Classical acoustic waves in damped media

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A Green function technique is employed to investigate the propagation of classical damped acoustic waves in complex media. The calculations are based on the linear response function approach, which is very convenient to deal with this kind of problem. Both the displacement and the gradient displacement Green functions are determined. All deformations in the media are supposed to be negligible, so the motions considered here are purely acoustic waves. The damping term γ is included in a phenomenological way into the wave vector expression. By using the fluctuation-dissipation theorem, the power spectrum of the acoustic waves is also derived and has interesting properties, the most important of them being a possible relation with the analysis of seismic reflection data.

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Wave propagation in complex media is a broad and interdisciplinary field of research, with many open questions that are scientifically sound and technologically important [1,2]. Acoustic propagation in the interfaces of structures that model the earth, is a good example of the interdisciplinary character of this general subject. It shares a number of common properties with other important topics in physics, such as electron transport in mesoscopic systems [3] and localization of photons [4,5] as well as phonons [6] in random media; the most important property being that they are all governed by wave equations. Besides, the analogies between the classical and quantum problems may lead to cross fertilization.

Another important motivation to study these structures comes from recognizing that the localization of electronic states, one of the most active fields in condensed matter physics, is essentially due to the wave nature of the electronic states and thus can be found in any wave phenomenon [7,8]. Furthermore, there are distinct advantages in studying localization using a classical wave equation instead of via the quantum mechanical electronic problem. Indeed, the latter usually deals with other types of interactions, such as the spin-orbit coupling, electron-phonon coupling, and electron-electron interactions, among others, which make the problem more complex.

Recently there has been a revival of interest in investigating the propagation of classical waves in complex media [9,10]. Much of the earlier work was on the Born-approximate forward modeling formula, whose basic approach is to use a cascade integral operator to produce a transformation from an input dataset at finite offset to an output dataset at zero offset [11]. The first member of the cascade is an inversion operator that creates an earth model from the input data. The second member is a modeling operator that creates the zero-offset data from the model or imaged data derived from migration. The application of this cascade operator was called seismic data mapping [12].

More recently, fundamental representations for the acoustic, isotropic, and anisotropic elastic cases were developed based on an integral representation for the wave field at a receiver point. These representations can be recast as modeling formulas for reflection from a transparent interface by exploiting the Kirchoff approximation, which expresses the unknown scattered field and its normal derivative in terms of the incident field. The result is called the Kirchoff-Helmholtz integral. Where the Born representation is linear in the perturbation of the medium parameters, the Kirchoff-Helmholtz representation is linear in the reflection coefficients, which, in turn, are nonlinear functions of the medium perturbation [13].

As an extension of these previous works, Schleicher *et al.* [14] used another mathematical model, based on a geometrical ray approximation (GRA) Green function formalism. Usually, the GRA Green function is expressed as a function of the phase velocities and the relative geometrical-spreading factor. This geometrical factor may be computed from mixed second-order travel time derivatives with respect to the phase-front coordinates, which are normal to the phase-velocity vectors. Instead, to take into account general anisotropic effects, they preferred to work with a GRA that is expressed by the group velocities and a relative geometrical-spreading factor of the acoustic wave.

It is the aim of this work to treat the classical problem of the propagation of acoustic waves in damped media considering a Green function formalism based on the *frequency distributions* of the acoustic waves's spectra. The frequency distribution of the acoustic waves are mainly determined by the power spectra of the thermally induced fluctuations in the degrees of freedom of the many scatterers found in the medium [16]. Instead of using the so-called recursive Green function technique based on the Dyson equation, frequently used to describe electronic conductance in mesoscopic systems [15], we believe that these power spectra, or correlation functions, are most conveniently calculated by using Green functions within the linear response function theory [17]. Taking into account the imaginary part of these Green functions, the required power spectra are obtained via the fluctuation-dissipation theorem [18].

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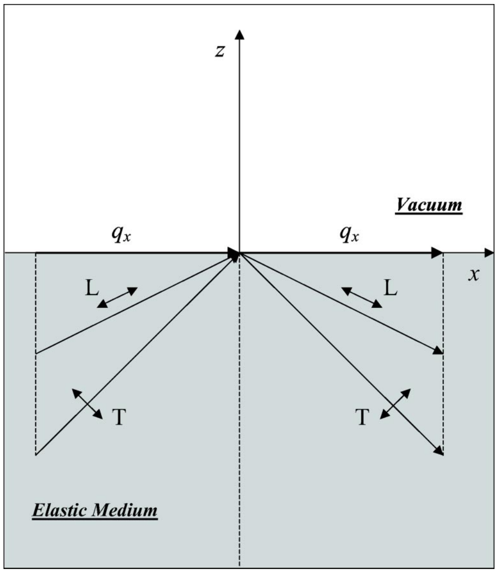

FIG. 1. The boundary between the elastic medium ($z<0$) and the adjacent vacuum medium ($z>0$). Here q_x is the common wave vector x component and the subscripts L and T mean longitudinal and transverse modes, respectively.

The propagation theory is mostly concerned with a medium of indefinite extent, where no thought needs to be given to the effects of sample boundaries. The frequency components present in the spectra are then the frequencies of bulk excitations, which extend uniformly through the medium. However, to make the propagation problem more realistic, it is necessary to take into account at least one boundary of the medium, where surface excitations may exist with amplitudes that fall off exponentially with distances from the sample surfaces. In the geometry considered in this work, with a single flat surface, the power spectrum shows the appearance of a surface wave excitation, named after Rayleigh, with interesting properties.

We consider a semi-infinite medium occupying the $z<0$ region, with an interface parallel to the xy plane and vacuum above it (see Fig. 1). The equation of motion for the propagation of an acoustic wave in such an elastic medium can be written as [19]

$$\rho \partial^2 u_i/\partial t^2 - \partial/\partial r_j[(C_{ijkl}/2)(\partial u_k/\partial r_l + \partial u_l/\partial r_k)] = 0, \tag{1}$$

where ρ is the density of the medium, u_i is the ith component of the displacement vector, and the summation convention is used. Also C_{ijkl} is the fourth-order elastic tensor, and $ijkl$ can be any Cartesian axis, i.e., x, y, or z.

Acoustic waves in elastic media can suffer spatial and/or temporal damping. It is sufficient for the present calculation to introduce the damping phenomenologically. With Cartesian superscripts removed, the stress(S)-strain(s) relation is replaced by $S = Cs + C\gamma(ds/dt)$. The second term on the right introduces a relaxation time γ into the strain caused by a time-varying stress. Its insertion into the equation of motion (1) produces wave vector versus frequency relation of form $q^2 = \omega^2/v^2(1 - i\omega\gamma)$. The phenomenology can be made more realistic by the assumption of branch and wave vector dependent relaxation times.

We now determine the Green functions by a classical linear response method [17]. The expressions to be derived are valid for any polarization, provided the appropriate velocity is replaced. In this sense, the displacement Green functions are obtained by calculating the effect of a fictitious external applied point force

$$F_z \exp(-i\omega t)\delta(z - z'), \tag{2}$$

which is parallel to the z axis and applied to a point z' in the medium. Here $\delta(z-z')$ is the Dirac delta function of the argument shown. The interaction energy between the force and the z component of the displacement is given by Hooke's law, i.e.,

$$H_{int} = -u_z(z')F_z \exp(-i\omega t). \tag{3}$$

This applied force produces displacement in both x and z directions, whose magnitudes are determined by insertion of Eq. (2) into the right-hand side of Eq. (1), i.e.,

$$\rho \partial^2 u_i/\partial t^2 - \partial/\partial r_j[(C_{ijkl}/2)(\partial u_k/\partial r_l + \partial u_l/\partial r_k)] = F_z \exp(-i\omega t)\delta(z - z'). \tag{4}$$

Assuming a harmonic time variation for the displacement (i.e., $\partial^2 u_i/\partial t^2 = -\omega^2 u_i$), the particular integral solution of the z component of the displacement in Eq. (4) is

$$u_z(\vec{q},z) = (iF_z/2\rho\omega^2)[q_L \exp(iq_L|z-z'|) + (q_x^2/q_T)\exp(iq_T|z-z'|)], \tag{5}$$

where

$$q_{L,T} = (|\vec{q}|^2 - q_x^2)^{1/2} = \{[\omega^2/v_{L,T}^2(1 - i\omega\gamma)]^2 - q_x^2\}^{1/2}, \tag{6}$$

q_x being the common wave vector x component, and the subscripts L and T mean longitudinal and transverse modes, respectively.

The homogeneous (or complementary function) solution of the z component of the displacement in Eq. (4) can be given by

$$u_z(\vec{q},z) = A\exp(-iq_L z) + B\exp(-iq_T z), \tag{7}$$

where A and B are constants to be found through the usual boundary conditions, i.e., the continuity of the z component of the displacement $u(\vec{q},z)$ and the stress S_{zz} at $z=0$.

The Green functions are now obtained by the application of the linear response theory. In view of the standard form (3) of the interaction energy, the displacement Green function is simply equal to

$$\langle\langle u_z(z); u_z(z')^* \rangle\rangle_\omega = u_z(\vec{q},z)/F_z, \tag{8}$$

where $\langle\langle \cdots \rangle\rangle_\omega$ is Zubarev's form [20] to express the Fourier transformed Green function of the arguments shown. The displacement gradient Green function can be easily found using

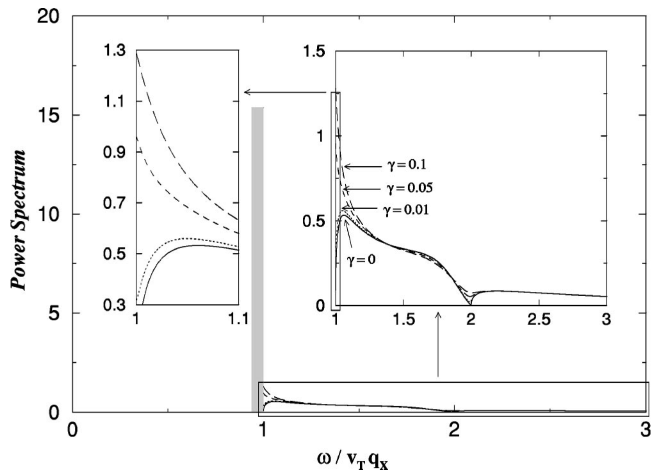

FIG. 2. The power spectrum for the damped acoustic waves propagating in an elastic medium, as a function of the dimensionless term $\omega/v_T q_x$.

$$\langle\langle u_{zz}(z);u_{zz}(z')^*\rangle\rangle_\omega = \partial^2/\partial z\partial z'\langle\langle u_z(z);u_z(z')^*\rangle\rangle_\omega . \tag{9}$$

The acoustic power spectrum at the interface ($z=0$) can now be determined by using the fluctuation-dissipation theorem at high temperature, i.e.,

$$\langle|u_z(0)|^2\rangle_\omega = (k_BT/\pi\omega)\mathrm{Im}\langle\langle u_z(0);u_z(0)^*\rangle\rangle_\omega . \tag{10}$$

Therefore, using Eqs. (5), (7), and (8),

$$\langle|u_z(0)|^2\rangle_\omega = \frac{k_BT}{\pi\rho v_T^3 q_x^2}\mathrm{Re}\left[\frac{\omega v_T^3 q_x^2 q_L}{4v_T^4 q_x^2 q_L q_T + (\omega^2 - 2v_T^2 q_x^2)^2}\right]. \tag{11}$$

In Eqs. (10) and (11), Im and Re means the imaginary and real part of the arguments shown.

We now discuss our analytical results in detail by applying them to the propagation of acoustic waves in a semi-infinite medium in which the ratio $v_L/v_T=2$. This corresponds to a medium whose Poisson's ratio σ is equal to 1/3.

Figure 2 shows the power spectrum, as described by the dimensionless term inside the bracket in Eq. (11), versus the reduced dimensionless frequency $\omega/v_T q_x$. In the inset of this figure, it is possible to see the dependence of the power spectrum on damping term as well as on frequency. The spectrum, regarding its frequency range, can be divided into three regions (see Fig. 3 for details). In the third region, which lies in the $\omega/v_T q_x > 2$ range, both the longitudinal and the transverse wave vectors q_L and q_T are real, in the absence of any damping term, and the spectrum has a continuous distribution. The damping effect is negligible in this region. For $v_T q_x < \omega < v_L q_x$, q_T is still real but q_L is imaginary (in the absence of the damping). The spectrum is characterized by another continuous distribution, which extends from 1 to 2 along the horizontal axis in Fig. 2. The two continua fall to zero at $\omega = v_T q_x$ and $\omega = v_L q_x$, respectively. The zero surface-fluctuation spectra at these frequencies result from cancellation of the surface-displacement contribution of the incident and reflected acoustic waves. The damping effects are more pronounced in this region. The spectrum increases proportional to the damping term γ as $\omega/v_T q_x$ goes from 1 to 1.4, with amplitudes equal to 0.32 for $\gamma=0.01$, to 0.96 for $\gamma=0.05$, and to 1.28 for $\gamma=0.1$ (see the left inset in Fig. 2). In the $1.4<\omega/v_T q_x<1.85$ range, the influence of the damping is reduced, and the power spectrum increases proportional to γ^{-1}. It is easy to see the existence of two points where the power spectrum is independent of the damping term. The first point is situated at $\omega/v_T q_x = 1.4$ and the second one at $\omega/v_T q_x = 1.85$.

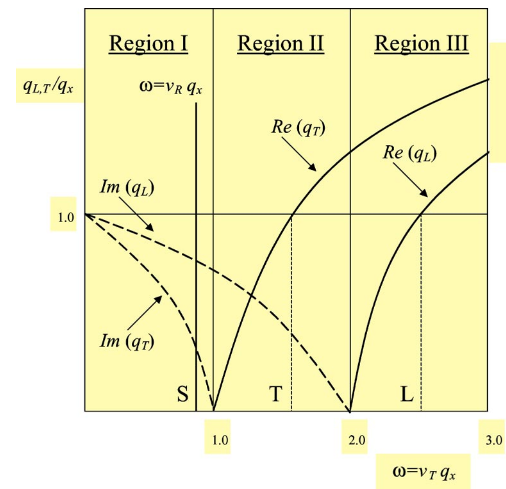

FIG. 3. The frequency dependence of the longitudinal and transverse wave vectors: solid line curves represent the real components; broken line curves are the imaginary parts. The thin vertical line shows the Rayleigh surface wave.

The first part of the spectrum, corresponding to $\omega < v_T q_x$, is the dominant one, and it is shown shaded for $\gamma=0$ in Fig. 2. Here both q_L and q_T are purely imaginary, and the factor inside the large brackets of (11) is also purely imaginary. However, the denominator

$$\Delta = 4v_T^4 q_x^2 q_L q_T + (\omega^2 - 2v_T^2 q_x^2)^2 \tag{12}$$

has a zero in this frequency range, the condition for a zero being the standard equation used to derive the frequency ω_R and velocity v_R of the Rayleigh surface waves, whose propagation in a semi-infinite and isotropic medium with inhomogeneities was recently reported [21]. Using the mathematical identity

$$(x+i\varepsilon)^{-1} = P(1/x) + i\pi\delta(x), \tag{13}$$

we have that the Rayleigh mode has amplitude equal to (without damping)

$$\Gamma = \pi x_R(1-x_R^2)^{1/2}\left|\frac{d\Delta}{dx}\right|_{x=x_R}^{-1}\delta(x-x_R) \tag{14}$$

with $x_R = v_R/v_T$. Therefore, the Rayleigh mode can be described by a δ-function, whose strength is given by its coefficient in Eq. (14). Considering a damping factor γ, we can use

$$\delta(x-x_R) = \gamma[\pi(x-|x_R|)^2 + \gamma^2]^{-1} \tag{15}$$

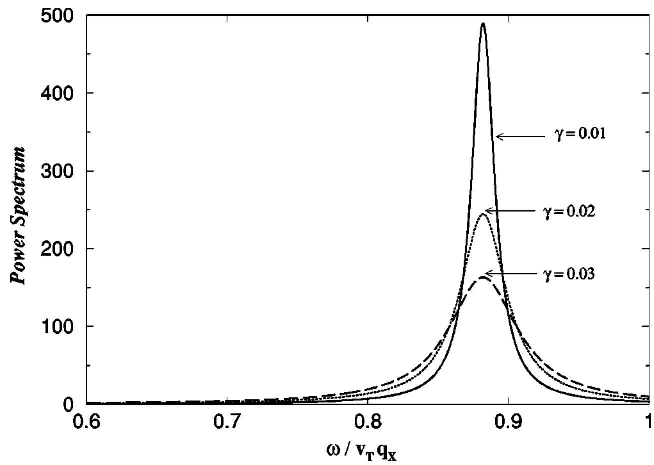

FIG. 4. Power spectrum for the damped Rayleigh surface wave contribution, as a function of the dimensionless term $\omega/v_T q_x$.

and henceforth the Rayleigh mode can have the Lorentzian profile centered at $\omega/v_T q_x \cong 0.88$, shown in Fig. 4 for three values of γ (0.01, 0.02, and 0.03). For small γ, the Lorentzian is tall and narrow; as γ increases, the Lorentzian broadens and its height decreases, keeping the surface area underneath constant.

The main motivation for writing this paper lies in the current inability to fully understand the unwanted noise behavior of seismic waves in damped media, which is very important in the analysis of seismic reflection data [12]. Seismic sources generate various types of surface waves, which in turn are a common source of these noises, depending on the near-surface environment and nature and position of the source [22]. Surface waves composed of dispersive Rayleigh waves (as those treated here), whose different frequency components travel at different velocities leading to complex wavetrains, can dominate near-source traces on seismic records [23]. They are such a problem in land seismic record acquisition that the design of acquisition parameters is dominated by the need to suppress them. Current processing methods of eliminating such surface waves from seismic records include frequency filtering, which may be conveniently tailored using the frequency-dependent power spectrum presented in Figs. 3 and 4.

Moreover, there are basically two sources of information about the seismic wave's propagation: travel times and amplitudes. The travel times of the various wave fronts in the wave field generally provide information about the low spatial-frequency components (the background) of the medium parameters, which can be described by correlations functions of the types related by the displacement, and displacement gradient Fourier transformed Green functions defined in Eq. (8) and (9). The amplitudes of the wave fronts, on the other hand, are most sensitive to the high spatial-frequency components (the reflectivity). The two types of information sample different aspects of the medium. For this case, the acoustic power spectrum may give the appropriate information, since it enables us to find a specifically chosen reflection term without the necessity of calculating other parameters that might be considered noise in a real situation (for instance, the presence of the Rayleigh singularity in the spectrum).

In summary, we have considered a rather simple model for the propagation of acoustic waves in damped media, looking for a better understanding of the unwanted noise in the seismic record. To have our model more realistically describe this problem, it is necessary to consider the presence of all singularities (not only the Rayleigh type described here) in the Earth's material properties that carry the waves. This means the introduction of algebraic singularities, not smooth differentiable functions, and therefore the consideration of a medium with fractal measurement, such as those studied previously for light waves propagation [24,25].

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