Exact solutions to nonlinear nonautonomous space-fractional diffusion equations with absorption

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We analyze a nonlinear fractional diffusion equation with absorption by employing fractional spatial derivatives and obtain some more exact classes of solutions. In particular, the diffusion equation employed here extends some known diffusion equations such as the porous medium equation and the thin film equation. We also discuss some implications by considering a diffusion coefficient $\mathcal{D}(x,t)=\mathcal{D}(t)|x|^{-\theta}$ ($\theta\in\mathcal{R}$) and a drift force $F=-k_1(t)x+k_\alpha x|x|^{\alpha-1}$. In both situations, we relate our solutions to those obtained within the maximum entropy principle by using the Tsallis entropy.

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I. INTRODUCTION

Recently the nonlinear and fractional diffusion equations have received a lot of attention. In fact, they have been applied in several situations such as percolation of gases through porous media [1], thin saturated regions in porous media [2], a standard solid-on-solid model for surface growth, thin liquid films spreading under gravity [3], modeling of nonMarkovian dynamical processes in protein folding [4], relaxation to equilibrium in a system (such as polymer chains and membranes) with long temporal memory [5], and anomalous transport in disordered systems [6]. A representative nonlinear diffusion equation that is usually employed in the above context is

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\partial}{\partial x}\left\{\mathcal{D}(x,\rho)\frac{\partial}{\partial x}\rho(x,t)\right\}. \tag{1}$$

For the particular case, $D(x,\rho)=\mathcal{D}\nu\rho^{\nu-1}$ is sometimes referred to as the *porous medium equation* and has been intensively studied in the literature [7,8] as well as its connection with the nonextensive statistics [9]. We may also have the high-order diffusionlike equation such as the thin film equation [10]

$$\frac{\partial}{\partial t}\rho(x,t)=-\mathcal{D}\frac{\partial}{\partial x}\left\{[\rho(x,t)]^{\gamma}\frac{\partial^3}{\partial x^3}\rho(x,t)\right\}, \tag{2}$$

which contains a fourth-order derivative. It can be applied to describe the lubrication models for thin viscous films, spreading droplets, and Hele-Shaw cells [11]. In addition to the context mentioned above, the fractional equations have also been employed to investigate the situations related to the anomalous diffusion [12–14]. By unifying the spatial fractional diffusion equation and the porous medium equation, we have that

$$\frac{\partial}{\partial t}\rho(x,t)=\mathcal{D}\frac{\partial^{\mu}}{\partial|x|^{\mu}}[\rho(x,t)]^{\nu}, \tag{3}$$

where $\partial^{\mu}/\partial|x|^{\mu}$ is a Riemann-Liouville fractional derivative [15]. For the particular case $\nu=1$, the Levy distributions are the solution to Eq. (3). For the case $\nu\neq1$, solutions and the connection between Eq. (3) and the nonextensive statistics have been investigated in Ref. [17].

The physical situations mentioned in above essentially concern anomalous diffusion of the correlated type (both subdiffusion and superdiffusion; see Ref. [18], and references therein) or of the Lévy type (superdiffusion; see Ref. [19], and references therein). Anomalous correlated diffusion has a finite second moment $\langle x^2\rangle\propto t^{\sigma}$ ($\sigma>1$, $\sigma=1$, and $0<\sigma<1$, respectively, correspond to superdiffusion, normal diffusion, and subdiffusion; $\sigma=0$ basically corresponds to localization). The second type is essentially characterized by Lévy distributions and, consequently, it has no finite second moment, i.e. $\langle x^2\rangle$ diverges.

Due to the broadness of the physical situations that these previous equations are able to describe, it is interesting to know more about equations related to various types of anomalous diffusion, their properties, solutions, and connections with extensive [20] or nonextensive [9] statistics. In this direction, we have, for example, complex systems such as the displacement of a viscous fluid by a less viscous one in a petroleum reservoir, which requires a more general approach in order to take the nonlinear behavior of the interface into account, and also the fractal or multifractal characteristics of porous rocks in which the oil is immersed. In particular, the geostatistics of these reservoirs are well described by a fractional Brownian motion and fractional Levy motion [21]. In order to accomplish the above situations in a unified scenario, we dedicate the present work to establish some classes of solutions of a general nonlinear fractional diffusion equation with absorption; we also investigate connections with the usual or generalized thermostatistics. More precisely, we focus our attention on the following generalized equation:

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\partial}{\partial|x|}\left\{\mathcal{D}(x,t)[\rho(x,t)]^{\gamma}\frac{\partial^{\mu-1}}{\partial|x|^{\mu-1}}[\rho(x,t)]^{\nu}\right\}-\frac{\partial}{\partial x}\{F(x)\rho(x,t)\}+\alpha(t)[\rho(x,t)]^{\mu'}, \tag{4}$$

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where $\nu,\gamma,\theta,\mu',\mu\in\mathcal{R}$, $\mathcal{D}(x,t)=\mathcal{D}(t)|x|^{-\theta}$ is a (dimensionless) diffusion coefficient, $F(x)\equiv-dV(x)/dx$ is a (dimensionless) external force (*drift*) associated with the potential $V(x)$, and $\alpha(t)$ plays the role of an absorbent $[\alpha(t)<0]$ (or source $[\alpha(t)>0]$) rate related to a reaction process. The presence of the reaction term as that in the above equation has been studied in several situations. Here, for example, we may recall the so-called one-species coagulation, that is, $A+A\to 0$ or $mA\to lA$ $(m>l)$, catalytic processes in regular, heterogeneous, or disordered systems [22]. Another example is an irreversible first-order reaction of the transported substance so that the rate of removal is $\alpha\rho$ [23]. This extra term may also appear when a tracer undergoing radioactive decay is transported through a porous medium [24] and in heat flow involving heat production [25]. In particular, in these situations and in solute transport through adsorbent samples, which are usually proportional to the concentration in the solution, Eq. (4) applies.

For $\alpha(t)=0$, it can be verified that $\int_{-\infty}^{\infty}dx\rho(x,t)$ is time independent (hence, if ρ is normalized at $t=0$, it will remain so for ever). Indeed, if we write the equation in the $\partial_t\rho=\partial_x\mathcal{J}$ form and assume the boundary conditions $\mathcal{J}(\pm\infty,t)\to 0$, it can be shown that $\int_{-\infty}^{\infty}dx\rho(x,t)$ is a constant of motion. Following Ref. [17], we use the Riemann-Liouville operator [12,13,15,16] and we work with the *positive x* axis. Later on, we will use symmetry to extend the results to the entire real axis (we are working, in other words, with $\partial^{\mu-1}/\partial|x|^{\mu-1}$). Also, we employ the initial condition $\rho(x,0)=\delta(x)$ and the boundary condition $\rho(x\to\pm\infty,t)\to 0$. Note that Eq. (4) recovers, for $(\mu,\gamma,\theta,\nu)=(2,0,0,1)$, the standard Fokker-Planck equation in the presence of a drift. The particular case $F(x)=0$ (no drift), $\mathcal{D}(x,t)=\text{const}$ and $(\mu,\theta,\gamma)=(2,0,0)$ has been considered by Spohn [8]. Other situations of $(\mu,\theta,\gamma)=(2,0,0)$ have also been considered in Refs. [26,27]. The $(\theta,\gamma)=(0,0)$ case without drift was investigated in Ref. [17]. Our present discussion involves extensions of these cases taking a wide variety of situations into account by employing the nonlinear diffusion equation, the fractional diffusion equation and the mixing of these cases. In Sec. II, we consider several situations for Eq. (4) as well as the connection of the solutions with the ones obtained within the maximum entropy principle. Later on, in Sec. III, we present our conclusions.

II. DIFFUSION EQUATION WITH ABSORPTION

Let us start by emphasizing that an essential point of our discussion is the scaled solutions of the type

$$\rho(x,t)=\frac{1}{\Phi(t)}\tilde{\rho}\left[\frac{x}{\Phi(t)}\right] \tag{5}$$

for Eq. (4) which satisfy the initial and the boundary conditions. For example, to reobtain the case discussed by Spohn [8], by using this ansatz, we insert Eq. (5) into Eq. (4) with $(\gamma,\theta,\mu)=(0,0,2)$, $F(x)=0$, $\alpha(t)=0$, and $\mathcal{D}(x,t)=\mathcal{D}=\text{const}$. This procedure leads to

$$-\Phi(t)^{\nu}\dot{\Phi}(t)\frac{d}{dz}[z\tilde{\rho}(z)]=\mathcal{D}\frac{d^2}{dz^2}[\tilde{\rho}(z)]^{\nu}, \tag{6}$$

with $z\equiv|x|/\Phi(t)$. A solution to this equation may be obtained if we choose

$$\frac{[\Phi(t)]^{\nu}}{\mathcal{D}}\frac{d}{dt}\Phi(t)=k, \tag{7}$$

hence

$$\Phi(t)=[(1+\nu)\mathcal{D}kt]^{1/(1+\nu)}, \tag{8}$$

where we have adopted the solution that satisfies $\Phi(0)=0$. This yields

$$\rho(x,t)=\frac{1}{\Phi(t)}\exp_q\left[-\frac{k}{2\nu}\left(\frac{x}{\Phi(t)}\right)^2\right]. \tag{9}$$

The constant k can be obtained from the normalization condition $\int_{-\infty}^{\infty}dx\rho(x,t)=1$. Furthermore, $q=2-\nu$ and $\exp_q(x)\equiv[1+(1-q)x]^{1/(1-q)}$ is the q-exponential function that arises within the nonextensive thermostatistical formalism by optimizing, under appropriate constraints, the entropic form [9,28]

$$S_q=\frac{1-\int dx[\rho(x,t)]^q}{q-1}. \tag{10}$$

Before continuing our discussion on the nonlinear fractional diffusion equation, it is convenient to make some comments about the nonextensive entropy S_q. This entropy (Tsallis entropy) was employed for the first time in connection with a nonextensive statistical mechanics by Tsallis [28]. It has a real parameter q that informs us the degree of nonextensivity and in the limit $q\to 1$ the usual entropy is recovered. By using Eq. (10), several situations have been investigated [9] focusing formal developments as well as applications.

Let us now extend the above result, Eq. (9), for a spatial and time dependent diffusion coefficient $D(x,t)$, i.e., we assume $D(x,t)=\mathcal{D}(t)|x|^{-\theta}$ $(\theta\in\mathcal{R})$, by considering a drift term $F(x,t)=-k_1(t)x$ and a source (or absorbent) term $\alpha(t)\rho(x,t)$. In this case, the solution to Eq. (4) is given by $\rho(x,t)=\exp[\int_0^t d\tilde{t}\alpha(\tilde{t})]\hat{\rho}(x,t)$, where $\hat{\rho}(x,t)$ can be expressed in terms of the stretched q exponential

$$\hat{\rho}(x,t)=\frac{1}{\tilde{\Phi}(t)}\exp_q\left[-\frac{k''}{\nu(2+\theta)}\left(\frac{|x|}{\tilde{\Phi}(t)}\right)^{2+\theta}\right], \tag{11}$$

with

$$\frac{\tilde{\Phi}(t)}{\tilde{\Phi}(0)}=\exp\left[-\int_0^t d\tilde{t}k_1(\tilde{t})\right]\left[1+\frac{(1+\nu+\theta)k''}{[\tilde{\Phi}(0)]^{1+\nu+\theta}}\times\int_0^t d\tilde{t}\mathcal{D}(\tilde{t})\exp\left\{-\int_0^{\tilde{t}}dt'[(1-\nu)\alpha(t')-(1+\nu+\theta)k_1(t')]\right\}\right]^{1/(1+\nu+\theta)}, \quad (12)$$

where k'', which plays the same role as k in Eq. (9), may be fixed by the normalization condition. We are interested in the physical solutions that decay at long distances; consequently, it must be $\theta>-2$. Furthermore, we verify, for $\mathcal{D}(t)$ constant, that the cases $\theta+\nu>1$, $\theta+\nu=1$, and $\theta+\nu<1$, respectively, correspond to the subdiffusive, normal, and superdiffusive regimes for $(k_1(t),\alpha(t))=(0,0)$, i.e. $\langle x^2\rangle \propto t^{2/(1+\nu+\theta)}$.

We can also extend solution (9) by assuming now $F(x)=-k_1(t)x+k_\alpha x|x|^{\alpha-1}$ and $D(x,t)=\mathcal{D}|x|^{-\theta}$ without the source term. We do not know what happens in the general (α,θ,ν) arbitrary case, but there is a special situation for which the scaled solution of the type indicated in Eq. (5) is still valid. This special case corresponds to $\alpha=q-\theta-2$, i.e., $\alpha+\theta+\nu=0$. If this condition is satisfied, we obtain

$$\rho(x,t)=\frac{1}{\bar{\Phi}(t)}\exp_q\left[-\frac{1}{\nu}\left\{\frac{k'}{2+\theta}\left(\frac{|x|}{\bar{\Phi}(t)}\right)^{2+\theta}-k_\alpha\ln_{2-q}\left(\frac{|x|}{\bar{\Phi}(t)}\right)\right\}\right],$$

$$\bar{\Phi}(t)=\exp\left[-\int_0^t d\tilde{t}k_1(\tilde{t})\right]\left[[\bar{\Phi}(0)]^{1+\nu+\theta}+(1+\nu+\theta)k'\mathcal{D}\times\int_0^t d\tilde{t}\exp\left[(1+\nu+\theta)\int_0^{\tilde{t}}dt'k_1(t')\right]\right]^{1/(1+\nu+\theta)}, \quad (13)$$

where $\ln_q x\equiv(x^{1-q}-1)/(1-q)$ is the q-logarithm function (the inverse function of the q exponential) and k' is a constant that plays a role analogous to k in Eq. (9), and is to be determined through the normalization condition. As a last comment, let us mention that the distributions obtained above are precisely of the type that is obtained by optimizing S_q with the constraint $\langle\langle O(|x|)\rangle\rangle_q=\text{const}$, where the *normalized q*-expectation value is defined as

$$\langle\langle O(|x|)\rangle\rangle_q\equiv\left[\int dx O(|x|)[p(x)]^q\right]\bigg/\left[\int dx[p(x)]^q\right]. \quad (14)$$

In this context, O is essentially the argument of $\exp_q[\cdots]$ in the optimal $\rho(x)$ $(\rho(x)\propto\exp_q[\cdots])$.

Now, we analyze the following (vanishing drift) equation:

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\partial}{\partial|x|}\left\{\mathcal{D}(t)|x|^{-\theta}[\rho(x,t)]^\gamma\frac{\partial^{\mu-1}}{\partial|x|^{\mu-1}}[\rho(x,t)]^\nu\right\}+\alpha(t)\rho(x,t), \quad (15)$$

which unifies the corresponding ones appearing in Refs. [17,29–31]. The procedure employed here is essentially the same as used in Ref. [17]; besides, the case discussed there corresponds to $\mathcal{D}(t)$ constant, $(\theta,\gamma)=(0,0)$ and $\alpha(t)=0$. In this direction, we take the generic property,

$$\frac{d^\delta}{dx^\delta}\mathcal{G}(ax)=a^\delta\frac{d^\delta}{d\bar{z}^\delta}\mathcal{G}(\bar{z})\quad(\delta\in\mathcal{R}), \quad (16)$$

with $\bar{z}=ax$ into account. This basic property holds not only for the ordinary derivative, but also for all fractional operators, in particular, for the Riemann-Liouville one. Thus, substituting $\rho(x,t)=\exp[\int_0^t dt\alpha(t)]\hat{\rho}(x,t)$ in Eq. (15) and employing Eq. (5) for $\hat{\rho}(x,t)$, we obtain

$$-[\Phi(t)]^{\xi-2}\frac{d}{dt}\Phi(t)=\bar{k}\mathcal{D}(t)\exp\left[-(1-\nu-\gamma)\times\int_0^{\tilde{t}}dt'\alpha(t')\right], \quad (17)$$

where $\xi=\nu+\mu+\theta+\gamma$, $\bar{k}$ is an arbitrary constant, and

$$\frac{d}{dz}\left\{z^{-\theta}[\tilde{\rho}(z)]^\gamma\frac{d^{\mu-1}}{dz^{\mu-1}}[\tilde{\rho}(z)]^\nu\right\}=\bar{k}\frac{d}{dz}[z\tilde{\rho}(z)]. \quad (18)$$

By solving Eq. (17), we find

$$\Phi(t)=\left[[\Phi(0)]^{\xi-1}+k'\int_0^t d\tilde{t}\mathcal{D}(\tilde{t})\exp\left[-(1-\nu-\gamma)\times\int_0^{\tilde{t}}dt'\alpha(t')\right]\right]^{1/(\xi-1)}, \quad (19)$$

with $k'=(1-\xi)\bar{k}$. And making an integration in Eq. (18), we have that

$$z^{-\theta}[\tilde{\rho}(z)]^\gamma\frac{d^{\mu-1}}{dz^{\mu-1}}[\tilde{\rho}(z)]^\nu=\bar{k}z\tilde{\rho}(z)+\mathcal{C}, \quad (20)$$

where $\mathcal{C}$ is another arbitrary constant. Also, we use the following general result:

$${}_0D_x^\delta[x^\alpha(a+bx)^\beta]=a^\delta\frac{\Gamma[\alpha+1]}{\Gamma[\alpha+1-\delta]}x^{\alpha-\delta}(a+bx)^{\beta-\delta}, \quad (21)$$

with ${}_0D_x^\delta\equiv d^\delta/dx^\delta$, $\delta\equiv\alpha+\beta+1$, $\alpha>-1$, and $\beta+\alpha<-1$. By defining $g(x)\equiv x^{\alpha/\nu}(a+bx)^{\beta/\nu}$ and $\lambda\equiv\alpha(1-1/\nu)-\delta$, and rearranging the indices, Eq. (21) can be rewritten as follows:

$$ {}_0D_x^{\delta}[g(x)]^{\nu}=\frac{\Gamma[\alpha+1]}{\Gamma[\alpha+1-\delta]}a^{\delta}x^{\lambda}g(x). \tag{22}$$

Using this property in Eq. (20) and, for simplicity, choosing $\mathcal{C}=0$, we find

$$\frac{\alpha}{\nu}=\frac{(\mu+\theta)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)},$$

$$\frac{\beta}{\nu}=-\frac{(1-\mu)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)},$$

$$\nu=\frac{(2-\mu)(1-\gamma)}{1+\mu+\theta}. \tag{23}$$

Note that the above results recover those obtained in Ref. [17] for $\theta=0$ and $\gamma=0$ and in Ref. [31] for $\gamma=0$. These results allow us to write the solution in the form

$$\tilde{\rho}(x,t)=\frac{\mathcal{N}}{\Phi(t)}\left[\frac{z^{(\mu+\theta)}}{(1+bz)^{(1-\mu)}}\right]^{(1+\mu+\theta)/[(1-2\mu-\theta)(1-\gamma)]}, \tag{24}$$

with

$$\mathcal{N}=\left[\overline{k}\frac{\Gamma(-\beta)}{\Gamma(\alpha+1)}\right]^{1/(\nu+\gamma-1)} \quad \text{and} \quad z\equiv\frac{|x|}{\Phi(t)}, \tag{25}$$

where b is an arbitrary constant (to be taken, later on, as ± 1 according to the specific solutions to be studied).

We can extend the previous achievement by incorporating a linear drift $F(x)=-k_1(t)x$ into Eq. (15). In this case, $\tilde{\rho}(z)$ remains unchanged and we need only to change Eq. (19) to

$$\Phi(t)=\exp\left[-\int_0^t dt k_1(t)\right]\left[[\Phi(0)]^{\xi-1}+\tilde{k}\int_0^t d\tilde{t}\mathcal{D}(\tilde{t})\exp\left\{\int_0^{\tilde{t}}dt'[(\xi-1)k_1(t')-(1-\nu-\gamma)\alpha(t')]\right\}\right]^{1/(\xi-1)}, \tag{26}$$

in which $\tilde{k}=(1-\xi)\overline{k}.$

Several regions can be considered in this case. For simplicity, we consider $\alpha(t)=0$ and illustrate two of them: $-\infty<\mu<-1-|\theta|-|\gamma|$ with $\theta\geqslant 0$ and $0\leqslant\gamma<1$, and $0<\mu<1/2$ with $0\leqslant\theta<1/2-\mu$ and $0\leqslant\gamma<1/3$. Let us start by considering the region $-\infty<\mu<-1-|\theta|-|\gamma|$. Without loss of generality, we can choose $b=-1$. Thus, the normalization condition

$$\mathcal{N}\int_{-1}^{1}\left[\frac{z^{(\mu+\theta)(1+\mu+\theta)}}{(1-z)^{(1-\mu)(1+\mu+\theta)}}\right]^{1/[(1-2\mu-\theta)(1-\gamma)]}dz=1 \tag{27}$$

implies

$$\mathcal{N}=\frac{\Gamma\left(2-\dfrac{1+\mu+\theta}{1-\gamma}\right)}{2\Gamma\left(1+\dfrac{(\mu+\theta)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)}\right)\Gamma\left(\dfrac{(\mu-1)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)}+1\right)} \tag{28}$$

(see Fig. 1) and the second moment is $\langle x^2\rangle\propto[\Phi(t)]^2$. Let us now illustrate the $0<\mu<1/2$ region (where $b=1$). In this situation, the normalization implies

$$\mathcal{N}=\frac{\Gamma\left(\dfrac{(1-\mu)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)}\right)}{2\Gamma\left(1+\dfrac{(\mu+\theta)(1+\mu+\theta)}{(1-2\mu-\theta)(1-\gamma)}\right)\Gamma\left(\dfrac{\mu+\theta+\gamma}{1-\gamma}\right)} \tag{29}$$

(see Fig. 2).

We return to Eq. (4) to consider two different particular cases, namely $\mu=0$ and $\mu=1$ for $\alpha(t)=0$. The $\mu=2$ case was addressed in Ref. [29]. Let us start with $\mu=0$ and arbitrary ν. The corresponding equation is

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\partial}{\partial|x|}\left\{\mathcal{D}(t)|x|^{-\theta}[\rho(x,t)]^{\gamma}\int_0^x[\rho(y,t)]^{\nu}dy\right\}. \tag{30}$$

To solve it, let us go back to Eq. (18) and, after some simplifications, we obtain

$$\overline{k}z^{1+\theta}\tilde{\rho}(z)=[\tilde{\rho}(z)]^{\gamma}\int_0^z d\overline{z}[\tilde{\rho}(\overline{z})]^{\nu}, \tag{31}$$

whose solution is given by

$$\tilde{\rho}(z)\propto\frac{1}{z^{(1+\theta)/(1-\gamma)}}(1+\tilde{\mathcal{C}}z^{1-\nu(1+\theta)/(1-\gamma)})^{1/(1-\nu-\gamma)}, \tag{32}$$

where $\tilde{\mathcal{C}}$ is a constant.

Let us now address the $\mu=1$ case. It corresponds to the equation

$$\frac{\partial}{\partial t}\rho(x,t)=\frac{\partial}{\partial|x|}\{\mathcal{D}(t)|x|^{-\theta}[\rho(x,t)]^{\gamma+\nu}\}. \tag{33}$$

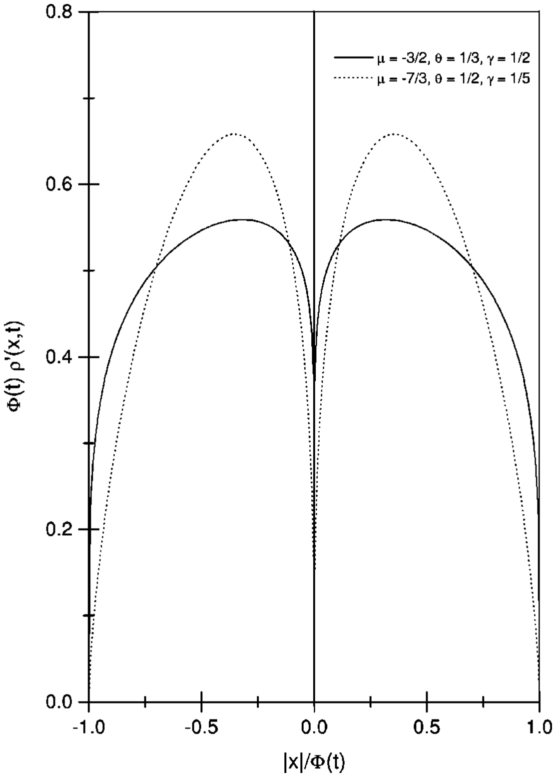

FIG. 1. Behavior of $\Phi(t)\tilde{\rho}(x,t)$ versus $x/\Phi(t)$, which illustrates Eq. (24) with typical values for μ, θ, and γ satisfying $\mu < -1-|\theta|-|\gamma|$, $\theta \geqslant 0$, and $0 \leqslant \gamma < 1$. We notice that the distribution vanishes at the abscissa equal to ± 1, and remains zero outside this interval.

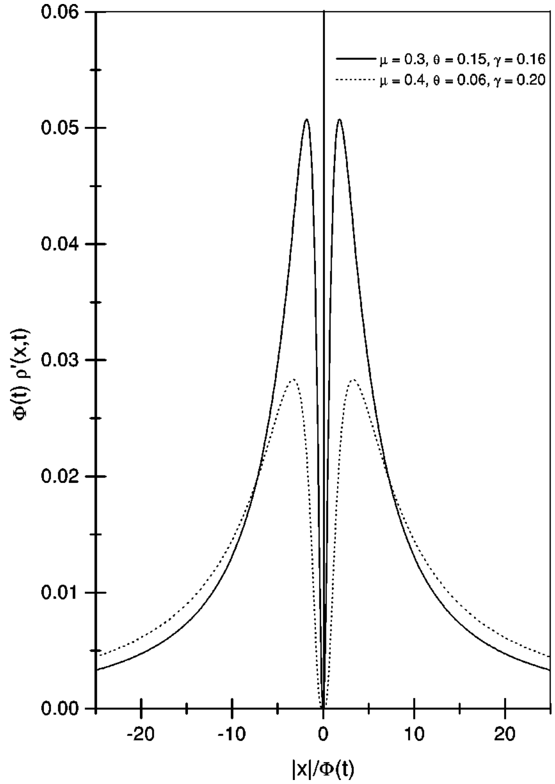

FIG. 2. Behavior of $\Phi(t)\tilde{\rho}(x,t)$ versus $x/\Phi(t)$, which illustrates Eq. (24) with typical values for μ, θ, and γ satisfying $0 < \mu < 1/2$, $0 \leqslant \theta < 1/2-\mu$, and $0 \leqslant \gamma < 1/3$.

To obtain its solution, it is convenient to go back to Eq. (18). It follows that

$$\overline{kz}\tilde{\rho}(z) = z^{-\theta}[\tilde{\rho}(z)]^{\gamma+\nu} + \overline{\mathcal{C}}, \tag{34}$$

which implicitly determines $\tilde{\rho}(z)$, where $\overline{\mathcal{C}}$ is a constant.

Now, we investigate the scaling behavior for the general case $\mu' \neq 1$ in Eq. (15) by considering the absence of external force and, for simplicity, $\alpha(t) = \alpha = \text{const}$ and $\mathcal{D}(x,t) = \mathcal{D}|x|^{-\theta}$, i.e., we analyze the following equation:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial |x|}\left\{ \mathcal{D}|x|^{-\theta}[\rho(x,t)]^{\gamma}\frac{\partial^{\mu-1}}{\partial |x|^{\mu-1}}[\rho(x,t)]^{\nu}\right\} + \alpha[\rho(x,t)]^{\mu'}. \tag{35}$$

To do this we consider another ansatz instead of Eq. (5), i.e., we employ the following ansatz: $\rho(x,t) = \varphi(t)\mathcal{P}(\zeta)$ with $\zeta = \phi(t)x$. Replacing this in Eq. (35) we obtain the functions $\varphi(t)$ and $\phi(t)$ as

$$\varphi(t) = [1+(1-\mu')\alpha t]^{1/(1-\mu')},$$

$$\phi(t) = \alpha^{1/(\theta+\mu)}[1+(1-\mu')\alpha t]^{(\mu'-\nu-\gamma)/[(1-\mu')(\theta+\mu)]}. \tag{36}$$

Therefore, Eq. (35) is reduced to an ordinary equation on the variable ζ:

$$\mathcal{P}(\zeta) + \frac{\mu'-\nu-\gamma}{\theta+\mu}\zeta\frac{d}{d\zeta}[\mathcal{P}(\zeta)] = \frac{d}{d|\zeta|}\left\{ \mathcal{D}|\zeta|^{-\theta}[\mathcal{P}(\zeta)]^{\gamma}\frac{d^{\mu-1}}{d|\zeta|^{\mu-1}}[\mathcal{P}(\zeta)]^{\nu}\right\} + [\mathcal{P}(\zeta)]^{\mu'}. \tag{37}$$

The above equation is complicated to be solved analytically; however, the nth moment of this distribution, when defined, is given by

$$\langle x^{2n}\rangle = \left[\int dx x^{2n}\rho(x,t)\right] \Big/ \left[\int dx \rho(x,t)\right] = \phi(t)^{-2n}\left[\int d\zeta \zeta^{2n}\mathcal{P}(\zeta)\right] \Big/ \left[\int d\zeta \mathcal{P}(\zeta)\right] \propto \phi(t)^{-2n} \tag{38}$$

with

$$\langle x^{2n+1}\rangle = 0, \tag{39}$$

yielding

$$\langle (x-\langle x\rangle)^2\rangle \propto \phi(t)^{-2} \sim t^{2(\nu+\gamma-\mu')/[(1-\mu')(\theta+\mu)]} \tag{40}$$

for long time and $(1-\mu')\alpha>0$. Note, in the above equation, that the diffusion can be subdiffusive normal, or superdiffusive, depending on the value of $2(\nu+\gamma-\mu')/[(1-\mu')(\theta+\mu)]$, to be less, equal or greater than 1.

Let us finally mention a connection between the results obtained for the fractional cases here and the solutions that arise from the optimization of the nonextensive entropy [9]. These distributions do not coincide for an arbitrary value of x. However, the comparison of the asymptotic behaviors ($|x|\to\infty$) enables us to identify the type of tails. By identifying the behavior exhibited in Eq. (24) with the asymptotic behavior $1/|x|^{2/(q-1)}$ that appears in Ref. [9] for the entropic problem, we obtain

$$q=\frac{3+\mu-2\gamma+\theta}{1+\mu+\theta}. \tag{41}$$

This relation recovers, for $\theta=0$ and $\gamma=0$, the one already established in Ref. [17] and extends that obtained in Ref. [31].

III. CONCLUSIONS

In summary, we have worked on a one-dimensional generalized diffusion equation [Eq. (4)] in several situations by considering some space and time dependent classes of external drifts and diffusion coefficients. We have shown that it admits exact solutions where space scales with a function of time. In particular, we have extended the results obtained in Refs. [17,26,27,29–32]. Whenever appropriate, we have also discussed the connection with nonextensive statistics, providing (through identification of the exact or at least asymptotic behaviors) the relation between the entropic index q and the exponents appearing in the diffusion equation. Finally, we hope that the results obtained here may be applied to physical systems exhibiting nontrivial forms of anomalous diffusion.

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