Harmonic oscillator with multiplicative noise: Nonmonotonic dependence on the strength and the rate of dichotomous noise

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The output signal of a undamped linear oscillator with a random frequency subject to a periodic force shows nonmonotonic dependence on the strength and the rate of color noise (stochastic resonance). The effect is absent for white noise.

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Stochastic resonance (SR) is an interesting phenomenon exhibited by nonlinear dynamic systems driven by a combination of a periodic signal and a random force [1]. Due to its many potential applications in biology, physics, and chemistry, along with its appeal to scientific curiosity, the number of publications in this field is growing steadily [2], which is reminiscent of the ''deterministic chaos'' boom in the 1970–1980s. Not by chance, the names of both these phenomena consist of half-deterministic and half-random terms. In fact, deterministic chaos denotes a random type of behavior in deterministic systems, while SR shows deterministiclike behavior in random systems. These peculiar features show that determinism and randomness are complementary, rather than contradictory phenomena [3].

In the broad sense, SR means the nonmonotonic dependence of the output signal or some function of it (moments, autocorrelation function, power spectrum, or a signal-to-noise ratio, or dynamic parameters) on the characteristics of noise (noise amplitude or the correlation time). The peculiarity of SR lies in the fact that noise, which usually appears as a destructive factor, may play a constructive role. Let us bring a partial list of versions of SR appearing under different headings, which show the ordered role of noise: noise-induced transition [4], noise-induced transport [5], noise-induced pattern formation [6], noise-induced resonances [7], noise-induced stabilization [8], noise-enhanced stability [9], noise-induced hypersensitivity [10], resonance activation [11], stochastic transport in ratchets [12], stochastic localization [13], self-organization and dissipative structures [14], coherent stochastic resonance [15], fluctuation barrier kinetics [16], amplification of weak signals via off-on intermittency [17], autonomous SR [19], aperiodic SR [20].

It first seemed that all three ingredients—nonlinearity, periodic, and random forces—are necessary for the onset of SR. However, it later became clear that SR may appear without a random force (replaced by a chaotic signal [18]), without a periodic force (autonomous SR [19], aperiodic SR [20]), or by replacing the characteristic frequency by some fluctuation rate [11]), and in linear systems (with multiplicative noise [21,22]).

SR in linear systems is the subject of the present analysis. The analysis of SR in linear systems was previously restricted to an overdamped oscillator with color multiplicative noise (Ornstein-Zernike [22], Gaussian [23], Poissonian [24], or composite [25] noise). The few examples of the analyses of SR in an underdamped oscillator either relate to additive noise [26] or involve no external field [27], i.e., describe autonomous SR.

We consider a forced, underdamped linear oscillator with random frequency

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+[\omega^2+\xi(t)]x=a\sin\Omega t. \qquad (1)$$

The random force $\xi(t)$ is a Gaussian variable with zero mean and Ornstein-Zernike correlator

$$\langle\xi(t)\xi(t_1)\rangle=\sigma\exp(-\lambda|t-t_1|). \qquad (2)$$

Fluctuations of external parameters [frequency in Eq. (1)] are expressed by multiplicative noise. The latter was widely introduced as a model to understand the different phenomena in physics (on-off intermittency [28], dye lasers [29], polymers in random field [30]), biology (population dynamics [31]), economics (stock market prices [32]), and so on.

The second-order differential equation (1) can be rewritten as two first-order differential equations

$$\frac{dx}{dt}=y, \qquad (3)$$

$$\frac{dy}{dt}=-\gamma y-\omega^2x-\xi x+a\sin\Omega t, \qquad (4)$$

which, after averaging, take the following form:

$$\frac{d}{dt}\langle x\rangle=\langle y\rangle, \qquad (5)$$

$$\frac{d}{dt}\langle y\rangle=-\gamma\langle y\rangle-\omega^2\langle x\rangle-\langle\xi(t)x\rangle+a\sin(\Omega t). \qquad (6)$$

The new correlator $\langle\xi(t)x\rangle$ has to be found separately. To this end, we use the well-known Furutzu-Novikov procedure [33], which, for exponentially correlated random functions, takes the Shapiro-Logunov form [34]

$$\frac{d\langle\xi(t)x\rangle}{dt}=\left\langle\xi(t)\frac{dx}{dt}\right\rangle-\lambda\langle\xi(t)x\rangle. \qquad (7)$$

Multiplying Eq. (3) by ξ, one gets after averaging,

$$\left\langle \xi(t)\frac{dx}{dt}\right\rangle = \langle \xi(t)y\rangle. \tag{8}$$

Inserting Eq. (8) into Eq. (7) results in

$$\frac{d\langle \xi(t)x\rangle}{dt} = \langle \xi(t)y\rangle - \lambda\langle \xi(t)x\rangle. \tag{9}$$

Using the procedure analogous to Eq. (7) for the correlator $\langle \xi(t)y\rangle$, one gets

$$\frac{d\langle \xi(t)y\rangle}{dt} = \left\langle \xi(t)\frac{dy}{dt}\right\rangle - \lambda\langle \xi(t)y\rangle. \tag{10}$$

Multiplying Eq. (4) by ξ and averaging, one obtains

$$\left\langle \xi\frac{dy}{dt}\right\rangle = -\gamma\langle \xi y\rangle - \omega^2\langle \xi x\rangle - \langle \xi^2 x\rangle. \tag{11}$$

Equation (11) contains the higher-order correlator $\langle \xi^2 x\rangle$, and one has to use a decoupling procedure. Another possibility is to consider the special case of the two-state Markov process (dichotomous noise) which is described by correlator (2), and $\xi^2 = \sigma$. For this special case, Eq. (11) can be rewritten as

$$\left\langle \xi\frac{dy}{dt}\right\rangle = -\gamma\langle \xi y\rangle - \omega^2\langle \xi x\rangle - \sigma\langle x\rangle. \tag{12}$$

Inserting Eq. (12) into Eq. (10) results in

$$\frac{d\langle \xi(t)y\rangle}{dt} = -\gamma\langle \xi y\rangle - \omega^2\langle \xi(t)x\rangle - \sigma\langle x\rangle - \lambda\langle \xi(t)y\rangle. \tag{13}$$

We thus obtain a system of four equations: Eqs. (5), (6), (9), and (13), for four variables, $\langle x\rangle$, $\langle y\rangle$, $\langle \xi x\rangle$, and $\langle \xi y\rangle$.

From these equations one can easily find the fourth-order differential equation for $\langle x\rangle$,

$$\begin{aligned}
&\frac{d^4\langle x\rangle}{dt^4} + 2(\lambda+\gamma)\frac{d^3\langle x\rangle}{dt^3} + (2\omega^2+\lambda^2+3\lambda\gamma+\gamma^2)\frac{d^2\langle x\rangle}{dt^2} \\
&\quad + [2\omega^2(\lambda+\gamma)+\lambda\gamma(\lambda+\gamma)]\frac{d\langle x\rangle}{dt} \\
&\quad + [\omega^2(\omega^2+\lambda^2+\lambda\gamma)-\sigma]\langle x\rangle \\
&= (\omega^2+\lambda^2+\lambda\gamma-\Omega^2)a\sin(\Omega t) + (2\lambda+\gamma)a\Omega\cos(\Omega t).
\end{aligned} \tag{14}$$

We seek the solution of Eq. (14) in the form

$$\langle x\rangle = \langle x\rangle_0 + \langle x\rangle_a, \tag{15}$$

where the output signal $\langle x\rangle_a$ is induced by an external field, $a\sin(\Omega t)$ and $\langle x\rangle_0$ is defined by internal dynamics. For purposes of this discussion we ignore the possible instability of an underdamped oscillator for fast fluctuations [35].

Let us write the solution $\langle x\rangle_a$ of Eq. (14) in the form

$$\langle x\rangle_a = A\sin(\Omega t+\phi). \tag{16}$$

Then, one easily finds

$$A = \left[\frac{f_1^2+f_2^2}{f_3^2+f_4^2}\right]^{1/2}; \quad \phi = \tan^{-1} = \frac{f_1 f_3 + f_2 f_4}{f_1 f_4 - f_2 f_3}; \tag{17}$$

where

$$\begin{gathered}
f_1 = (2\lambda+\gamma)a\Omega, \quad f_2 = (\Omega^2-\omega^2-\lambda^2-\lambda\gamma)a, \\
f_3 = (\Omega^2-\omega^2)(\Omega^2-\omega^2-\lambda^2) - \sigma - (3\lambda\gamma+\gamma^2)\Omega^2 + \lambda\gamma\omega^2, \\
f_4 = \Omega(\lambda+\gamma)[2(\omega^2-\Omega^2)+\lambda\gamma].
\end{gathered} \tag{18}$$

In the absence of friction, $\gamma = 0$, Eqs. (17) and (18) take the following forms:

$$A = a\left[\frac{[(\Omega^2-\omega^2-\lambda^2)^2+4\lambda^2\Omega^2]}{[(\Omega^2-\omega^2)(\Omega^2-\omega^2-\lambda^2)-\sigma]^2+4\lambda^2\Omega^2(\Omega^2-\omega^2)^2}\right]^{1/2} \tag{19}$$

and

$$\phi = \tan^{-1}\frac{2\Omega\lambda\sigma}{(\Omega^2-\omega^2)[(\Omega^2-\omega^2-\lambda^2)^2+4\Omega^2\lambda^2] - (\Omega^2-\omega^2-\lambda^2)\sigma}. \tag{20}$$

For small noise strength σ, Eq. (19) reduces to Eq. (8.6) of [36] found in a different context by perturbation theory.

Prior to the analysis of Eqs. (19) and (20), let us consider the limiting case of white Gaussian noise, which, according to Eq. (2), corresponds to $\sigma\to\infty$ and $\lambda\to\infty$ with a constant ratio. Then,

$$A = \frac{a}{(\omega^2-\Omega^2)}; \quad \phi = 0 \tag{21}$$

as it should be, since for white noise $\langle x\rangle$ satisfies the following equation [37]:

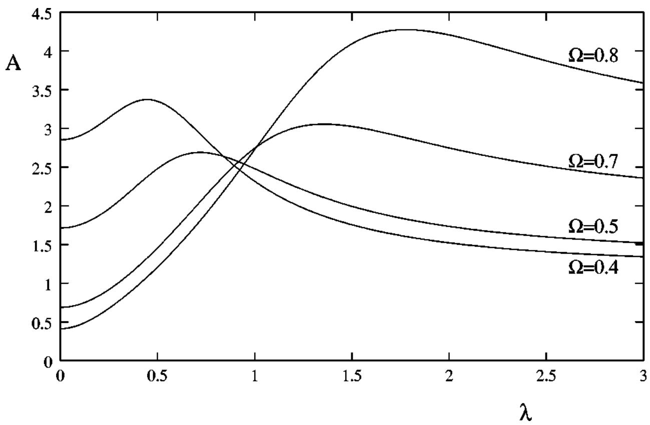

FIG. 1. The amplitude A of a stationary signal as a function of the correlation rate λ for $a=\sigma=\omega=1$, and $\gamma=0$. The curves displayed correspond to different values of the frequency of an external field $\Omega=0.4$, 0.5, 0.7, and 0.8.

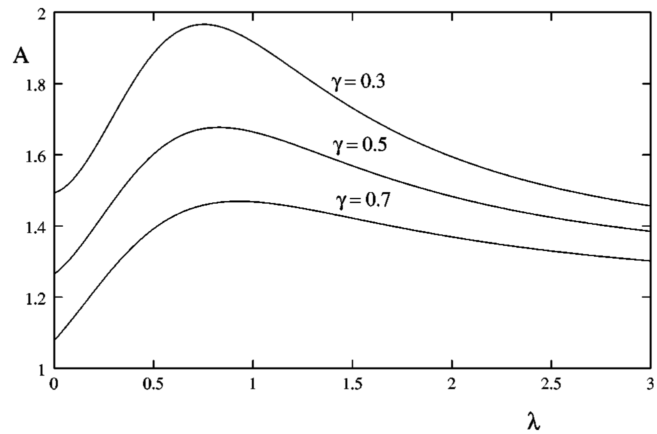

FIG. 2. The amplitude A of a stationary signal as a function of the correlation rate λ for $a=\sigma=\omega=1$, $\Omega=0.5$. The curves displayed correspond to different values of the friction $\gamma=0.3$, 0.5, and 0.7.

$$\frac{d^2\langle x\rangle}{dt^2}+\omega^2\langle x\rangle=a\sin\Omega t. \tag{22}$$

However, for color noise (dichotomous in the present case) the output signal (19) shows nonmonotonic dependence on the noise strength σ and the correlation rate λ (stochastic resonance). Indeed, the amplitude of the output signal A reaches a maximum at

$$\sigma=(\Omega^2-\omega^2)(\Omega^2-\omega^2-\lambda^2). \tag{23}$$

In Fig. 1 we show the dependence of the amplitude of a stationary signal A on the correlation rate λ for $a=\sigma=\omega=1$, $\gamma=0$, and different frequencies Ω of the external field. This graph shows typical SR nonmonotonic behavior for $\Omega<\omega$. However, the heights of the maxima are nonmonotonic functions of Ω. Indeed, the maximal value of the amplitude A for $\Omega=0.5$ is lower than those for both $\Omega=0.4$ and $\Omega=0.7$, whereas the positions of the maxima are monotonically shifted to higher λ with a rise in Ω.

Note that the resonance amplitude of a nondamped harmonic oscillator [Eq. (19) with $\Omega=\omega$] remains restricted in the presence of colored noise (effective damping).

In Fig. 2 we show the influence of friction on the amplitude of a stationary signal A. As expected, an increase in damping decreases the value of the output signal.

Finally, we have found that SR appears in a underdamped, forced linear oscillator with multiplicative color noise. For dichotomous noise, one can easily find the higher moments of $x(t)$ [38] along with the first moment considered above.

[1] R. Benzi, S. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); C. Nicolis, Tellus **34**, 1 (1982).
[2] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[3] M. Gitterman, Eur. J. Phys. **23**, 119 (2002).
[4] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin 1984).
[5] P. Hanggi and R. Bartussec, *Lecture Notes in Physics* (Springer, Berlin 1996), Vol. 476, p. 294.
[6] J.M.R. Parrondo, C. Van den Broeck, and F.J. de la Rubia, Physica A **224**, 153 (1996).
[7] P. Jung and P. Talkner, Phys. Rev. E **51**, 2640 (1995).
[8] M. Lucke and F. Schank, Phys. Rev. Lett. **54**, 1465 (1985).
[9] I. Dayan, M. Gitterman, and G.H. Weiss, Phys. Rev. A **46**, 757 (1992).
[10] S.L. Ginzburg and M.A. Pustovoit, Phys. Rev. Lett. **80**, 4840 (1998).
[11] C.R. Doering and J.C. Gadoua, Phys. Rev. Lett. **69**, 2318 (1992).
[12] F. Marchesoni, Phys. Lett. A **237**, 126 (1998).
[13] D.W. Brown, L. Bernstein, and K. Lindenberg, Phys. Rev. E **54**, 3352 (1996).
[14] G. Nicolis and I. Prigogine, *Self-Organization in Non-Equilibrium Systems* (Wiley, New York, 1977).
[15] J.E. Fletcher, S. Havlin, and G.H. Weiss, J. Stat. Phys. **51**, 215 (1988).
[16] P. Reimann and P. Hanggi, *Lecture Notes in Physics* (Springer, Berlin, 1997), Vol. 484, p. 127.
[17] C. Zhou and C-H. Lai, Phys. Rev. E **59**, R6243 (1999).
[18] S. Sinha, Physica A **270**, 204 (1999).
[19] H. Gang, T. Ditzinger, C.Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993); J.L. Cabrera, J. Corronogoitia, and F.J. de la Rubia, *ibid.* **82**, 2816 (1999).
[20] J.J. Collins, C.C. Chow, A.C. Capela, and T.T. Imhoff, Phys. Rev. E **54**, 5575 (1996).
[21] A. Fulinski, Phys. Rev. E **52**, 4523 (1995).
[22] V. Berdichevsky and M. Gitterman, Europhys. Lett. **36**, 161 (1996).
[23] A.V. Barzykin and K. Seki, Europhys. Lett. **40**, 117 (1997).
[24] V. Bezak, Czech. J. Phys. **48**, 529 (1998).

[25] A.V. Barzykin, K. Seki, and F. Shibata, Phys. Rev. E **57**, 6555 (1998).
[26] L. Alfonsi, L. Gammaitoni, S. Santucci, and A.R. Bulsara, Phys. Rev. E **62**, 299 (2000).
[27] A. Fulinski and P.F. Gora, J. Stat. Phys. **101**, 483 (2000).
[28] S.C. Venkataramani, T.M. Antonsen, Jr., E. Ott, and J.C. Sommerer, Physica D **96**, 66 (1996).
[29] S.Q. Zhu, Phys. Rev. A **45**, 8148 (1992).
[30] J.M. Deutsch, Physica A **208**, 445 (1994).
[31] M. Turelli, *Theoretical Population Biology* (Academic, New York, 1977), Vol. 12, p. 140.
[32] A-H. Sato and H. Takayasu, Physica A **250**, 231 (1998).
[33] K. Furutsu, J. Res. Natl. Bur. Stand., Sect. D **67**, 303 (1963); E.A. Novikov, Sov. Phys. JETP **20**, 1290 (1965).
[34] Y.E. Shapiro and V.M. Logunov, Physica A **91**, 563 (1978).
[35] K. Mallick and P. Marcq, Phys. Rev. E **66**, 041113 (2002).
[36] M. F. Dimentberg, *Nonlinear Stochastic Problems of Mechanical Oscillations* (Nauka, Moscow, 1980), in Russian.
[37] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).
[38] V. Berdichevsky and M. Gitterman, Phys. Rev. E **60**, 1494 (1999).