than one local minimum even if the original potential has only one. Under such circumstances, an assembly of Brownian particles would tend to segregate in two separate collections. Moreover, the fact that the effective potential depends on physical properties, such as the mass of the particles, can be exploited to promote segregation of two sets of Brownian particles that differ from each other in mass or some other physical attribute.

The layout of the paper is as follows. In the following section, we define the problem and discuss the different time scales involved and their interplay, and discuss various regimes qualitatively. In Sec. III, we develop the necessary formalism required to address rapid periodic drive and arrive at a perturbative scheme. In Sec. IV, we use this scheme to determine the asymptotic distribution. In Sec. V, we briefly discuss the effect of slow periodic driving so as to compare with that of rapid driving. Finally, in Sec. VI, we conclude with a discussion of possible directions in which our results may be generalized, possible applications (particle segregation, particle sifting), and an example (the simple pendulum) that demonstrates the significance of the new effects found.

## II. PERIODICALLY DRIVEN BROWNIAN PARTICLE

We shall consider a one-dimensional Brownian particle in a potential well, subjected to a periodic force along with a damping force and random noise. The equation of motion of the driven Brownian particle moving in a viscous environment is

$$m\ddot{x} = -\gamma\dot{x} - \frac{\partial}{\partial x}U(x) + F(x,t) + \eta(t), \quad (1)$$

where $m$ is the mass, $\gamma$ is the coefficient of viscosity, $U(x)$ is a static confining potential, $F(x,t)$ is the periodic driving force with a period $T$, $F(x,t)=F(x,t+T)$, and $\eta(t)$ is a Gaussian random noise with $\langle \eta(t)\rangle_\eta=0$ and $\langle \eta(t)\eta(t')\rangle_\eta=2\gamma\beta^{-1}\delta(t-t')$, where $\beta^{-1}$ is the temperature of the surrounding heat bath.

The probability distribution of the Brownian particle is defined as $P(x,v,t)=\langle \delta(x-x_\eta(t))\delta(v-\dot{x}_\eta(t))\rangle_\eta$, where $x_\eta(t)$ and $\dot{x}_\eta(t)$ are the position and the velocity at time $t$ for a particular history of $\{\eta(t)\}$ over a time $t$. The time evolution of $P(x,v,t)$ is described by the following Fokker-Planck (FP) equation (also referred to as the Kramers equation),

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}(vP)-\frac{\partial}{\partial v}\left[\frac{1}{m}\left\{-\gamma v-\frac{\partial}{\partial x}U(x)+F(x,t)\right\}P\right]$$

$$+\frac{\gamma}{\beta m^2}\frac{\partial^2 P}{\partial v^2}. \tag{2}$$

The driving force $F(x,t)$ that is oscillating with a frequency $\omega=2\pi/T$ is chosen to be

$$F(x,t)=f(x)\cos(\omega t)+g(x)\sin(\omega t) \tag{3}$$

in the domain $L_1<x<L_2$, where the amplitude functions $f(x)$ and $g(x)$ vanish at the boundaries, $x=L_1$ and $L_2$, and outside the domain. This choice of $F(x,t)$ is made for convenience; choosing a more general periodic function will not hinder our analysis. The generalization to higher dimensions and to many interacting particles is also straightforward. Our aim is to find the large-time distribution that we denote by $P_\infty(x,v,t)=\lim_{t\to\infty}P(x,v,t)$.

In the absence of a driving force, all solutions $P(x,v,t)$ of the FP equation, corresponding to various arbitrary initial distributions, tend to a unique distribution after a long enough time [8]. This distribution $P_\infty(x,v,t)$ for the Brownian particles takes the equilibrium canonical form $P_{eq}(x,v)=(1/Z_0)\exp\{-\beta[\frac{1}{2}mv^2+U(x)]\}$. When a periodic driving force is present, $P_\infty(x,v,t)$ approaches a periodic function of time which is unique up to a phase [9,3]. In brief, the argument for the periodicity goes as follows. When the FP operator is periodic, $\mathcal{L}(t)=\mathcal{L}(t+T)$, the solution to the FP equation $[\partial_t-\mathcal{L}(t)]P(x,v,t)=0$ can be expanded in terms of Floquet-type functions $p_\mu(x,v,t)\exp(-\mu t)$. The functions $p_\mu(x,v,t)=p_\mu(x,v,t+T)$ are periodic and are the right eigenfunctions of $\partial_t-\mathcal{L}(t)$ with eigenvalues $\mu$. It is known that for an $N$-dimensional FP equation, the real parts of these eigenvalues $\mathrm{Re}(\mu)$ are positive semidefinite and hence in the large-time limit, for typical confining potentials, only $p_0(x,v,t)$ survives. We are interested in finding this large-time distribution $P_\infty(x,v,t)$ for a given $F(x,t)$. Since no analytic solution of the FP equation is known for an arbitrary time periodic $F(x,t)$, even when it involves only the fundamental frequency, we shall restrict our attention to certain regimes of the driving frequency while solving for $P_\infty(x,v,t)$.

In the absence of the driving force there are two important time scales in the system; one, $\tau_v=m/\gamma$, is introduced by the

viscous medium and the other, $\tau_w=2\pi/\omega_0$, is related typically to the curvature at the bottom of the potential well, $\omega_0=\sqrt{U''(x_{min})/m}$. The velocity variable equilibriates in a time scale set by $\tau_v$ and hence for larger times it gets described by a stationary distribution. Thus, in a highly viscous medium, where $\tau_v\ll\tau_w$ holds, after a time $t\gg\tau_v$ the distribution can be written as $P(x,v,t)=P_{eq}(v)P(x,t)$, where $P_{eq}(v)\sim\exp[-\beta\frac{1}{2}mv^2]$ is the canonical distribution for velocity and $P(x,t)=\int dv\,P(x,v,t)$ is the marginal distribution involving only position. This marginal distribution satisfies an FP equation (often called the Smoluchowski equation) obtained by dropping the inertial term in Eq.(1), as this term becomes insignificant in comparison with the viscous term once the time exceeds $\tau_v$.

If, however, there is a driving force with a high enough frequency, then this reduction to the Smoluchowski limit does not take place. The oscillating driving force introduces one other important time scale associated with the time-period $T$ whose existence restricts the domain of validity of the Smoluchowski equation even in the high-friction limit. It is useful to demarcate different regimes of the driving frequency in this limit: (a) $\tau_v\ll\tau_w<T$ and (b) $\tau_v<T\ll\tau_w$ or $T<\tau_v\ll\tau_w$. In case (a), the velocity decouples from the position and one can still use the Smoluchowski equation to obtain the large-time distribution, while in case (b), the driving force has a time scale comparable to that of the velocity relaxation time and hence it is necessary to retain the Kramers equation.

If the time scale of measurement exceeds the time period $T$, then the relevant quantity is the large-time distribution averaged over a time period $\overline{P_\infty(x,v,t)}=(1/T)\int_0^T P_\infty(x,v,t)$. Hence we shall also determine $\overline{P_\infty(x,v,t)}$.

## III. RAPIDLY OSCILLATING FORCES: FORMALISM

In a mechanical system, in the absence of a viscous force and random noise, it is known that for $\omega\gg\omega_0$, the particle executes small amplitude oscillations of frequency $\omega$ about a smooth mean path [2]. This motion can be described by separating it into slow and fast variables; the fast variable decouples from the slow one while the slow variable sees a static effective potential, modified due to the oscillations of the fast variable. In this section, we develop the necessary formalism that accommodates an analogy with this separation into slow and fast variables and enables us to solve the problem when viscous and random forces are included.

### A. Transformation of FP equation

In this subsection, we transform the FP equation under a specific coordinate transformation which enables it to be solved perturbatively. For the perturbative treatment to be valid a sufficient condition, though not necessary, is that $\omega$ be large, while no assumption is made about the amplitude of the driving force in comparison with the static potential.

We make the coordinate transformation $\{x,v,t\}\to\{X,V,\tau\}$ under which the distribution is made to behave like a scalar function: $P(x,v,t)\to\widetilde{P}(X,V,\tau)=P(x,v,t)$. The old and new coordinates are related as follows,

$$x = X + \xi(X,\tau), \quad v = V + \frac{\partial}{\partial \tau}\xi(X,\tau), \quad t = \tau. \qquad (4)$$

The explicit form of $\xi(X,\tau)$ will be specified later. Note that the volume element $dx\,dv = dX\,dV\,J(X,V,\tau)$, where $J(X,V,\tau) = (1 + \partial\xi/\partial X)$ and hence, if $P$ is the probability density in $(x,v)$ space, then $\widetilde{P}J$ becomes the the probability density in $(X,V)$ space. Also, under the above coordinate transformation the derivatives transform as follows:

$$\frac{\partial}{\partial x} = \frac{1}{1+\xi'}\frac{\partial}{\partial X} - \frac{\dot{\xi}'}{1+\xi'}\frac{\partial}{\partial V},$$

$$\frac{\partial}{\partial v} = \frac{\partial}{\partial V},$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{\dot{\xi}}{1+\xi'}\frac{\partial}{\partial X} + \left(\frac{\dot{\xi}\dot{\xi}'}{1+\xi'} - \ddot{\xi}\right)\frac{\partial}{\partial V}, \qquad (5)$$

where the *dots* and *primes* on $\xi$ stand for derivatives with respect to $\tau$ and $X$, respectively. Making use of Eqs. (4) and (5) in Eq. (2), we get

$$\frac{\partial \widetilde{P}}{\partial \tau} = -\frac{\partial}{\partial X}(V\widetilde{P}) - \frac{\partial}{\partial V}\bigg[\frac{1}{m}\{-\gamma V + f_U(X+\xi)$$

$$+ F(X+\xi,\tau) - F(X,\tau)\}\widetilde{P}\bigg] + \frac{\gamma}{\beta m^2}\frac{\partial^2 \widetilde{P}}{\partial V^2}$$

$$+ V\left[\frac{\xi'}{1+\xi'}\frac{\partial \widetilde{P}}{\partial X} + \frac{\dot{\xi}'}{1+\xi'}\frac{\partial \widetilde{P}}{\partial V}\right]$$

$$+ \frac{1}{m}[m\ddot{\xi} + \gamma\dot{\xi} - F(X,\tau)]\frac{\partial \widetilde{P}}{\partial V}, \qquad (6)$$

where $f_U(X) = -\partial U(X)/\partial X$.

Now choose $\xi(X,\tau)$ such that it is a solution of the following equation,

$$m\ddot{\xi} = -\gamma\dot{\xi} + F(X,\tau). \qquad (7)$$

For $F(X,\tau)$ as given in Eq. (3), the solution to the above equation is

$$\xi(X,\tau) = \frac{-1}{m\left(\omega^2 + \dfrac{\gamma^2}{m^2}\right)}\left[\left(f(X) + \frac{\gamma}{m\omega}g(X)\right)\cos(\omega\tau)\right.$$

$$\left. + \left(g(X) - \frac{\gamma}{m\omega}f(X)\right)\sin(\omega\tau)\right]. \qquad (8)$$

Since $\xi$ is small for large values of $(\omega^2 + \gamma^2/m^2)$, we may expand Eq. (6) perturbatively in $\xi$. The reason for choosing the above dynamics [Eq. (7)] for $\xi$ is that the last term in Eq. (6) becomes zero and further the explicitly (time)

$\tau$-dependent term becomes small if $\xi$ is small, thus making the equation amenable to perturbative analysis.

Note that when the driving force is $x$ independent, Eq. (6) reduces to the usual FP equation with the static force $f_U(X)$ being replaced by $f_U(X+\xi)$.

### B. The perturbative scheme

We now find the large-time solution of Eq. (6) perturbatively in powers of $\xi$. Upon substituting $\xi$ as given in Eq. (8), we see that Eq. (6) takes the form

$$\frac{\partial}{\partial \tau}\widetilde{P}(X,V,\tau) = [\mathcal{L} + \Delta\mathcal{L}]\widetilde{P}(X,V,\tau)$$

$$\equiv \sum_{n=0}^{\infty}[\mathcal{L}^{(n)} + \Delta\mathcal{L}^{(n)}]\widetilde{P}(X,V,\tau), \qquad (9)$$

where $\mathcal{L}$ and $\Delta\mathcal{L}$ are static and time-dependent operators, respectively. The superscript on the operators $\mathcal{L}^{(n)}$ and $\Delta\mathcal{L}^{(n)}$ indicates that they are of $O(\xi^n)$; the explicit forms of the first few operators are

$$\mathcal{L}^{(0)} = -V\frac{\partial}{\partial X} + \frac{1}{m}\frac{\partial}{\partial V}[\gamma V + U'(X)] + \frac{\gamma}{\beta m^2}\frac{\partial^2}{\partial V^2},$$

$$\mathcal{L}^{(1)} = -\frac{1}{m}\overline{\xi F'(X,\tau)}\frac{\partial}{\partial V},$$

$$\mathcal{L}^{(2)} = \frac{1}{2m}\overline{\xi^2}U'''(X)\frac{\partial}{\partial V} - V\overline{\xi'^2}\frac{\partial}{\partial X},$$

$$\Delta\mathcal{L}^{(0)} = 0,$$

$$\Delta\mathcal{L}^{(1)} = \frac{1}{m}[\xi U''(X) - \xi F'(X,\tau) + \overline{\xi F'(X,\tau)}]\frac{\partial}{\partial V}$$

$$+ V\xi'\frac{\partial}{\partial X} + V\dot{\xi}'\frac{\partial}{\partial V},$$

$$\Delta\mathcal{L}^{(2)} = \frac{1}{2m}[(\xi^2 - \overline{\xi^2})U'''(X) - \xi^2 F''(X,\tau)]\frac{\partial}{\partial V}$$

$$- V(\xi'^2 - \overline{\xi'^2})\frac{\partial}{\partial X} - V\dot{\xi}'\xi'\frac{\partial}{\partial V}, \qquad (10)$$

where the *bar* over the terms indicates an average over a time period. The perturbative asymptotic solution of Eq. (9) can be formally written as follows:

$$\widetilde{P}_\infty(X,V,\tau) = Q_\infty(X,V,\tau) + \frac{1}{\partial_\tau - \mathcal{L}}\Delta\mathcal{L}\widetilde{P}_\infty(X,V,\tau), \qquad (11)$$

where $Q_\infty(X,V,\tau)$ is the right eigenfunction of $\mathcal{L}$ with eigenvalue zero. Since the asymptotic distribution is periodic and the nonzero eigenvalues of $\mathcal{L}$ have a nonvanishing real part, it follows from Eq. (9) that $\Delta\mathcal{L}\widetilde{P}_\infty$ has no overlap with the eigenfunction of $\partial_\tau - \mathcal{L}$ with zero eigenvalue. Hence the op-

eration of the inverse of $\partial_\tau - \mathcal{L}$ is well defined as it acts only on the space of functions orthogonal to that eigenfunction.

It is not possible to obtain an explicit form for the inverse of $\partial_\tau - \mathcal{L}$ for arbitrary $U(X)$ and $F(X,\tau)$, in which case numerical or variational methods might be adopted to determine the eigenfunctions and eigenvalues of $\mathcal{L}$ or $\mathcal{L}_0$ and to obtain $(\partial_\tau - \mathcal{L})^{-1}$. If for some form of the potential $U(X)$, we are able to find all the eigenfunctions of, say, $\mathcal{L}_0$ then this inverse can be conveniently expanded to any order in $\xi$ as follows

$$\frac{1}{\partial_\tau - \mathcal{L}} = \sum_{n=0}^{\infty} \left[ \frac{1}{\partial_\tau - \mathcal{L}_0} (\mathcal{L} - \mathcal{L}_0) \right]^n \frac{1}{\partial_\tau - \mathcal{L}_0}. \quad (12)$$

But a more suitable series expansion can be given for this operator when either $\omega$ or $\gamma/m$ is large. If $\omega$ is also much larger than $\gamma/m$, then this suitable expansion is

$$\frac{1}{\partial_\tau - \mathcal{L}} = \sum_{n=0}^{\infty} \left( \frac{1}{\partial_\tau} \mathcal{L} \right)^n \frac{1}{\partial_\tau}. \quad (13)$$

In the case when $\gamma/m$ is comparable to $\omega$, we can write the inverse in terms of an operator $\mathcal{L}_V$ containing terms of $O(\gamma/m)$ only and which is given as

$$\mathcal{L}_V \equiv \frac{\gamma}{m} \frac{\partial}{\partial V} V + \frac{\gamma}{\beta m^2} \frac{\partial^2}{\partial V^2} = e^{-\beta m V^2/4} \left( -\frac{\gamma}{m} a^\dagger a \right) e^{\beta m V^2/4}, \quad (14)$$

where the operators $a$ and $a^\dagger$ follow the commutation relation $[a, a^\dagger] = 1$ and are defined as

$$a = \frac{1}{\sqrt{\beta m}} \frac{\partial}{\partial V} + \frac{1}{2} \sqrt{\beta m} V, \quad a^\dagger = -\frac{1}{\sqrt{\beta m}} \frac{\partial}{\partial V} + \frac{1}{2} \sqrt{\beta m} V. \quad (15)$$

So in this case the convenient expansion of the inverse operator is

$$\frac{1}{\partial_\tau - \mathcal{L}} = \sum_{n=0}^{\infty} \left[ \frac{1}{\partial_\tau - \mathcal{L}_V} (\mathcal{L} - \mathcal{L}_V) \right]^n \frac{1}{\partial_\tau - \mathcal{L}_V}$$

$$= e^{-\beta m V^2/4} \left[ \sum_{n=0}^{\infty} \left( \frac{1}{\partial_\tau + \frac{\gamma}{m} a^\dagger a} (\mathcal{L} - \mathcal{L}_V) \right)^n \right.$$

$$\left. \times \frac{1}{\partial_\tau + \frac{\gamma}{m} a^\dagger a} \right] e^{\beta m V^2/4}. \quad (16)$$

The idea of writing this operator in terms of $a$ and $a^\dagger$ is that its action on $h(V) \exp(-\beta m V^2/2)$, where $h(V)$ is some polynomial of $V$, can be determined more easily since it involves the action of a specific function of $a^\dagger a$ on a series made of eigenfunctions of $a^\dagger a$.

The calculational scheme is thus reduced to the following: Find the right eigenfunction $Q_\infty$ of $\mathcal{L}$ with eigenvalue zero to

the desired order in $\xi$. Then use one of the truncated series (12), (13), or (16), such that the truncation is consistent with the chosen order, keeping in mind that $\xi$ depends on $\omega$. Next substitute this truncated inverse operator in Eq. (11) and from it extract $\tilde{P}_\infty(X,V,\tau)$ order by order.

## IV. ASYMPTOTIC DISTRIBUTION: A PERTURBATIVE ANALYSIS

### A. First-order perturbation

We now find the asymptotic distribution $P_\infty(x,v,t)$, to first order in $\xi$, with the condition that it vanishes at the boundary of $(x,v)$ space. We will also show that the time average of this distribution $\overline{P_\infty(x,v,t)}$ is the canonical equilibrium distribution with $U(x)$ replaced by $U_{eff}(x) = U(x) + U^{(1)}(x)$, where $U^{(1)}(x)$ will be explicitly evaluated.

To this end we need to determine $Q_\infty(X,V,\tau) = \Sigma_n Q_\infty^{(n)}(X,V,\tau)$ to the same order from the equation $\mathcal{L} Q_\infty(X,V,\tau) = 0$. As before, the superscript on $Q_\infty^n$ indicates the corresponding order in $\xi$. The zeroth order $Q_\infty^{(0)}$ is the solution to the equation $\mathcal{L}^{(0)} Q_\infty^{(0)} = 0$, which yields the equilibrium distribution

$$\tilde{P}_\infty^{(0)}(X,V,\tau) = Q_\infty^{(0)}(X,V,\tau) = \frac{1}{Z_0} \exp\{-\beta[\tfrac{1}{2} m V^2 + U(X)]\}. \quad (17)$$

The first order $\tilde{P}_\infty^{(1)}$ is obtained from Eqs. (11) and (16),

$$\tilde{P}_\infty^{(1)} = Q_\infty^{(1)} + e^{-\beta m V^2/4} \frac{1}{\partial_\tau + \frac{\gamma}{m} a^\dagger a} e^{\beta m V^2/4} \Delta \mathcal{L}^{(1)} \tilde{P}_\infty^{(0)}, \quad (18)$$

where $Q_\infty^{(1)}$ is the solution to the equation $\mathcal{L}^{(0)} Q_\infty^{(1)} + \mathcal{L}^{(1)} Q_\infty^{(0)} = 0$. This solution is straightforward to determine since $\mathcal{L}^{(1)} Q_\infty^{(0)}$ has the same form as the right-hand side of the identity: $\mathcal{L}^{(0)}[h(X) Q_\infty^{(0)}] = -V h'(X) Q_\infty^{(0)}$ for any arbitrary function $h(X)$. We get

$$Q_\infty^{(1)}(X,V,\tau) = -\beta U^{(1)}(X) \tilde{P}_\infty^{(0)},$$

$$\frac{\partial}{\partial X} U^{(1)}(X) = -\overline{\xi(X,\tau) F'(X,\tau)}. \quad (19)$$

In Eq. (18) we have used inverse operator (16) truncated right after the first term which amounts to neglecting $O(\xi/\omega)$ terms. To this approximation, it is then sufficient to keep only the term proportional to $\dot{\xi}$ in $\Delta \mathcal{L}^{(1)}$, which is rewritten as follows:

$$\Delta \mathcal{L}^{(1)} = e^{-\beta m V^2/4} \left[ -\sqrt{\frac{\beta}{m}} (\xi U'' - \xi F' + \overline{\xi F'}) a^\dagger \right.$$

$$\left. + \xi' \frac{\partial}{\partial X} \frac{a + a^\dagger}{\beta m} - \dot{\xi}' (a + a^\dagger) a^\dagger \right] e^{\beta m V^2/4}. \quad (20)$$

Hence Eq. (18) upon neglecting terms of $O(\xi/\omega)$ reduces to

$$\tilde{P}_\infty^{(1)}(X,V,\tau) = -\beta U^{(1)}(X)\tilde{P}_\infty^{(0)} - e^{-\beta m V^2/4}\frac{1}{\partial_\tau + \dfrac{\gamma}{m}a^\dagger a}$$

$$\times[\dot{\xi}(a+a^\dagger)a^\dagger]e^{-\beta m V^2/4}\frac{1}{Z}e^{-\beta U(X)}$$

$$= -\beta[U^{(1)}(X)+K^{(1)}(V;\xi)]\tilde{P}_\infty^{(0)}(X,V,\tau),\tag{21}$$

where

$$K^{(1)}(V;\xi) = \frac{1}{\omega^2+4\dfrac{\gamma^2}{m^2}}\left[\frac{2\gamma}{\beta m}\left(\frac{2\gamma}{m}\xi' - \dot{\xi}'\right)\right.$$

$$\left.+\left(\omega^2\xi' + \frac{2\gamma}{m}\dot{\xi}'\right)mV^2\right],\tag{22}$$

which simplifies to $K^{(1)}(V;\xi)\approx\xi' m V^2$ when $\omega\gg\gamma/m$.

We now get the asymptotic distribution from Eqs. (4), (17), and (21),

$$P_\infty(x,v,t) = \frac{1}{Z}\exp\{-\beta[\tfrac{1}{2}mv^2 + U(x) + U^{(1)}(x)$$

$$+K^{(1)}(v;\xi) - \xi U'(x) - \dot{\xi}mv]\}.\tag{23}$$

Unlike in the distributions for static potentials, here the velocity $v$ gets coupled to the position $x$ through $\xi$. The averaged large-time distribution is given as

$$\overline{P_\infty(x,v,t)} = \frac{1}{Z}\exp\{-\beta[\tfrac{1}{2}mv^2 + U(x) + U^{(1)}(x)]\},\tag{24}$$

where the explicit expression for $U^{(1)}(x)$ is obtained upon substituting $\xi$ from Eq. (8) in Eq. (19) and then integrating,

$$U^{(1)}(x) = \frac{1}{4m\left(\omega^2+\dfrac{\gamma^2}{m^2}\right)}\left[(f^2(x)+g^2(x))\right.$$

$$\left.+\frac{2\gamma}{m\omega}\int^x dy\{g(y)f'(y)-f(y)g'(y)\}\right].\tag{25}$$

Thus the time-averaged large-time behavior of the Brownian particles is described by the canonical distribution at a temperature $\beta^{-1}$ with an effective potential $U_{eff} = U + U^{(1)}$ that depends on the frequency and space dependence of the driving force in addition to the properties of the particle. Note that a nontrivial contribution to the effective potential arises to this order only if $f$ or $g$ are space dependent, and that it can be tuned by varying $f$, $g$, or $\omega$.

The additional term $U^{(1)}$ is the average energy associated with the rapid motion. This can be seen easily upon substituting for $F$ in terms of $\xi$ using Eq. (7) and rewriting $U^{(1)}$ as follows:

$$U^{(1)}(x) = -\int^x dy\,\overline{[\xi(y,t)F'(y,t)]}$$

$$= \frac{1}{2}m\overline{[\dot{\xi}(x,t)]^2} - \int^x dy\,\overline{\left(-\gamma\dot{\xi}(y,t)\frac{\partial}{\partial y}\xi(y,t)\right)}.\tag{26}$$

Thus $U^{(1)}$ is sum of the average kinetic energy and the work done against the damping on the fast variable $\xi$.

A first-order perturbation treatment is justified, provided $\mathcal{L}^{(2)}$ and $\Delta\mathcal{L}^{(2)}$ are negligible when compared to $\mathcal{L}^{(1)}$ and $\Delta\mathcal{L}^{(1)}$, respectively. The criterion for this is $|\xi\partial f/\partial x|\gg|U'''\xi^2|$. For instance, suppose that the length scales over which $U(x)$ and $F(x,t)$ vary are comparable and $\alpha\omega_0$ is a typical frequency associated with anharmonic terms. Then the above criterion reduces to $(\omega^2+\gamma^2/m^2)\gg\alpha^2\omega_0^2$, which is consistent with $\xi$ being small.

It should be remarked that the restriction of periodicity of $F(x,t)$ to the fundamental frequency $\omega$ is not essential; one can include the higher harmonics as well. Also by considering generalizations to higher dimensions or to many interacting particles one does not encounter any additional computational difficulties in the evaluation of the distribution. The only additional assumption needed to write down $U_{eff}$ is that $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ are curl free.

### B. Second-order perturbation

We now calculate the second-order corrections to the asymptotic distribution. Further, in this subsection, we restrict our treatment to frequencies that satisfy the condition $\omega\gg\gamma/m$, in which case the first-order term $\tilde{P}_\infty^{(1)}$ is as given in Eq. (21) with $K^{(1)}(V;\xi)=\xi' m V^2$. The second-order term of Eq. (11) is

$$\tilde{P}_\infty^{(2)} = Q_\infty^{(2)} + \frac{1}{\partial_\tau - \mathcal{L}}[\Delta\mathcal{L}^{(1)}\tilde{P}_\infty^{(0)} + \Delta\mathcal{L}^{(1)}\tilde{P}_\infty^{(1)} + \Delta\mathcal{L}^{(2)}\tilde{P}_\infty^{(0)}]$$

$$= Q_\infty^{(2)} + \left\|\frac{1}{\partial_\tau - \mathcal{L}^{(0)}}\Delta\mathcal{L}^{(1)}\tilde{P}_\infty^{(0)}\right\| + \frac{1}{\partial_\tau}\Delta\mathcal{L}^{(1)}\tilde{P}_\infty^{(1)}$$

$$+\frac{1}{\partial_\tau}\Delta\mathcal{L}^{(2)}\tilde{P}_\infty^{(0)} + O\left(\frac{1}{\omega}\xi^2\right).\tag{27}$$

Though the second term in the above expression is of $O(\xi)$, it has been included here because when calculated earlier, to first order in $\xi$, the terms of $O(\xi/\omega)$ were neglected. This term is written within *pipes* to indicate that only terms of $O(\xi/\omega)$ and $O(\xi/\omega^2)$ have to be retained and not the $O(\xi/\omega^0)$ term that has already been included in $\tilde{P}_\infty^{(1)}$.

We now evaluate the terms on the right-hand side of Eq. (27). The first term $Q_\infty^{(2)}$ is the solution to the equation $\mathcal{L}^{(0)}Q_\infty^{(2)} + \mathcal{L}^{(1)}Q_\infty^{(1)} + \mathcal{L}^{(2)}Q_\infty^{(0)} = 0$. This, as in the case of $Q_\infty^{(1)}$, is straightforward to determine and we get

$$Q_\infty^{(2)}(X,V,\tau) = -\beta \mathcal{U}_1(X)\widetilde{P}_\infty^{(0)} + \frac{\beta^2}{2}(U^{(1)}(X))^2\widetilde{P}_\infty^{(0)},$$

$$\frac{\partial}{\partial X}\mathcal{U}_1(X) = -\overline{(\xi')^2}U'(X) + \frac{1}{2}\overline{(\xi)^2}U'''(X). \qquad (28)$$

The second term is

$$\left\|\frac{1}{\partial_\tau - \mathcal{L}^{(0)}}\Delta\mathcal{L}^{(1)}\widetilde{P}_\infty^{(0)}\right\|$$

$$= \left\|\left(\frac{1}{\partial_\tau} + \frac{1}{\partial_\tau^2}\mathcal{L}^{(0)} + \frac{1}{\partial_\tau^3}(\mathcal{L}^{(0)})^2 + \cdots\right)\Delta\mathcal{L}^{(1)}\widetilde{P}_\infty^{(0)}\right\|$$

$$= -\beta\mathcal{K}(X,V;\xi)\widetilde{P}_\infty^{(0)}, \qquad (29)$$

where

$$\mathcal{K}(X,V;\xi)\widetilde{P}_\infty^{(0)} = -\frac{\partial_\tau}{\omega^2}H(X,V;\xi) - \frac{1}{\omega^2}\mathcal{L}^{(0)}H(X,V;\xi),$$

$$H(X,V;\xi) = (\xi U'' + \xi'U' - \tfrac{1}{4}\xi F' + \tfrac{1}{4}\overline{\xi F'})V\widetilde{P}_\infty^{(0)}$$

$$+ \mathcal{L}^{(0)}(mV^2\xi'\widetilde{P}_\infty^{(0)}). \qquad (30)$$

The third and the fourth terms are

$$\frac{1}{\partial_\tau}\Delta\mathcal{L}^{(1)}\widetilde{P}_\infty^{(1)} = -\beta mV^2[(\xi')^2 - \overline{(\xi')^2}]\widetilde{P}_\infty^{(0)} + \beta^2[U^{(1)}mV^2\xi'$$

$$+ \tfrac{1}{2}(mV^2\xi')^2]\widetilde{P}_\infty^{(0)},$$

$$\frac{1}{\partial_\tau}\Delta\mathcal{L}^{(2)}\widetilde{P}_\infty^{(0)} = \frac{1}{2}\beta mV^2[(\xi')^2 - \overline{(\xi')^2}]\widetilde{P}_\infty^{(0)} - \beta[\mathcal{U}_2(X)$$

$$+ C(X,V)]\widetilde{P}_\infty^{(0)}, \qquad (31)$$

where $(\mathcal{U}_2)' = \overline{\xi\xi'}U'' - 3\overline{\xi'\xi'}U'$ and $C(X,V)$ is an arbitrary $\tau$-independent function. These arbitrary and $\tau$-independent terms are included since the action of $\partial_\tau^{-1}$ allows this ambiguity. This ambiguity is removed from the condition obtained by substituting $\widetilde{P}_\infty$ in Eq. (9) and averaging over a period. Thus $\widetilde{P}_\infty^{(2)}$ should satisfy

$$\mathcal{L}^{(0)}\overline{\widetilde{P}_\infty^{(2)}} + \overline{\Delta\mathcal{L}^{(1)}\widetilde{P}_\infty^{(1)}} = 0, \qquad (32)$$

which implies, after some calculation, that $C$ is the solution of the equation

$$\mathcal{L}^{(0)}[C\widetilde{P}_\infty^{(0)}] + 4m\overline{\xi'\xi''}V^3\widetilde{P}_\infty^{(0)} = 0. \qquad (33)$$

In the high-viscous limit, $\gamma/m \gg \omega_0$, this function $C(X,V)$ $\approx (4m/3\gamma)\overline{\xi'\xi''}(mV^3 + 6V/\beta)$. Using Equations (27)-(31) and Eq. (21) we get the asymptotic distribution to second order

$$P_\infty(x,v,t) = \frac{1}{Z}\exp\{-\beta[\tfrac{1}{2}mV^2 + U(X) + U^{(1)}(X) + U^{(2)}(X)$$

$$+ R(X,V,t)]\}, \qquad (34)$$

where $\overline{R(X,V,t)} = K^{(1)}(V;\xi) + \mathcal{K}(X,V;\xi)$ $+ \tfrac{1}{2}mV^2\{[\xi'(X,t)]^2 - \overline{(\xi'(X,t))^2}\} + C(X,V);$ $U^{(2)}(X)$ $= \mathcal{U}_1(X) + \mathcal{U}_2(X);$ and, to second order, $X = x - \xi(x,t)$ $- \xi(x,t)\xi'(x,t)$ and $V = v - \dot{\xi}(x,t) - \xi(x,t)\dot{\xi}'(x,t)$. It is clear from the above equation that $\overline{P_\infty(x,v,t)}$ will contain terms with both $v$ and $x$ dependence in addition to purely $x$-dependent and $v$-dependent terms.

The explicit form of $U^{(2)}(X) = U^{(2)}(x)$ which contributes at second order to the effective static potential is

$$U^{(2)}(x) = \frac{1}{4m^2\omega^2\left(\omega^2 + \dfrac{\gamma^2}{m^2}\right)}\left[[f(x)^2 + g(x)^2]U''(x)\right.$$

$$\left. - 8\int^x dy\{[f'(y)]^2 + [g'(y)]^2\}U'(y)\right]. \qquad (35)$$

We notice here that a nontrivial contribution arises even for $x$-independent driving, provided $U(x)$ is anharmonic. Also note that this second-order correction to the effective potential depends on $U(x)$ while this is not the case at the first order.

We recover the results of Devoret *et al.* [4] and Jung [5] when $f(x)$ and $g(x)$ are independent of the position coordinate $x$. In the former reference the inertial term was considered while in the latter it was not, but their analysis of the high-frequency limit is tantamount to assuming that the driving frequency is larger than all typical frequencies of the system including that of noise. This assumption restricts the validity of the answer, and clearly does not hold for white noise. In fact, this assumption would lead to the absence of the term $\int^x[(f')^2 + (g')^2]U'$ in the effective potential for $x$-dependent driving. That the error, when this assumption is made, shows up at $O(\xi^2)$ is also evident from the presence of the term $V\overline{\xi'^2}\partial/\partial X$ in $\mathcal{L}^{(2)}$ operator.

We reiterate that there is a nontrivial contribution to the effective potential from the position-dependent driving forces. First, it shows up at first order itself and, second, it shows up at second order even for harmonic potentials; both of which are absent for position-independent driving forces.

## V. SLOWLY OSCILLATING FORCES

In this section, we will consider the opposite extreme, namely, $\omega$ small compared to both $\gamma/m$ and $\omega_0$. The aim is to compare with the asymptotic behavior we found in the previous section under rapid driving. Under slow driving, the Brownian particle sees an unchanging potential within its relaxation time and so the Boltzmann-like distribution corresponding to the instantaneous potential is a good zeroth-order starting point for perturbation.

The large-time distribution $P_\infty(x,v,t)$ will acquire the same periodicity as that of the driving force. Hence the left-

hand side of Eq. (2) is of $O(\omega)$ while the right-hand side has two terms, one of $O(\gamma/m)$ and the other of $O(\omega_0)$. Thus $P_\infty(x,v,t)$ can be evaluated perturbatively in $(m\omega/\gamma)$ and $(\omega/\omega_0)$. To the leading (zeroth) order, this distribution is

$$P_\infty^{(0)}(x,v,t) = \frac{1}{Z(t)}\exp\{-\beta[\tfrac{1}{2}mv^2 + U(x) + U_f(x)\cos(\omega t)$$
$$+ U_g(x)\sin(\omega t)]\}, \qquad (36)$$

where $U_f(x) = -\int^x dy f(y)$ and $U_g(x) = -\int^x dy g(y)$; $Z(t)$ is the instantaneous normalization constant determined by the normalization condition: $\int dx P_\infty(x,v,t) = 1$ at any time $t$, and evidently satisfies the periodicity condition $Z(t+T) = Z(t)$.

We now estimate the averaged large-time distribution $\overline{P_\infty(x,v,t)}$ at low and high temperatures separately. To determine the average distribution at high temperature, it is convenient to rewrite this equation as follows.

$$P_\infty(x,v,t) = \frac{1}{Z(t)}e^{-\beta[mv^2/2 + U(x)]}\mathcal{I}_0(\beta V(x))$$
$$\times\left[1 + \sum_{n\neq 0} e^{in\omega t}\frac{\mathcal{I}_n(\beta V(x))}{\mathcal{I}_0(\beta V(x))}\right], \qquad (37)$$

where $\mathcal{I}_n(\alpha)$ is the modified Bessel function and $V(x) = \sqrt{[U_f(x)]^2 + [U_g(x)]^2}$. The ratio $\mathcal{I}_n(\alpha)/\mathcal{I}_0(\alpha)$ lies between 0 and 1 for $0 \leq \alpha < \infty$, and decreases very rapidly for small $\alpha$; $\mathcal{I}_n(\alpha)/\mathcal{I}_0(\alpha) \sim \alpha^n/2^n n!$. Thus it suffices to keep a few terms, and to leading order we obtain $\overline{P_\infty(x,v,t)} = (1/Z)\exp\{-\beta[\tfrac{1}{2}mv^2 + U_{eff}(x)]\}$, where

$$U_{eff}(x) = U(x) - \frac{1}{\beta}\ln\{\mathcal{I}_0(\beta\sqrt{[U_f(x)]^2 + [U_g(x)]^2})\}. \qquad (38)$$

Thus the Brownian particles, when observed over a time of $O(T)$, get described by the canonical distribution at a temperature $\beta^{-1}$ with an effective potential that depends on the temperature.

At low temperature the saddle-point approximation can be used to evaluate $\overline{P_\infty(x,v,t)}$. To the leading order, this will be $\overline{P_\infty(x,v,t)} = P_{eq}(v)[1/N(t)]\sum_{i=1}^{N(t)}\delta(x - x_{min}^{(i)}(t))$, where $\{x_{min}^{(i)}(t)\}$ are the minima at time $t$ of the function $U(x) + U_f(x)\cos(\omega t) + U_g(x)\sin(\omega t)$ and $N(t)$ is the number of minima at that time.

It might appear that $U_f(x)$ and $U_g(x)$, obtained by integrating $f(x)$ and $g(x)$, respectively, have different constants of integration in regions where the driving forces are present from those where they are absent. However, they get fixed by the condition that $P(x,v,t)$ and $\partial P(x,v,t)/\partial x$ are single-valued functions of $x$.

## VI. DISCUSSION

In this section, we review the results of this paper for a general position-dependent periodic driving force with a focus on the high-frequency regime. We then discuss effective potentials for specific forms of the driving functions, and use these results to point out some possible applications. The nature of these effective potentials can be drastically different from that of the original potentials, and we illustrate this fact using the example of a simple pendulum.

The principal results of this paper concern the form of the asymptotic distribution of a Brownian particle subjected to a driving force, which is periodic in time but is an arbitrary function of position. In the limit of high-frequency driving, the particle makes small, rapid excursions around a smooth path along which the motion is relatively slow. This affords the possibility of a systematic perturbative treatment in powers of the excursion amplitude. The result for the distribution averaged over a cycle is described in terms of an effective potential whose form we derived. Interestingly, the leading contribution $U^{(1)}$, which is second order in the amplitude of the applied driving force, is present only if the driving force is position dependent. This $U^{(1)}$ can be interpreted as the average energy of the excursion variable. In the next-order contribution to the effective potential as well, this position dependence of the driving force has an interesting effect even for purely harmonic confining potentials.

Central to our discussion of rapid periodic driving is the separation into slow and fast variables. It should be noted that this demarcation of slow and fast is based on whether or not the variable varies considerably over a time period. This is different from the distinction made in discussions of fast and slow variables that are so categorized according to whether they relax in small or large times. In this latter case, many methods have been developed to eliminate the fast variables and obtain an effective dynamics for the relaxation of the slow variables [8,10].

The effective potential that we find can be qualitatively different in the low- and high-frequency regimes. When $\omega$ is small, the leading additional potential felt by the particles, $\Delta U(x) \equiv U_{eff}(x) - U(x)$, is always nonpositive for any choice of $f(x)$ and $g(x)$. On the other hand for large $\omega$, $\Delta U(x)$ can be either positive or negative in general, though if one of $f(x)$ or $g(x)$ is identically zero then it is always non-negative. Also to the leading order when $\omega$ is small, $\Delta U(x)$ depends on temperature but not on $\omega$, whereas in large $\omega$ limit it only depends on $\omega$ and not on temperature.

Certain choices of the driving force can lead to interesting outcomes. Suppose we choose $U(x)$ to be a confining potential which is a monotonically increasing function of $|x|$ while $f(x)$ is a monotonically decreasing function of $|x|$ which vanishes at $|x| = L$, and $g(x)$ vanishes identically: for example, $U(x) = \tfrac{1}{2}m\omega_0^2 x^2$ and $f(x) = f_0\sin(\pi x/L)$ for $-L \leq x \leq L$ and zero otherwise. When $\omega$ is small the effective potential continues to have a single minimum at the origin. But when $\omega$ is large, it develops two additional minima; one close to $L$, between 0 and $L$ and the other close to $-L$, between $-L$ and 0. Thus, a system of Brownian particles would cluster in a region near the origin for low driving frequency, while at high frequency these particles would segregate into two clusters which are separated by a distance of $O(L)$.

The parameters specifying the particles also enter the effective potential. Hence the minima of the effective potential

as seen by different species of particles are different. One can make use of this fact, for instance, to separate different species of Brownian particles in a situation where they are initially mixed, by driving them with a space-dependent periodic force.

To get an idea of the magnitude of the qualitative change the effective potential introduces, it is instructive to examine an example. Consider a rigid pendulum: a massless rod of length $l$ with a bob of mass $m$ attached to it at the end and oscillating in a gravitational field $g$. The potential has extrema at $\theta=0$ and $\theta=\pi$ which are, respectively, stable and unstable points, where $\theta$ is the angle the rod makes with the negative $y$ axis (along the direction of gravity). Now oscillate the point of suspension along the $y$ axis with a frequency $\omega$ and amplitude $a$. The angle $\theta$ then evolves according to the equation $l\ddot{\theta}+g\sin\theta=a\omega^2\sin\theta\cos(\omega t)$. The effective potential $V_{eff}(\theta)$ has extrema at $\theta=0,\pi$ and $\theta_\pm$, where $\theta_\pm=\pi\pm\cos^{-1}(\lambda)$ with $\lambda=2gl/a^2\omega^2$. The stability of these points is as follows: $\theta=0$ is stable; $\theta=\theta_\pm$ exist only if $\lambda<1$ and when they exist they are unstable; $\theta=\pi$ is stable when $\theta_\pm$ exist and is unstable otherwise. In a nutshell, when $\lambda\geqslant1$ the pendulum shows no qualitative change in its behavior upon oscillating the point of suspension whereas when $\lambda<1$ then $\theta=\pi$ also becomes a stable point and hence the pendulum can make small oscillations about this point too. This dramatic change in the behavior of the pendulum was experimentally demonstrated by Kapitza [1].

Interestingly, if we oscillate the point of suspension along the $x$ axis instead of the $y$ axis, the pendulum will exhibit a different behavior. In this case, the effective potential has extrema at $\theta=0$, $\pi$, and $\pm\cos^{-1}\lambda$ if $\lambda<1$; the points $\pm\cos^{-1}\lambda$ are stable when they exist; $\theta=0$ is unstable if $\lambda<1$ and stable otherwise; $\theta=\pi$ is always unstable. In other words, the pendulum now oscillates about points that do not lie on the $y$ axis. The nature of the extrema of the effective potential does not change in the presence of the viscous term and noise except that the value of $\lambda$ will now be different [with $\omega^2$ replaced by $\omega^4/(\omega^2+(\gamma/m)^2)$].

To conclude, we have developed a perturbative calculational scheme to study the asymptotic behavior of Brownian particles under the influence of rapidly oscillating forces. When these forces are position dependent, nontrivial effects are seen in the large-time behavior. The formalism developed here can be generalized straighforwardly to interacting Brownian particles in any dimensions. It can also be extended to study the behavior of fluctuating fields when subjected to periodic driving.

[1] P.L. Kapitza, Zh. Eksp. Teor. Fiz. **21**, 588 (1951); Usp. Fiz. Nauk **44**, 7 (1951); both papers are reprinted in *Collected Papers of P. L. Kapitza*, edited by D. ter Haar (Pergamon Press, New York, 1965), Vol. 2.

[2] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Butterworth-Heinemann, Oxford, 1976).

[3] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).

[4] M.H. Devoret, D. Esteve, J.M. Martinis, A. Cleland, and J. Clarke, Phys. Rev. B **36**, 58 (1987).

[5] P. Jung, Z. Phys. B: Condens. Matter **76**, 521 (1989).

[6] C. Presilla, F. Marchesoni, and L. Gammaitoni, Phys. Rev. A **40**, 2105 (1989).

[7] Hu Gang, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).

[8] H. Risken, *The Fokker-Planck Equation*, Springer Series in Synergetics Vol. 18 (Springer-Verlag, Berlin, 1984); N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, 2nd ed. (North-Holland, Elsevier, 1992).

[9] P. Jung, Phys. Rep. **234**, 175 (1993).

[10] F. Marchesoni and P. Grigolini, Physica A **121**, 269 (1983); Z. Phys. B: Condens. Matter **55**, 257 (1984); Adv. Chem. Phys. **62**, 29 (1985); T. Fonseca and P. Grigolini, Phys. Rev. A **33**, 1122 (1986); A.N. Drozdov and P. Talkner, Phys. Rev. E **54**, 6160 (1996).