Magnetization distribution in the transverse Ising chain with energy flux

V. Eisler,[1] Z. Rácz,[1,2] and F. van Wijland[2]
[1]*Institute for Theoretical Physics, Eötvös University, Pázmány sétány 1/a, 1117 Budapest, Hungary*
[2]*Laboratoire de Physique Théorique, Bâtiment 210, Université de Paris–Sud, 91405 Orsay Cedex, France*
(Received 28 January 2003; published 27 May 2003)

The zero-temperature transverse Ising chain carrying an energy flux j_E is studied with the aim of determining the nonequilibrium distribution functions, $P(M_z)$ and $P(M_x)$ of its transverse and longitudinal magnetizations, respectively. An exact calculation reveals that $P(M_z)$ is a Gaussian both at $j_E=0$ and at $j_E\neq 0$, and the width of the distribution decreases with increasing energy flux. The distribution of the order-parameter fluctuations, $P(M_x)$, is evaluated numerically for spin chains of up to 20 spins. For the equilibrium case ($j_E=0$), we find the expected Gaussian fluctuations away from the critical point, while the critical order-parameter fluctuations are shown to be non-Gaussian with a scaling function $\Phi(x)=\Phi(M_x/\langle M_x\rangle)=\langle M_x\rangle P(M_x)$ strongly dependent on the boundary conditions. When $j_E\neq 0$, the system displays long-range, oscillating correlations but $P(M_x)$ is a Gaussian nevertheless, and the width of the Gaussian decreases with increasing j_E. In particular, we find that, at critical transverse field, the width has a $j_E^{-3/8}$ asymptotic in the $j_E\to 0$ limit.

DOI: 10.1103/PhysRevE.67.056129 PACS number(s): 05.50.+q, 05.60.Gg, 05.70.Ln, 75.10.Jm

I. INTRODUCTION

Nonequilibrium steady states (NESS) have been much studied but a description of some generality has not emerged so far. Among the many approaches tried, there is one that continues to receive particular attention. It is an attempt to understand the general features of NESS through studies of nonequilibrium phase transitions [1–3]. The basic assumption here is that the universality displayed by equilibrium phase transitions carries over to critical phenomena in NESS, as well. Thus, by investigating the similarities and differences from equilibrium, one may gain an understanding of the role of various components of the competing dynamics generating the steady state. For example, one may find by observing the universality classes of various nonequilibrium phase transitions that dynamical anisotropies often yield dipolelike effective interactions [4–6] or that competing nonlocal dynamics (anomalous diffusion) generates long-range, power-law interactions [7].

The extension of concepts of critical phenomena to phase transitions in NESS also implies that the distribution functions of macroscopic quantities (such as the order parameter) are nontrivial (non-Gaussian). They are universal, however, and characterize the given nonequilibrium universality class, too. The advantage of studying the scaling functions associated with the distribution functions is that building these functions does not involve any fitting procedure and thus they allow for a fit-free comparison with experiments. Indeed, using these distribution functions, a number of interesting results have been derived for surface growth as well as for other nonequilibrium processes [8–13].

In this paper, we continue our studies of nonequilibrium distribution functions by investigating the effects of a nonequilibrium constraint on a well-known *quantum* phase transition; namely, we take the transverse Ising chain that has an order-disorder transition as the transverse field h is varied, and drive it by a field to produce an energy flux j_E through the system. The resulting steady states have been described in Ref. [14] and it has been found that, in addition to the equilibrium phases, a flux-carrying nonequilibrium phase appears, which is distinct by its correlations decaying with distance as a power law. We shall be concerned with the distribution function in the various phases of the above system. More precisely, we shall determine the steady-state distribution functions $P(M_z)$ and $P(M_x)$ of the M_z (nonordering field) and M_x (ordering field) components of the macroscopic magnetization in all three phases of the system at and near its critical point.

The results are surprisingly simple. The distribution functions are Gaussian in the equilibrium phases away from the critical point. This is expected since we have macroscopic quantity and the correlations decay exponentially. The distribution of the nonordering field remains a Gaussian at the critical point of the equilibrium system, as well. The reason for this is that although the appropriate correlations decay with distance n as a power law but the exponent in the power is large ($1/n^2$), so that the fluctuations $\langle M_z^2\rangle-\langle M_z\rangle^2$ do not diverge at $h=h_c$. The distribution function of the ordering field becomes nontrivial at h_c and our numerical calculations demonstrate that $P(M_x)$ depends strongly on the boundary conditions taken to be periodic, antiperiodic, and free. The unexpected simplicity is in the current-carrying phase where the energy flux generates long-range correlations decaying as a power law ($1/\sqrt{n}$) but, nevertheless, the distributions are Gaussian. The mathematical reason for this lies in the oscillating character of the correlations, which prevents the divergence of the spatial sum of the correlations which in turn are proportional to the fluctuations. Physically, the oscillations in the correlations can be traced to the form of energy flux [see Eq. (3) below], which suggest that the consecutive x and y components of the spins are more and more rigidly interconnected as j_E is increased and thus fluctuations decrease with increasing j_E. This picture will be seen to be valid near the nonequilibrium phase boundaries where the fluctuations as a function of j_E can be explicitly calculated.

The above results are presented in the following order. Section II contains a review of the transverse Ising model with energy flux, including the setup of the formalism convenient for calculating the distribution functions. Next (Sec. III), distribution $P(M_z)$ is calculated exactly. Numerical work on $P(M_x)$ and preliminary analytic work on correlations are presented in Sec. IV, followed by concluding remarks in Sec. V.

II. TRANSVERSE ISING MODEL WITH ENERGY FLUX

The transverse Ising chain is one of the simplest systems displaying a critical order-disorder transition [14–19]. It is defined by the Hamiltonian

$$\hat{H}_I=-J\sum_i s_i^x s_{i+1}^x-\frac{h}{2}\sum_i s_i^z, \tag{1}$$

where $\vec{s}_i=\frac{1}{2}\vec{\sigma}_i$ and σ_i^α ($\alpha=x,y,z$) denotes the three Pauli matrices at sites $i=1,2,\ldots,N$ of a $d=1$ chain, and h is the transverse field in units of the Ising coupling ($J=1$ is set in the rest of the paper). We shall mainly consider periodic boundary conditions ($s_{N+1}^\alpha=s_1^\alpha$), which are the simplest ones allowing for a nonzero energy current to flow through the chain (free boundary conditions imply a zero steady-state current in the above model and as it turns out the same holds for antiperiodic boundary conditions).

The order parameter of this system is $M_x=\Sigma_i s_i^x$, and the ground state of this Hamiltonian changes from being disordered $\langle M_x\rangle=0$ for $h>1$ to ordered $\langle M_x\rangle\neq 0$ for $h<1$. The transition point $h=h_c$ is a critical point in the universality class of the two-dimensional Ising model.

We would like to investigate the nonequilibrium states of $\hat{H}_I$ which carry a given energy flux. At zero temperature, this amounts to finding the lowest energy state of $\hat{H}_I$ with a prescribed energy flux. This can be accomplished [14–17] by introducing a Lagrange multiplier λ conjugate to the energy current $\hat{J}_E$. Thus one should find the ground state of the following Hamiltonian:

$$\hat{H}=-\sum_i s_i^x s_{i+1}^x-\frac{h}{2}\sum_i s_i^z-\lambda\hat{J}_E. \tag{2}$$

Here the driving field λ is again measured in units of J, and the current operator $\hat{J}_E$ is the sum of local energy fluxes given by the following expression:

$$\hat{J}_E=\frac{h}{4}\sum_i (s_i^x s_{i+1}^y-s_i^y s_{i+1}^x). \tag{3}$$

Note that the time evolution of the chain is still governed by $\hat{H}_I$ through which $\hat{J}_E$ was initially defined. $\hat{H}$ is just a mathematical intermediary to determine the nonequilibrium steady state of $\hat{H}_I$.

The driven system defined by Eqs. (2) and (3) can be solved [14] and one finds that the ground state does not change and the energy flux is zero up to a critical value $\lambda=\lambda_c(h)$ of the driving field. The ground-state expectation value of energy flux $j_E=\langle\hat{J}_E\rangle/N$ becomes nonzero for $|\lambda|>\lambda_c$. The resulting phase diagram is depicted in Fig. 1.

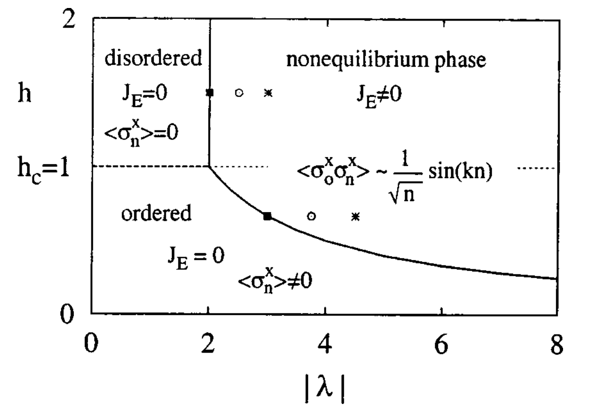

FIG. 1. Phase diagram of the driven transverse Ising model in the $h-|\lambda|$ plane where h is the transverse field while λ is the effective field that drives the flux of energy. Pairs of dual-conjugate points are shown by filled squares, circles, and stars; and line $h=1$ is self-dual as discussed in the text. Power-law correlations are present in the nonequilibrium phase, $J_E\neq 0$, and on the Ising critical line in the equilibrium phase, $J_E=0$ ($h=1$, $0\leq|\lambda|\leq 2$).

Here we note a hitherto unnoticed property of the Hamiltonian (2); namely, the duality properties of the transverse Ising model [18] have an appropriate generalization to the full nonequilibrium phase diagram of $\hat{H}$. Indeed, let us denote the action of duality transformation by $s_i^\alpha\to s_i^{\alpha *}$ and define the transformation as is done for the Ising model in a transverse field

$$s_i^{z*}=2s_i^x s_{i+1}^x, \quad s_i^{x*}s_{i+1}^{x*}=\frac{1}{2}s_i^z. \tag{4}$$

It can be verified then that the Hamiltonian $\hat{H}[h,\lambda,\{s_i^\alpha\}]$ (2) characterized by the two couplings (h,λ) transforms into an identical Hamiltonian with couplings $(h,\lambda)^*=(h^{-1},-\lambda h)$,

$$\hat{H}[h,\lambda,\{s_i^\alpha\}]=h\hat{H}[h^{-1},-\lambda h,\{s_i^{\alpha *}\}]. \tag{5}$$

Hence the duality transformation leaves the whole $|\lambda|=\lambda_c$ curve globally invariant and, furthermore, it leaves the $h=1$ line pointwise invariant (examples of dual-conjugate points are given in Fig. 1). In order to keep the formulas simple, from now on we shall restrict our analysis to $\lambda\geq 0$, that is to $j_E\geq 0$.

The self-dual $h=1$ line is expected to display special properties. For example, quantities such as λ_c/λ, or the wave vectors $q_\pm$ where the excitation spectrum is gapless, are left invariant by the duality transformation. Furthermore, the functional form of various physical quantities (dispersion, energy flux, fluctuations) considerably simplify on this line and thus the knowledge of self-duality helps in locating limits where exact calculation can be carried out.

Our main goal is to calculate distribution functions $P(M_z)$ and $P(M_x)$ in various regions in the above phase diagram. These functions are defined as

$$P(M_\alpha)=\left\langle \delta\left(M_\alpha-\sum_i s_i^\alpha\right)\right\rangle, \tag{6}$$

where brackets denote the quantum mechanical expectation value in the ground state [note that we have omitted the index α in $P_\alpha(.)$, i.e., the argument of the function defines which distribution is considered].

There are two parts to our calculations. Function $P(M_x)$ is evaluated numerically by diagonalizing $\hat{H}$ for chains containing up to $N=18$–20 spins, while $P(M_z)$ is found analytically. The exact calculation is possible because $M_z=\Sigma_q(c_q^\dagger c_q-1/2)$ is a quadratic form in the fermion operators $(c_q^\dagger,\ c_q)$ in which the Hamiltonian $\hat{H}=\hat{H}_I-\lambda\hat{J}_E$ is quadratic as well. Thus the calculation of the generating function

$$G(s)=\langle e^{-sM_z}\rangle \tag{7}$$

becomes a problem of evaluating Gaussian integrals. We begin with this part of the problem.

A. Formalism

Hamiltonian $\hat{H}$ can be diagonalized [14] by first introducing creation-annihilation operators, then employing the Jordan-Wigner transformation [19] to transform them into fermion operators $(c_\ell, c_\ell^\dagger)$ and, finally, using a Bogoliubov transformation on the c_q and $c_{-q}^\dagger$ components of the Fourier transforms of c_ℓ-s. The calculation of $P(M_z)$ becomes relatively simple if after using the Jordan-Wigner transformation one passes to a path-integral formulation (see, e.g., Ref. [20] for a pedagogical account). One then finds that the system is described by the following quadratic action:

$$S[\bar{c},c]=\int \frac{d\omega}{2\pi}\int \frac{dq}{2\pi}\left[\frac{1}{2}[\bar{c}_q(\omega)c_{-q}(-\omega)]B_{q,\omega}\begin{pmatrix}\bar{c}_{-q}(-\omega)\\ c_q(\omega)\end{pmatrix}\right], \tag{8}$$

where the Grassmann fields $\bar{c}_q(\omega)$ and $c_q(\omega)$ are related to $c_q^\dagger$ and c_q correspondingly, while the scattering matrix $B_{q,\omega}$ has the following inverse

$$B_{q,\omega}^{-1}=\begin{pmatrix}\langle \bar{c}_{-q}(-\omega)\bar{c}_q(\omega)\rangle & \langle \bar{c}_{-q}(-\omega)c_{-q}(-\omega)\rangle\\ \langle c_q(\omega)\bar{c}_q(\omega)\rangle & \langle c_q(\omega)c_{-q}(-\omega)\rangle\end{pmatrix}. \tag{9}$$

Here the correlators are given by

$$\langle c_q(\omega)\bar{c}_q(\omega)\rangle=\langle c_q(\omega)\bar{c}_q(\omega)\rangle^*=\frac{1}{2\Lambda_q}\left[\frac{\frac{h}{2}+\frac{1}{2}\cos q-\Lambda_q}{i\omega-\Lambda_q^+}-\frac{\frac{h}{2}+\frac{1}{2}\cos q+\Lambda_q}{i\omega-\Lambda_q^-}\right] \tag{10}$$

and

$$\langle \bar{c}_{-q}(-\omega)\bar{c}_q(\omega)\rangle=\langle c_q(\omega)c_{-q}(-\omega)\rangle^*=\frac{i\frac{1}{2}\sin q}{2\Lambda_q}\left(\frac{1}{i\omega-\Lambda_q^-}-\frac{1}{i\omega-\Lambda_q^+}\right), \tag{11}$$

where Λ_q and $\Lambda_q^\pm$ are the dispersion relations for $\hat{H}_I$ and $\hat{H}$, respectively,

$$\Lambda_q=\frac{1}{2}\sqrt{1+h^2+2h\cos q}, \tag{12}$$

$$\Lambda_q^\pm=\pm\Lambda_q+\frac{\lambda h}{4}\sin q. \tag{13}$$

In order to return to the time variables, we must first study the two branches $\Lambda_q^\pm$ of the spectrum.

B. Spectrum and energy flux

Flux of energy is present in the system only above a critical drive $\lambda>\lambda_c$ [14] (see Fig. 1), where

$$\lambda_c(h)=\begin{cases}2 & \text{if}\quad h\geqslant 1,\\ \dfrac{2}{h} & \text{if}\quad h\leqslant 1.\end{cases} \tag{14}$$

Indeed, it is not hard to see that

$$\Lambda_q^+(\lambda\leqslant\lambda_c)>0,\quad \Lambda_q^-(\lambda\leqslant\lambda_c)<0. \tag{15}$$

Hence the ground state is unchanged with respect to the pure Ising model ground state as long as the driving field does not exceed $\lambda_c(h)$. This means that all observables will assume their Ising model values and no energy current will be flowing through the chain.

However, for $\lambda\geqslant\lambda_c$ one may see that Λ_q^+ (Λ_q^-) changes sign over the interval $I_2=[q_-,q_+]$ $(I_4=[-q_+,-q_-])$. The explicit expression for the wave vectors $q_\pm$ is deduced from

$$\cos q_\pm=\frac{-4\pm\sqrt{(\lambda^2h^2-4)(\lambda^2-4)}}{\lambda^2 h},\quad -\pi\leqslant q_\pm\leqslant 0. \tag{16}$$

Beyond the critical drive, the excitation spectrum gives rise to a ground state that breaks the left-right symmetry and, indeed, it may be verified [14] that a nonzero energy flux is present in the chain with the explicit form of the flux given by

$$j_E=\frac{h}{4\pi\lambda^2}\sqrt{\left(\lambda^2-\frac{4}{h^2}\right)(\lambda^2-4)}. \tag{17}$$

For later applications, we specify the $j_E = j_E(\lambda)$ function in the vicinity of the critical drive, as $\lambda \to \lambda_c^+$,

$$j_E \approx \begin{cases} \dfrac{\sqrt{h^2-1}}{4\pi}\sqrt{\lambda-2} & \text{for } h>1, \quad \lambda_c=2, \\ \dfrac{1}{4\pi}(\lambda-2) & \text{for } h=1, \quad \lambda_c=2, \\ \dfrac{h^{3/2}\sqrt{1-h^2}}{4\pi}\sqrt{\lambda-\dfrac{2}{h}} & \text{for } h<1, \quad \lambda_c=\dfrac{2}{h}. \end{cases} \tag{18}$$

Besides, as they will naturally arise in the upcoming discussion, we further define here intervals $I_1=[-\pi,q_-]$, $I_3=[q_+,-q_+]$, and $I_5=[-q_-,\pi]$ which are complementary to I_2 and I_4 in $[-\pi,\pi]$.

C. Inverse of the scattering matrix in the absence or presence of energy flux

When no current flows through the system, $j_E=0$, the equal-time transform of $(B_{q,\omega})^{-1}$, denoted by B_q^{-1}, reads

$$B_q^{-1} = \begin{pmatrix} \dfrac{i\frac{1}{2}\sin q}{2\Lambda_q} & \dfrac{\frac{h}{2}+\frac{1}{2}\cos q+\Lambda_q}{2\Lambda_q} \\ -\dfrac{\frac{h}{2}+\frac{1}{2}\cos q+\Lambda_q}{2\Lambda_q} & -\dfrac{i\frac{1}{2}\sin q}{2\Lambda_q} \end{pmatrix}. \tag{19}$$

For $j_E \neq 0$, on the other hand, B_q^{-1} is given by Eq. (19) only for $q \in I_1 \cup I_3 \cup I_5$. If $q \in I_2 \cup I_4$, its expression is changed to

$$I_2 : B_q^{-1} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad I_4 : B_q^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

Having the expressions for B_q^{-1}, we can start the calculation of the distribution function $P(M_z)$.

III. DISTRIBUTION FUNCTION FOR THE TRANSVERSE MAGNETIZATION

A. Calculation of the generating function and its moments

The generating function (7) of $P(M_z)$ can be expressed through fermionic operators as

$$G(s) = \langle e^{-sM_z}\rangle = e^{Ns/2}\left\langle \exp\left(-s\sum_q c_q^\dagger c_q\right)\right\rangle. \tag{21}$$

After normal ordering $e^{-sc_q^\dagger c_q}$ and using the Grassmann fields, we are left with evaluating the following expression:

$$G(s) = e^{N\frac{s}{2}}\left\langle \exp\left(-\tilde{s}\sum_q \bar{c}_q c_q\right)\right\rangle, \quad \tilde{s}=1-e^{-s} \tag{22}$$

and the fields in the exponentials are evaluated at some fixed time. In order to evaluate Eq. (22), we recall the following result for Grassmann integrals [20]: the vacuum expectation value of observables of form $\exp(-\Sigma_q z_q \bar{c}_q c_q)$ can be obtained as

$$\left\langle \exp\left(-\sum_q z_q \bar{c}_q c_q\right)\right\rangle = \prod_q \sqrt{\det\begin{pmatrix} -\sqrt{z_q z_{-q}}(B_q^{-1})_{11} & -1+z_{-q}(B_q^{-1})_{12} \\ 1+z_q(B_q^{-1})_{21} & -\sqrt{z_q z_{-q}}(B_q^{-1})_{22} \end{pmatrix}}. \tag{23}$$

As one can see $G(s)$ is a special case of Eq. (23) and it can be evaluated by using the appropriate expressions (19) or (20) for B_q^{-1}.

If no current flows, that is, for $\lambda \leq \lambda_c$, we find that $\ln G$ (the cumulant generating function) is given by

$$\ln G(s) = \frac{N}{2}\int_{-\pi}^{\pi}\frac{dq}{2\pi}\ln[(1-n_q)e^s + n_q e^{-s}], \tag{24}$$

where

$$n_q = (h+\cos q + 2\Lambda_q)/(4\Lambda_q). \tag{25}$$

One can verify that the normalization condition $G(0)=1$ is satisfied, and one can also recover the well-known result found by Pfeuty [21] for the magnetization

$$-\frac{\partial \ln G(s)}{\partial s}\Big|_0 = \langle M_z\rangle = N\int_{-\pi}^{\pi}\frac{dq}{2\pi}\left(n_q - \frac{1}{2}\right). \tag{26}$$

As we shall see below, $P(M_z)$ is a Gaussian thus, in addition to $\langle M_z\rangle$, the variance of the transverse magnetization $N\sigma^2(h) = \langle M_z^2\rangle - \langle M_z\rangle^2$ will characterize the distribution. It can be obtained from the second derivative of $\ln G(s)$ as

$$\sigma^2(h) = 2\int_{-\pi}^{\pi}\frac{dq}{2\pi}n_q(1-n_q). \tag{27}$$

It is interesting to note that the fluctuations in M_z are independent of the magnetic field in the ordered phase

$$\sigma^2(h) = \begin{cases} 1/4 & \text{for } |h| \leq 1, \\ 1/(4h^2) & \text{for } |h|>1. \end{cases} \tag{28}$$

In the presence of nonzero energy flux ($\lambda > \lambda_c$), the generating function is more complicated only because of the limits of integration in Eq. (24),

$$\ln G_\lambda(s) = \frac{N}{2}\int_{I_1\cup I_3\cup I_5}\frac{dq}{2\pi}\ln[(1-n_q)e^s + n_q e^{-s}]. \tag{29}$$

Accordingly, the first and second cumulants of the magnetization are given by

$$\langle M_z\rangle_\lambda = \frac{N}{2}\int_{I_1\cup I_3\cup I_5}\frac{dq}{2\pi}\frac{h+\cos q}{2\Lambda_q}, \tag{30}$$

$$\sigma_\lambda^2(h)=\frac{1}{2}\int_{I_1\cup I_3\cup I_5}\frac{dq}{2\pi}\,\frac{\sin^2 q}{1+h^2+2h\cos q}, \tag{31}$$

where we have used Eq. (25) to write out the integrands explicitly. Note that we added a λ subscript to G, $\langle M_z\rangle$, and σ^2 in order to indicate that these quantities do depend on λ for $\lambda>\lambda_c$.

B. The transverse magnetization distribution is a Gaussian

In order to show that the transverse magnetization distribution is a Gaussian, let us consider the nth cumulant $\langle M_z^n\rangle_c$ which is the coefficient in front of $(-s)^n/n!$ in the expansion of $\ln G(s)$. The latter coefficient is N times an integral of a polynomial in n_q, both in the current-carrying and current-free phases and, furthermore, n_q is a nonsingular, strictly positive function of q (note that n_q is finite even at $h=1$). Hence each cumulant depends linearly on N. In particular, as we have seen, the variance of M_z denoted by $N\sigma^2$ and σ^2 [in Eq. (31) or Eq. (27)] is finite.

It follows from the linear N dependence of the cumulants that

$$\frac{\langle M_z^n\rangle_c}{\langle M_z\rangle_c^{n/2}}\sim\frac{N}{N^{n/2}}\sim N^{(2-n)/2}, \tag{32}$$

thus the above ratio goes to zero for all $n>2$ as $N\to\infty$. We may therefore conclude that the limiting form of the distribution function of the transverse magnetization is a Gaussian of variance $N\sigma^2$:

$$P(M_z)=\frac{1}{\sqrt{2\pi N\sigma^2}}e^{-(M_z-\langle M_z\rangle)^2/2N\sigma^2}. \tag{33}$$

The above result applies over the whole phase diagram and it is useful to check the numerical procedure employed in Sec. IV by evaluating $P(M_z)$ for finite-size systems. As can be seen in Fig. 2, there is a convergence to the limiting form with increasing N and, furthermore, the nearly Gaussian fluctuations of M_z are observed already at small sizes ($N=16-20$). It is remarkable that the deviations from the Gaussian are very close to those of an N step random walk with a drift determined from a correspondence between the left (right) moves and the up (down) spins generating the average $\langle M_z\rangle$.

C. Width of the Gaussian

Recalling the expression of the average and the variance, Eqs. (26) and (27), we calculate them in the limit of vanishing flux ($\lambda\to\lambda_c^+$). Let us define

$$\delta M_z=\langle M_z\rangle_\lambda-\langle M_z\rangle,\quad \delta\sigma^2=\sigma_\lambda^2(h)-\sigma^2(h). \tag{34}$$

The above quantities exhibit singular behavior as one enters the current-carrying phase. For the magnetization we find

$$\delta M_z=-N\int_{q_-}^{q_+}\frac{dq}{2\pi}\,\frac{h+\cos q}{\Lambda_q}, \tag{35}$$

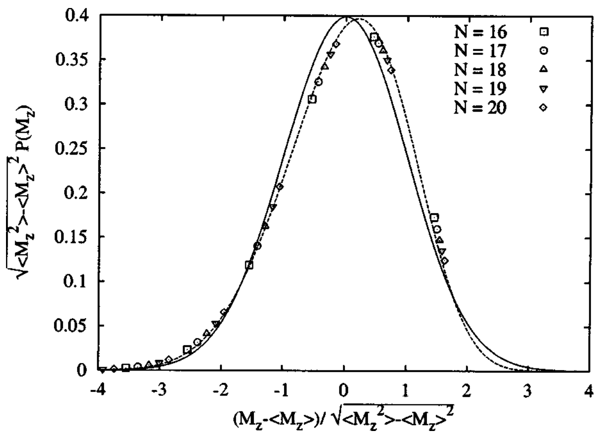

FIG. 2. Distribution function $P(M_z)$ for the transverse magnetization M_z on the critical line $h=h_c=1$, $\lambda\leq 2$. Results for periodic boundary conditions are displayed. The solid line is the asymptotic Gaussian while the dashed line is the displacement distribution of a 20-step random walk of step length 1/2 having a drift generating an average displacement equal to $\langle M_z\rangle$.

$$\delta M_z=\begin{cases}\dfrac{-1}{\pi}(1-h^{-2})(\lambda-\lambda_c)^{1/2} & \text{for}\quad h>1,\quad \lambda_c=2,\\[2mm] \dfrac{-1}{\pi}(\lambda-\lambda_c) & \text{for}\quad h=1,\quad \lambda_c=2,\\[2mm] \dfrac{-1}{\pi}h^{5/2}(\lambda-\lambda_c)^{3/2} & \text{for}\quad h<1,\quad \lambda_c=\dfrac{2}{h},\end{cases} \tag{36}$$

and for the variance

$$h\geq 1,\quad \lambda_c=2,\quad \delta\sigma^2\simeq-\frac{1}{\pi h^2}\sqrt{\lambda-\lambda_c},$$

$$h\leq 1,\quad \lambda_c=\frac{2}{h},\quad \delta\sigma^2\simeq-\frac{\sqrt{h}}{\pi}\sqrt{\lambda-\lambda_c}. \tag{37}$$

Using Eq. (18) we find, as $j_E\to 0$,

$$h\neq 1,\quad \delta\sigma^2\simeq -j_E, \tag{38}$$

$$h=1,\quad \delta\sigma^2\simeq -j_E^{1/2}.$$

As can be seen, the variance of the transverse magnetization is smaller in the current-carrying phase than in the current-free phase. This supports the view that imposing a current stiffens the system, and thus decreases fluctuations.

IV. DISTRIBUTION FUNCTION FOR THE LONGITUDINAL MAGNETIZATION

The exact evaluation of $P(M_x)$ appears to be a nontrivial task and we have been able to calculate it only numerically for finite-size chains. Since expression (6) for $P(M_x)$ is a ground-state expectation value, we had to find the ground-state wave function and, due to the sparseness of the Hamiltonian matrix, the Lanczos algorithm [22] could be used ef-

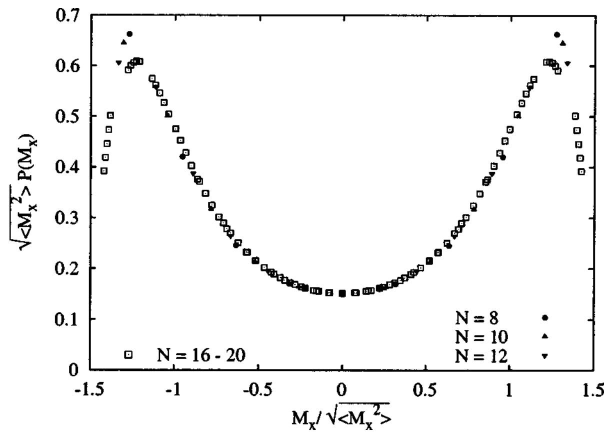

FIG. 3. Scaling function for the distribution $P(M_x)$ of the longitudinal magnetization M_x on the critical line $h=h_c=1$, $\lambda \leqslant 2$, for periodic boundary conditions. In order to demonstrate the smallness of finite-size effects, the small systems ($N=8,10,12$) are displayed by full symbols while the larger systems ($N=16-20$) are all shown by a single empty symbol.

fectively. Since the ground-state wave function is needed with precision, we were able to accomplish this task for chain lengths of up to $N=20$ with the results displayed on Figs. 3–5.

A. Equilibrium distribution

In the equilibrium system ($j_E=0$), the correlation length is infinite only at the critical point. Thus one expects $P(M_x)$ to be a Gaussian for $h>1$, a sum of two Gaussians for $h<1$, and a nontrivial distribution emerges only at criticality ($h=h_c=1$). This is indeed what we observe, apart from the finite-size effects showing up in non-Gaussian corrections close to $h=1$. At the critical point itself, $P(M_x)$ shows fast convergence to the asymptotic form as can be seen in Fig. 3 where the $N \geqslant 16$ points appear to have settled on the asymptotic curve. This means that the N dependence of $P(M_x)$ is almost all in the scaling variable $M_x/\sqrt{\langle M_x^2 \rangle}$,

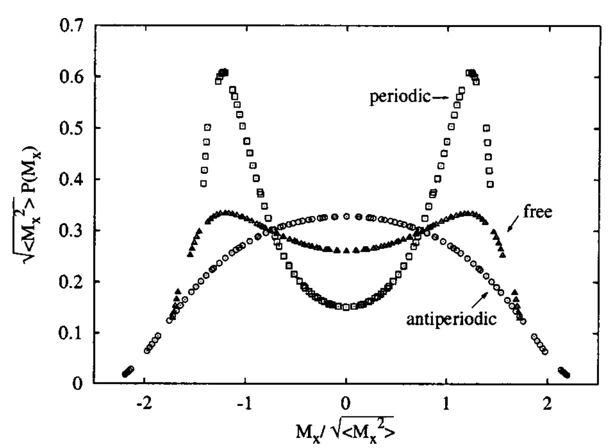

FIG. 4. Distribution function $P(M_x)$ for the longitudinal magnetization M_x at the critical line $h=h_c=1$, $\lambda \leqslant 2$. Results for periodic, free, and antiperiodic boundary conditions are displayed for system sizes $N=16$–20.

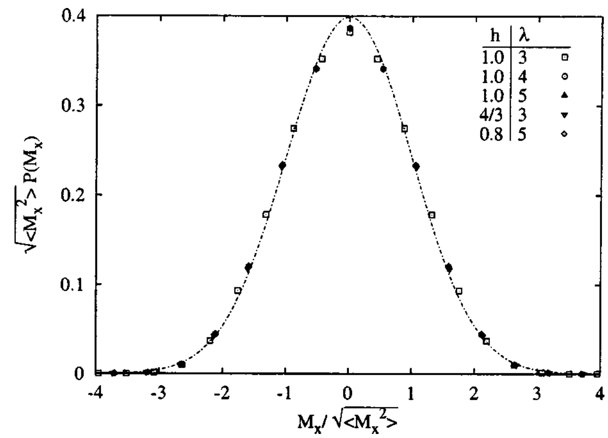

FIG. 5. Distribution function $P(M_x)$ for the longitudinal magnetization M_x away from the critical line $h=h_c=1$, $\lambda \leqslant 2$.

a remarkable feature that has been observed in a series of equilibrium- and nonequilibrium-critical states [8,9,11,23,24]. Note also that the finite-size effects show up mainly in the large $M_x/\sqrt{\langle M_x^2 \rangle}$ region. This is in accord with the general observation that the large-argument region of the scaling function is related to the long-wavelength properties of the system.

Figure 4 shows the critical point scaling functions for various boundary conditions (periodic $s_{N+1}^{\alpha}=s_1^{\alpha}$, antiperiodic $s_{N+1}^{\alpha}=-s_1^{\alpha}$, and free). One can observe here not only the strongly non-Gaussian character of the distributions, but also the fact that scaling functions do vary with changing the boundary conditions. The boundary condition dependence of the critical scaling functions is known [25–28]. It is also known that the scaling functions depend on the shape of the system as well. In case of the $d=2$ Ising model, this means that the scaling function depends on the aspect ratio a of a rectangular sample. Since the transverse Ising model has its origin in the transfer matrix of the $d=2$ Ising model in an anisotropic limit [26], we speculate that the distribution functions displayed in Fig. 4 are equal to the $d=2$ critical order parameter distributions in the $a\to 0$ limit with the boundary conditions in the "short" direction being in the $d=1$ and $d=2$ systems. Implicit in this belief is the assumption that the boundary conditions in the "long" direction do not affect the scaling function provided $a\to 0$.

B. Nonequilibrium distribution

In the nonequilibrium case ($j_E \neq 0$), we find that similarly to M_z, the fluctuations of the longitudinal magnetization M_x are also Gaussian (Fig. 5). Remarkably, the finite-size, non-Gaussian corrections are small even near the $h=1$, $\lambda=2$ point.

The Gaussian result is somewhat surprising since the flux generates power-law correlations in $\langle s_0^x s_n^x \rangle$ decaying with distance as $1/\sqrt{n}$ [14]. Thus one may presume that the current-carrying states are effectively critical. This is not so, however, since the correlations are oscillating and the fluctuations $\langle M_x^2 \rangle$ (given by the integral of the correlations) are finite. Actually $\langle M_x^2 \rangle$ is decreasing with increasing j_E (see below) and thus we see here another example of fluxes generating power-law correlations but, nevertheless, making the system more rigid.

The decrease of $\langle M_x^2\rangle$ with increasing j_E can be seen in the numerical studies of the finite spin chains. On the self-dual line ($h=1$), however, this can be shown analytically in the limit of vanishing flux ($\lambda\to\lambda_c=2$), where one finds

$$\langle M_x^2\rangle \propto N j_E^{-3/8}. \tag{39}$$

The derivation of the above result is possible because the ($h=1$, $\lambda=\lambda_c$) point is a critical point with infinite correlation length. Approaching this point along the $h=1$, $\lambda\to\lambda_c^+$ line, one can observe from numerical studies [14] that the wavelength κ^{-1} of the oscillation of the correlation function $C_x(r)=\langle s_i^x s_{i+r}^x\rangle \propto \cos\kappa r$ diverges as $\kappa^{-1}\propto(\lambda-\lambda_c)^{-1/2}$. This diverging wavelength allows one to take a continuum limit and to establish [33] to all orders in perturbation theory, that the order-parameter correlations possess the following scaling form:

$$C_x(r)\approx \frac{\mathcal{A}}{r^{1/4}}\mathcal{F}(\kappa r), \tag{40}$$

with the small argument limit of the scaling function explicitly given by $\mathcal{F}(x)=1-x^2/2+x^4/16+O(x^6)$.

The derivation of the above results follows the ideas [29–32] used for the calculation of the order-parameter fluctuations in the equilibrium transverse Ising model; namely, the correlations are expressed as a Pfaffian of a block Toeplitz matrix constructed of 2×2 matrices. Then, in the equilibrium case, the analysis of the Toeplitz matrices in the asymptotic limit of $r\to\infty$, and $h\to1$ with $r(h-1)$ kept fixed, yields $C_x^{eq}(r)\sim r^{-1/4}F(r(h-1))$. A similar asymptotic analysis in the current-carrying phase, using the scaling limit $r\to\infty$, and $\kappa\sim(\lambda-\lambda_c)^{1/2}\to0$ with κr kept finite, results in Eq. (40). The derivation is rather technical and we shall present it, together with the analysis of other scaling limits, in a separate publication [33].

Once the correlations are known, the fluctuations can be calculated from

$$\langle M_x^2\rangle = N\left[1+2\sum_{r\geqslant1} C_x(r)\right]\propto N\int_0^{+\infty} dr\, C_x(r), \tag{41}$$

where changing the sum into integral is again allowed because of the diverging characteristic length scale κ^{-1}. Using now the scaling form (40), we find that

$$\langle M_x^2\rangle \propto N\kappa^{-3/4}\propto N j_E^{-3/8}. \tag{42}$$

The above expression demonstrates the decrease of fluctuations with increasing flux and it also tells us how the Gaussian distribution crosses over to the nontrivial shape observed at the critical point.

V. FINAL REMARKS

Returning to the problems discussed in the Introduction, we can see that the connection between NESS and critical states in terms of (universal) distribution functions is not straightforward. NESS is generated by fluxes, and fluxes may or may not generate long-range correlations. There are numerous examples [1–3] where the fluxes are spatially localized and long-range correlations do not develop (unless the system is at a special point in the parameter space). Clearly, in such cases, one cannot hope for a general description to emerge. If the fluxes are global, as is the case for the model treated in the present paper, long-range correlations do emerge frequently [1–3]. Even in this case, however, it is far from trivial whether these correlations drive the system to an effectively critical state or whether they make the system more rigid.

The driven transverse Ising model treated above is an example where a global flux of energy generates long-range correlations but the resulting state becomes more rigid in the sense that the fluctuations are decreased due to the presence of the flux. Driven diffusive systems [1] provide other examples [34] where the fluctuations decrease while the fluxes induce power-law correlations. Thus we should conclude that, in general, the power-law correlations generated by global fluxes cannot be the source of possible universality of nonequilibrium distribution functions. It remains, however, an intriguing question whether the weak long-range interactions supported by global fluxes can underlie a kind of "weak" universality classification of distributions in NESS.

ACKNOWLEDGMENTS

We thank G. Györgyi, V. Hunyadi, and L. Sasvári for helpful discussions. This research was partially supported by the Hungarian Academy of Sciences (Grant No. OTKA T029792).

[1] B. Schmittmann and R.K.P. Zia, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic Press, New York, 1996).
[2] J. Marro and R. Dickman *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999).
[3] Z. Rácz, e-print cond-mat/0210435.
[4] B. Schmittmann, Europhys. Lett. **24**, 109 (1993).
[5] K.E. Bassler and Z. Rácz, Phys. Rev. Lett. **73**, 1320 (1994); Phys. Rev. E **52**, 9 (1995).
[6] U.C. Täuber, V.K. Akkineni, and J.E. Santos, Phys. Rev. Lett. **88**, 045702 (2002).
[7] M. Droz, Z. Rácz, and P. Tartaglia, Phys. Rev. A **41**, 6621 (1989); B. Bergersen and Z. Rácz, Phys. Rev. Lett. **67**, 3047

(1991).

[8] G. Foltin, K. Oerding, Z. Rácz, R.L. Workman, and R.K.P. Zia, Phys. Rev. E **50**, 3589 (1994); Z. Rácz and M. Plischke, *ibid.* **50**, 3530 (1994).

[9] T. Antal, M. Droz, G. Györgyi, and Z. Rácz, Phys. Rev. Lett. **87**, 240601 (2001); Phys. Rev. E **65**, 046140 (2002).

[10] S.T. Bramwell, P.C.W. Holdsworth, and J.-F. Pinton, Nature (London) **396**, 552 (1998).

[11] E. Marinari, A. Pagnani, G. Parisi, and Z. Rácz, Phys. Rev. E **65**, 026136 (2002).

[12] G. Korniss, Z. Toroczkai, M.A. Novotny, and P.A. Rikvold, Phys. Rev. Lett. **84**, 1351 (2000).

[13] G. Tripathy and W. van Saarloos, Phys. Rev. Lett. **85**, 3556 (2000).

[14] T. Antal, Z. Rácz and L. Sasvári, Phys. Rev. Lett. **78**, 167 (1997).

[15] T. Antal, Z. Rácz, A. Rákos, and G.M. Schütz, Phys. Rev. E **57**, 5184 (1998).

[16] T. Antal, Z. Rácz, A. Rákos, and G.M. Schütz, Phys. Rev. E **59**, 4912 (1999); Z. Rácz, J. Stat. Phys. **101**, 273 (2000).

[17] J. Cardy and P. Suranyi, Nucl. Phys. B **565**, 487 (2000).

[18] H.A. Kramers and G.H. Wannier, Phys. Rev. **60**, 252 (1941).

[19] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) **16**, 403 (1961).

[20] J.W. Negele and H. Orland, *Quantum Many-Particle Systems* (Perseus Books, Cambridge, MA, 1998).

[21] P. Pfeuty, Ann. Phys. (N.Y.) **57**, 79 (1970).

[22] C. Lanczos, J. Res. Natl. Bur. Stand. **45**, 255 (1950); J.K. Cullum and R.A. Willoughby, *Lanczos Algorithms for Large Symmetric Eigenvalue Computations* (Birkhäuser, Boston 1985).

[23] P. Archambault, S.T Bramwell, and P.C.W. Holdsworth, J. Phys. A **35**, 1231 (2002).

[24] S.T. Bramwell, J.-Y. Fortin, P.C.W. Holdsworth, S. Peysson, J.-F. Pinton, B. Portelli, and M. Sellitto, Phys. Rev. E **63**, 041106 (2001).

[25] V. Privman, in *Finite Size Scaling and Numerical Simulation of Statistical Physics*, edited by V. Privman (World Scientific, Singapore, 1990).

[26] B.M. McCoy and T.T. Wu, *The Two Dimensional Ising Model* (Harvard University Press, Cambridge, MA, 1973).

[27] Exotic boundary conditions (Moebius strip, Klein bottle) for the $d=2$ Ising model have been studied in K. Kaneda and Y. Okabe, Phys. Rev. Lett. **86**, 2134 (2001); W.T. Lu and F.Y. Wu, Phys. Rev. E **63**, 026107 (2001).

[28] C.-K. Hu, C.-Y. Lin, and J.-A. Chen, Phys. Rev. Lett. **75**, 193 (1995).

[29] E. Barouch and B.M. McCoy, Phys. Rev. A **2**, 786 (1971).

[30] M. Bander and C. Itzykson, Phys. Rev. D **15**, 463 (1977).

[31] T.T. Wu, B. McCoy, C.A. Tracy, and E. Barouch, Phys. Rev. B **13**, 316 (1976).

[32] T.T. Wu, Phys. Rev. **149**, 380 (1966).

[33] V. Eisler and F. van Wijland (unpublished).

[34] B. Derrida, J.L. Lebowitz, and E.R. Speer, J. Stat. Phys. **107**, 599 (2002).