PHYSICAL REVIEW E **67**, 046614 (2003)

Inverse scattering theory: Renormalization of the Lippmann-Schwinger equation for acoustic scattering in one dimension

Donald J. Kouri
Departments of Chemistry, Mathematics, and Physics, University of Houston, Houston, Texas 77204-5005

Amrendra Vijay
Department of Chemistry, University of Houston, Houston, Texas 77204-5005
(Received 2 December 2002; published 25 April 2003)

The most robust treatment of the inverse acoustic scattering problem is based on the reversion of the Born-Neumann series solution of the Lippmann-Schwinger equation. An important issue for this approach to inversion is the radius of convergence of the Born-Neumann series for Fredholm integral kernels, and especially for acoustic scattering for which the interaction depends on the square of the frequency. By contrast, it is well known that the Born-Neumann series for the Volterra integral equations in quantum scattering are absolutely convergent, independent of the strength of the coupling characterizing the interaction. The transformation of the Lippmann-Schwinger equation from a Fredholm to a Volterra structure by renormalization has been considered previously for quantum scattering calculations and electromagnetic scattering. In this paper, we employ the renormalization technique to obtain a Volterra equation framework for the inverse acoustic scattering series, proving that this series also converges absolutely in the entire complex plane of coupling constant and frequency values. The present results are for acoustic scattering in one dimension, but the method is general. The approach is illustrated by applications to two simple one-dimensional models for acoustic scattering.

DOI: 10.1103/PhysRevE.67.046614 PACS number(s): 02.90.+p

I. INTRODUCTION

The inverse scattering problem has enormous importance both for practical and theoretical applications. The former include hydrocarbon exploration and production, medical imaging of many varieties, nondestructive testing, target identification and location, etc. The latter include relating interactions governing atomic and molecular systems to experimental measurements, determination of the structure of surfaces and condensed matter systems, imaging of nanostructures, etc. In much of the literature, the focus has been on determining the conditions under which the data inversion will yield a unique result and precisely what information is required to make an inversion possible. In terms of algorithms employed for various types of imaging, an important practical tool is the first Born approximation, which assumes that all scattering is direct, involving a single interaction of the probe with the target. Of course, this is known to be incorrect. Indeed, most imaging procedures or algorithms typically make use of some assumed model for the propagation of the probe signal or disturbance in the scattering medium. Generally, inversion is practical only in the circumstance where there is a sufficiently small difference between the propagation of the probe signal within the target and its ''reference propagation'' (low contrast between the target and the reference medium). Over the last decade, Weglein and co-workers [1] have pioneered inverse acoustic scattering methods that do not require an assumed propagation velocity model within the medium. Their approach is based on the early work of Jost and Kohn [2], Moses [3], and Razavy [4], who used the Born-Neumann power series solution of the acoustic Lippmann-Schwinger equation and a concommittant expansion of the interaction in ''orders of the data.'' Reversion of the Born-Neumann series leads to an order-by-order scheme for evaluating the terms of the series representation of the scattering interaction in terms of the measured data, e.g., only the on-shell reflection amplitude is required to invert for a local interaction. In principle, the method is completely general and requires no prior information about the target or the propagation details of the probe signal within the target. The *only* fundamental limitation of the approach appears to be the finite radius of convergence of the Born-Neumann series solution of the acoustic Lippmann-Schwinger equation. This is generally analyzed using the ''spectral radius'' of the Fredholm kernel of this equation (Morse and Feshbach [5]; Newton [6]), and in particular by the $\mathcal{L}^2$ norm of this kernel. References and very clear discussions on the issues, involved in the convergence of the Born-Neumann forward scattering series, can be found in works of Goldberger and Watson [7], and Newton [6]. Despite this limitation, Weglein and co-workers [1] have made significant progress using this approach by introducing the idea of ''subseries'' within the Born-Neumann expansion, which are associated with specific inversion tasks. This expresses the inversion series in terms of a set of subtasks, which can be carried out separately from one another. A particularly significant benefit of this approach is the fact that the convergence properties of the subseries studied to date are much more favorable than those of the full Born-Neumann series. Indeed, empirical evidence has been very encouraging regarding the convergence of the inverse series. However, the nature of the kernel of the Lippmann-Schwinger equation, viewed as an equation for the interaction in terms of the T operator, is such that its maximum eigenvalue always depends on the explicit nature of the on- and off-shell T matrix, and general statements regarding convergence are difficult to obtain (Prosser [8]).

Another, more robust approach towards solving integral equations is given by Fredholm [9], which can be viewed as a generalization of the well-known Cramer's method for solving systems of linear simultaneous algebraic equations. Consequently, fundamental to the approach is a continuous generalization of the determinant of coefficients and its minors. Under the circumstances where the integral equation is of the Volterra type, the "Fredholm determinant" can be shown to be equal to one and the Fredholm solution reduces to a Born-Neumann expansion, allbeit one that converges absolutely independent of the scattering interaction strength. Consequently, for such Volterra equations, the Born-Neumann expansion possesses the most robust convergence properties for which one can hope.

Some years ago, Sams and Kouri [10] (for noniterative computations in quantum scattering) and Kouri [11] (for electromagnetic scattering) showed that one could carry out a renormalization transformation of the Lippmann-Schwinger equation into a Volterra equation form. Although the Volterra equations for quantum scattering were well known (Goldberger and Watson [7]; Newton [6]), previous studies had focused almost exclusively on their use for studying the analytic structure of the S matrix and the scattering state. The work of Kouri and co-workers concentrated on making use of the Volterra form of the scattering equations to create a noniterative computational algorithm. Their approach, however, made essential use of the "triangular" character of the Volterra equation kernel, which in one dimension (1D) is

$$K(z,z')=0, \quad z\geqslant z' \quad \text{or} \quad K(z,z')=0, \quad z\leqslant z', \tag{1}$$

combined with a Newton-Cotes quadrature to solve the equations by a noniterative recursion. However, it is also well known that the property (1) underlies the extremely robust nature of the convergence of these Volterra equations with respect to an iterative solution (Morse and Feshbach [5]; Newton [6]). Indeed, the Born-Neumann series solution of the Volterra equation converges absolutely, irrespective of the magnitude of the (in general complex) coupling strength of the interaction. Furthermore, the convergence depends on the global behavior of the interaction (essentially whether it is measureable in a particular sense) and not on its smoothness. For 1D interactions having compact support (and for even more general interactions in the case of 3D scattering), the iterative solution of the Volterra equation converges uniformly on any closed domain of definition in the scattering position variable. Again, under certain relatively weak conditions on the interaction, the iterative solution is an entire function of the scattering wave number k (Newton [6]).

Thus, the possible benefits of formulating acoustic scattering in terms of Volterra kernels appear substantial. The infinitely large radius of convergence of the Born-Neumann series solution of the Volterra equation is of special interest from the standpoint of the inverse acoustic scattering approach of Weglein and co-workers [1]. It seems natural, therefore, to investigate the possible benefits of using the renormalization technique as a framework for developing an inverse scattering series. In fact, we shall show that it is possible to establish general rigorous convergence properties for the inverse acoustic scattering series, and in the process show that its radius of convergence is also infinite. We shall restrict our discussion here to 1D scattering, but our approach is completely general and extends to higher dimensions.

This paper is organized as follows. In Sec. II, we discuss renormalization of the Lippmann-Schwinger equation for acoustic scattering and introduce an auxilliary transition operator $\tilde{T}$. This is used as the framework to analyze the convergence of the forward scattering Born-Neumann series. The approach is illustrated by applying it to scattering by a Dirac δ-function model interaction. In Sec. III, we show the relationship between the interaction as a function of the physical T operator and as a function of the auxilliary $\tilde{T}$ operator. We next analyze the nonlocal nature of $\tilde{T}$ in the coordinate representation, and then use the results to establish the convergence properties of the Volterra-based Born-Neumann inverse series for the interaction. We include in this section an application to the Dirac δ-function interaction. Next, in Sec. IV, the Volterra inverse series is applied to the case of sound scattering by either a square well or barrier. Our conclusions are given in Sec. V.

II. RENORMALIZATION OF THE LIPPMANN-SCHWINGER EQUATION

A. Derivation of the renormalization transformation and auxilliary transition operator $\tilde{T}$

We assume that the reader is familiar with the acoustic scattering Lippmann-Schwinger equation for the transition operator T, given by Razavy [4], Goldberger and Watson [5], and Newton [6]:

$$T=\gamma V+\gamma V G_{0k}^{+}T, \tag{2}$$

where G_{0k}^{+} is the causal free Green's operator, multiplied by a factor of k^2,

$$G_{0k}^{+}=\frac{k^2}{E-H_0+i\epsilon}, \tag{3}$$

$k^2=E$ (i.e., k is the frequency associated with the incident acoustic wave), H_0 governs the "free propagation" of the acoustic wave, and γV is the interaction responsible for the scattering, with γ being the coupling parameter characterizing the strength of the interaction. In general, γ is complex. The additional factor of k^2 results from the fact that in acoustic scattering (as in general for scattering governed by a Helmholtz-type wave equation), the interaction responsible for scattering depends on k^2. The full acoustic wave propagation (scattering process) is thus governed by the operator H,

$$H=H_0+k^2\gamma V. \tag{4}$$

The present 1D acoustic scattering problem in the coordinate representation leads to

$$T(z,z')=\gamma V(z,z') + \int_{-\infty}^{\infty} dz'' \gamma V(z,z'') \int_{-\infty}^{\infty} dz''' G_{0k}^{+}(z'',z''') T(z''',z'). \tag{5}$$

By incorporating this factor of k^2 into the Green's function, we are able to treat the remaining portion of the interaction, which depends purely on the spatial variation of the scattering interaction. Initially, we restrict ourselves to "local scattering media," so that $V(z,z')=V(z)\delta(z-z')$ and therefore

$$T(z,z')=\gamma V(z)\delta(z-z') + \gamma V(z)\int_{-\infty}^{\infty} dz'' G_{0k}^{+}(z,z'')T(z'',z'). \tag{6}$$

The nonlocal character of the causal free Green's function $G_{0k}^{+}(z,z'')$, reflected in its noncommutation with γV, is responsible for the fact that $T(z,z'')$ is also generally nonlocal, i.e., it is *never* diagonal in the coordinate representation [except for a local, Dirac δ-function interaction, $V(z,z')=V(z)\delta(z-z')=\lambda\delta(z-z')\delta(z-z_0)$]. For 1D causal scattering boundary conditions, $G_{0k}^{+}(z,z'')$ is explicitly

$$G_{0k}^{+}(z,z'')=-\frac{ik}{2}e^{ik|z-z''|}. \tag{7}$$

The general scattering amplitude is determined by the matrix elements of the T operator, usually computed in the momentum representation $T(k',k'')$, given by

$$T(k',k'')=\langle k'|T|k''\rangle, \tag{8}$$

where in general k',k'', and the on-energy-shell wave number $k=\sqrt{E}$ need not be equal to one another. The physical "reflection scattering amplitude," denoted $r(k)$, results when $|k'|=|k''|=|k|$ and $k'=-k$:

$$r(k)=(-ik\pi)\langle -k|T|k\rangle. \tag{9}$$

In 1D scattering, one can also identify the transmission amplitude $t(k)$ given by

$$t(k)=1+(-ik\pi)\langle k|T|k\rangle. \tag{10}$$

In the work of Sams and Kouri [10], the renormalization transformation to a Volterra equation results from eliminating the $|z-z''|$ argument in the free Green's function in Eq. (6). This is done by dividing the integration over z'' into segments from $-\infty$ to z and from z to ∞:

$$T(z,z')=\gamma V(z)\delta(z-z')-\frac{ik}{2}\gamma V(z)\int_{-\infty}^{z}dz'' e^{ik(z-z'')}T(z'',z')-\frac{ik}{2}\gamma V(z)\int_{z}^{\infty}dz'' e^{-ik(z-z'')}T(z'',z'). \tag{11}$$

One then adds and subtracts $-(ik/2)\gamma V(z)\int_z^\infty dz''\exp[ik(z-z'')]T(z'',z')$, and after simple manipulation, one obtains

$$T(z,z')=\gamma V(z)\left[\delta(z-z')-\frac{ik}{2}e^{ikz}\int_{-\infty}^{\infty}dz'' e^{-ikz''}T(z'',z')\right] - \frac{ik}{2}\gamma V(z)\int_{z}^{\infty}dz''[e^{-ik(z-z'')} - e^{ik(z-z'')}]T(z'',z'). \tag{12}$$

It is easily verified that this is equivalent to writing $G_{0k}^{+}(z,z'')$ as

$$G_{0k}^{+}(z,z'')=\tilde{G}_{0k}(z,z'')-\frac{ik}{2}e^{ik(z-z'')}, \tag{13}$$

so that

$$\tilde{G}_{0k}(z,z'')=-\frac{ik}{2}[e^{ik(z''-z)}-e^{-ik(z''-z)}] \equiv k\sin[k(z''-z)], \quad z<z'' \tag{14}$$

$$=0, \quad z\geqslant z''. \tag{15}$$

In abstract operator notation, this is

$$G_{0k}^{+}=\tilde{G}_{0k}-ik\pi|k\rangle\langle k|. \tag{16}$$

This relation is extremely useful in our subsequent analysis and we shall make much use of it. Notice that the Green's operator $\tilde{G}_{0k}$ differs from the usual causal one, G_{0k}^{+}, by a solution of the homogeneous equation (Newton [6]):

$$(E-H_0)G_{0k}^{+}=k^2, \tag{17}$$

$$(E-H_0)\tilde{G}_{0k}=k^2, \tag{18}$$

$$(E-H_0)[-ik\pi|k\rangle\langle k|]=[-ik\pi|k\rangle\langle k|](E-H_0)=0. \tag{19}$$

The abstract version of Eq. (12) results from substituting Eq. (16) into Eq. (2):

$$T=\gamma V[1-ik\pi|k\rangle\langle k|T]+\gamma V\tilde{G}_{0k}T. \tag{20}$$

Next we note that the action of T on the initial state $|k\rangle$ is of the form

$$T|k\rangle=\gamma V[1-ik\pi\langle k|T|k\rangle]|k\rangle+\gamma V\tilde{G}_{0k}T|k\rangle. \tag{21}$$

Defining the (unknown) *constant* c_k as

$$c_k = 1 - ik\pi\langle k|T|k\rangle \equiv t(k), \tag{22}$$

we see that

$$T|k\rangle = \gamma V c_k|k\rangle + \gamma V\tilde{G}_{0k}T|k\rangle. \tag{23}$$

The relationship between $T|k\rangle$ and the Lippmann-Schwinger pressure state $|P_k^+\rangle$ is

$$\sqrt{2\pi}T|k\rangle = \gamma V|P_k^+\rangle, \tag{24}$$

and thus

$$|P_k^+\rangle = \sqrt{2\pi}c_k|k\rangle + \tilde{G}_{0k}\gamma V|P_k^+\rangle. \tag{25}$$

Clearly, the factor c_k is simply a *normalization constant*, and one can define an auxiliary pressure state vector $|p_k\rangle$, in relation to $|P_k^+\rangle$, according to

$$|P_k^+\rangle = c_k|p_k\rangle, \tag{26}$$

$$|p_k\rangle = \sqrt{2\pi}|k\rangle + \tilde{G}_{0k}\gamma V|p_k\rangle. \tag{27}$$

The coordinate representation $\langle z|p_k\rangle = p_k(z)$ satisfies

$$p_k(z) = e^{ikz} + k\int_z^\infty dz''\sin[k(z''-z)]\gamma V(z'')p_k(z''), \tag{28}$$

which is recognized as an inhomogeneous Volterra integral equation of the second kind. We remark here that Volterra equations involving improper limits (i.e., $\pm\infty$) still converge absolutely for $|\gamma|<\infty$, but they must satisfy additional restrictions on the z dependence of the interaction. This is especially true in order for their iterative solutions to converge uniformly on any closed interval $[z_1,z_2]$. It is sufficient that the interaction $V(z)$ has compact support and $|V(z)|$ is measurable. It remains true even for infinite ranged interactions as long as they decay sufficiently rapidly and are not too singular. This is discussed for similar Volterra equations by Goldberger and Watson [5], and Newton [6]. Throughout our discussion, we assume that such conditions are met. By Eq. (26), $|p_k\rangle$ results from renormalizing $|P_k^+\rangle$ according to

$$|p_k\rangle = \frac{|P_k^+\rangle}{c_k}; \tag{29}$$

in fact, c_k is essentially the inverse of the Jost function (Newton [6]). We remark that the above expression also provides the physical interpretation of the "Volterra pressure wave" $p_k(z)$ [12]. Clearly, it represents a wave produced by an incident plane wave having an amplitude equal to $1/c_k \equiv 1/t(k)$. This leads to a reflected wave with the amplitude $r(k)/t(k)$, and a transmitted wave with the amplitude exactly equal to 1. Of course, such an incident wave cannot, in general, be created experimentally since it requires advance knowledge of the effect of the scatterer in the form of $1/t(k)$. However, this does not alter the interpretation of the wave $p_k(z)$.

Let us now return to Eq. (20) and define an auxilliary transition operator $\tilde{T}$ according to

$$T = \tilde{T}[1 - ik\pi|k\rangle\langle k|T]. \tag{30}$$

It is easily verified that

$$\tilde{T} = \gamma V + \gamma V\tilde{G}_{0k}\tilde{T}, \tag{31}$$

and this is the fundamental equation that will be used to analyze the inverse series for γV. (Note that the operator inverse $[1 - ik\pi|k\rangle\langle k|T]^{-1}$ should always exist. This essentially requires that the operator $ik\pi|k\rangle\langle k|T$ does not have any eigenvalues equal to $+1$. A worst case would correspond to the inverse of T being equal to $ik\pi|k\rangle\langle k|$, which cannot occur since T does *not* commute with H_0 while $|k\rangle\langle k|$ does.) It is instructive to evaluate explicitly the normalization constant c_k in terms of the solution of the Volterra equation. This is quite easily done by combining Eqs. (21), (22), and (30) to write

$$T|k\rangle = c_k\tilde{T}|k\rangle. \tag{32}$$

Then Eq. (22) can be expressed as

$$c_k = 1 - ik\pi\langle k|\tilde{T}|k\rangle c_k, \tag{33}$$

so that

$$c_k = \frac{1}{1 + ik\pi\langle k|\tilde{T}|k\rangle}. \tag{34}$$

Thus, the renormalized or auxiliary pressure state $|p_k\rangle$ is given by

$$|p_k\rangle = |P_k^+\rangle[1 + ik\pi\langle k|\tilde{T}|k\rangle]. \tag{35}$$

The physical reflection amplitude $r(k)$ is given by

$$r(k) = -ik\pi\langle -k|T|k\rangle = -t(k)ik\pi\langle -k|\tilde{T}|k\rangle. \tag{36}$$

These relations provide us with the necessary tools to express auxilliary amplitudes in terms of the physical amplitudes.

B. Convergence of the Born-Neumann series for $|p_k\rangle$ and $\tilde{T}$

On one hand, the convergence of the Born-Neumann series for either $|P_k^+\rangle$ or T is well known to depend critically on the size of the coupling constant γ (or equivalently, on the size of the "contrast" between the propagation under H_0 and that under $H = H_0 + k^2\gamma V$) (Goldberger and Watson [5]; Newton [6]). On the other hand, it is also well known that iterative solutions of either Eq. (28) or (31) converge absolutely for $|\gamma|<\infty$ (Newton [6]). Furthermore, the iteration of Eq. (28) converges uniformly on any closed domain of z (for a wide class of interactions). It is useful to stress the origin of this robustness since it turns out to be the basis of the convergence of the Volterra-based inverse series for γV. The kernel of Eq. (28) can be written (for all z,z'') as

$$\gamma K(z,z'') = k\gamma \sin[k(z''-z)]V(z''), \quad z < z'' \tag{37}$$

$$\equiv 0, \quad z \geqslant z''. \tag{38}$$

According to the discussion, (Newton [6]; see also those in Rodberg and Thaler [13] and Mathews and Walker [14]), one characterizes the convergence in terms of Fredholm's method of solution. This method is the continuum analog of solving a linear system of algebraic equations, and it expresses the inverse of the integral kernel in terms of the ratio of the first Fredholm minor to the Fredholm determinant Δ. The determinant Δ can be expressed as an infinite series of the form (for the acoustic case)

$$\Delta = \sum_{n=0}^{\infty} (\gamma)^n \kappa_n, \tag{39}$$

where

$$\kappa_n \equiv \mathrm{Tr}(K^n). \tag{40}$$

It is not difficult to verify that for the Volterra kernel $K(z,z'')$ above,

$$\kappa_n = \delta_{n0}, \tag{41}$$

and consequently, for such kernels,

$$\Delta \equiv 1, \tag{42}$$

regardless of the strength of the scatterer, γ. Furthermore, by use of Hadamard's theorem [6,13], it is easily proved that the infinite series for the first Fredholm minor converges absolutely and uniformly for γ in the entire complex plane (more details are given in the Appendix to this paper). It also has been established (Mathews and Walker [14]) that when the Fredholm determinant equals 1, the Fredholm solution is identical to the Born-Neumann iterative solution of the integral equation. We conclude that iterative solutions of Volterra integral equations possess the most robust convergence possible. While it is true that these convergence properties are independent of the strength of γ, there are conditions on the analytical structure allowed for the scattering interaction. These have to do with the integrability of any singularities and the behavior at infinity. They are discussed by Newton [6] in some detail. If the interaction has compact support, and is not too singular, then the convergence is of the strongest character (i.e., absolute and uniform, leading to entire functions of wave number k and coupling γ).

It is therefore clear that the essential property of the Volterra kernel is that it satisfies Eqs. (37) and (38); as noted by Newton [6], this is the continuous version of the "triangular" property of matrices. It is equivalent to the property that the Fredholm determinant of Eq. (28) or (31) is identically 1. Furthermore, it ensures that the Born-Neumann series for $p_k(z)$, obtained from Eq. (31), is uniformly convergent on any closed domain of z for a wide class of interactions.

We stress that this is all well known. We have included it explicitly here because its implications for the inverse scattering series determining γV have never been explicated. In addition, it is perhaps not appreciated that the original Lippmann-Schwinger equation itself can be directly iterated in a fashion that is also everywhere absolutely convergent. Obviously, such an iteration must differ from the straightforward iteration of the Lippmann-Schwinger equation $|P_k^+\rangle = \sqrt{2\pi}|k\rangle + G_{0k}^+\gamma V|P_k^+\rangle$, which leads to

$$|P_k^+\rangle = \sqrt{2\pi}\sum_{n=0}^{\infty} (G_{0k}^+\gamma V)^n|k\rangle, \tag{43}$$

the proof of whose convergence depends on the $\mathcal{L}^2$ norm of the kernel, $\|G_{0k}^+\gamma V\|_2$. In fact, we can simply iterate Eq. (25) for the Lippmann-Schwinger state $|P_k^+\rangle$:

$$|P_k^+\rangle = \sqrt{2\pi}\sum_{n=0}^{\infty} (\tilde{G}_{0k}\gamma V)^n c_k|k\rangle = \sqrt{2\pi}c_k\sum_{n=0}^{\infty} (\tilde{G}_{0k}\gamma V)^n|k\rangle. \tag{44}$$

We stress that even though c_k is unknown in Eq. (44), it is simply a number and can be calculated directly from the known iterate vectors, $(\tilde{G}_{0k}\gamma V)^n|k\rangle$. Thus,

$$c_k = \frac{1}{1 + ik\pi\sum_{n=0}^{\infty}\langle k|\gamma V(\tilde{G}_{0k}\gamma V)^n|k\rangle} = t(k). \tag{45}$$

Obviously, this is equivalent to the iterative solution for $|p_k\rangle$, but the point we wish to stress is that the standard physical Lippmann-Schwinger equation can be iterated in an absolutely convergent fashion, independent of the strength of the interaction. Of course, this is simply a reflection of the fact that the Lippmann-Schwinger equation is *neither* purely a Fredholmn or a Volterra equation. Therefore, it can manifest the convergence characteristics of either, depending on the manner in which it is written and iterated.

C. Illustrative example

It is helpful to consider an example problem in order to appreciate better the vast difference in convergence between the Born-Neumann series, based on the Lippmann-Schwinger Fredholm equation, and the renormalized Lippmann-Schwinger Volterra equation. A convenient and simple model scattering interaction is the Dirac δ function:

$$\gamma V(z) = \gamma\delta(z - z_0). \tag{46}$$

The solution to the Lippmann-Schwinger equation is easily found from noting that

$$P_k^+(z) = e^{ikz} - \frac{ik\gamma}{2}e^{ik|z-z_0|}P_k^+(z_0). \tag{47}$$

Obviously, this implies that

$$P_k^+(z_0) = \frac{e^{ikz_0}}{\left(1 + \frac{ik\gamma}{2}\right)} \tag{48}$$

so the exact solution is

$$P_k^+(z) = e^{ikz} - \frac{ik\gamma/2}{1+ik\gamma/2} e^{ikz_0} e^{ik|z-z_0|}. \tag{49}$$

The Born-Neumann series solution is given by

$$P_k^+(z) = e^{ikz} - \left(\frac{ik\gamma}{2}\right) e^{ikz_0} e^{ik|z-z_0|} \left[1 - \frac{ik\gamma}{2} + \left(\frac{ik\gamma}{2}\right)^2 + \cdots\right]. \tag{50}$$

It is clear that the convergence of this series is determined by the requirement $|k\gamma/2| < 1$, which is just the condition for the convergence of a power series expansion of $(1+ik\gamma/2)^{-1}$. *It is also evident that the Born-Neumann series is convergent only at low energies in this case since $k=\sqrt{E}$, and therefore only for sufficiently low E will the convergence condition will be satisfied*. This is the opposite of the usual situation that applies to quantum scattering (Goldberger and Watson [5]). Of course, in this simple example, one can easily recognize that the series can be analytically summed to yield the exact result valid at all k and γ. In general, that will not be the case.

We next consider the Born-Neumann series for $p_k(z)$; Eq. (28) then becomes

$$\begin{aligned} p_k(z) = e^{ikz} &+ k\int_z^\infty dz'' \sin[k(z''-z)]\gamma\delta(z''-z_0)e^{ikz''} + k^2 \\ &\times \int_z^\infty dz'' \sin[k(z''-z)]\gamma\delta(z''-z_0) \\ &\times \int_{z''}^\infty dz''' \sin[k(z'''-z'')]\gamma\delta(z'''-z_0)e^{ikz'''} + \cdots. \end{aligned} \tag{51}$$

We see that all terms higher than first order in the interaction vanish identically due to the appearance of the factor $\sin[k(z_0-z_0)]$. Thus, for $z \geqslant z_0$, we obtain exactly

$$p_k(z) = e^{ikz} \tag{52}$$

and for $z < z_0$, we obtain exactly

$$p_k(z) = e^{ikz} + k\gamma \sin[k(z_0-z)]e^{ikz_0}. \tag{53}$$

This does not complete the analysis since we must also evaluate c_k using only information generated by the iterative solution for $p_k(z)$. This is simple using Eqs. (32)–(36) and yields

$$c_k = \frac{1}{1 + \frac{ik\gamma}{2}\int_{-\infty}^\infty dz e^{-ikz}\delta(z-z_0)p_k(z)} \tag{54}$$

$$= \frac{1}{1+\frac{ik\gamma}{2}e^{-ikz_0}p_k(z_0)}. \tag{55}$$

But by Eq. (52), this gives

$$c_k = \frac{1}{1+\frac{ik\gamma}{2}}. \tag{56}$$

Therefore the Born-Neumann series for $P_k^+(z)$ based on the renormalized Lippmann-Schwinger equation results in

$$P_k^+(z) = \frac{e^{ikz}}{1+\frac{ik\gamma/2}{2}}, \quad z \geqslant z_0 \tag{57}$$

$$= e^{ikz} - \frac{ik\gamma/2}{1+ik\gamma/2} e^{2ikz_0} e^{-ikz}, \quad z < z_0. \tag{58}$$

We conclude that the Volterra-based iteration converges to the *exact* answer with just the first-order term, all higher terms being zero. The fundamental difference between the Volterra and the Fredholm iterated expressions is that the former does *not* involve a power series expansion of the normalization factor c_k, whose convergence would have required that $|k\gamma/2|$ be less than 1. Instead, c_k has been factored out by renormalizing from $P_k^+(z)$ to $p_k(z)$. We emphasize that this renormalization follows for *any* scattering problem that is expressible in terms of Green's functions $G_{0k}^\pm$, since it is true in general (for 1D scattering) that

$$G_{0k}^\pm = \tilde{G}_{0k} \mp ik\pi|k\rangle\langle k|. \tag{59}$$

We also note that analogous relationships have been derived for 3D scattering Green's functions. We now turn to consider the inverse scattering series for the interaction γV.

III. THE INVERSE SCATTERING SERIES FOR γV

A. Fredholm and Volterra Born-Neumann series for γV

We begin by establishing that distinct Born-Neumann series for γV can be obtained from Eq. (2) for T or Eq. (31) for $\tilde{T}$. We solve Eq. (2) for γV as

$$\gamma V = T(1+G_{0k}^+T)^{-1} \tag{60}$$

$$= T(1+\tilde{G}_{0k}T - ik\pi|k\rangle\langle k|T)^{-1}. \tag{61}$$

But by Eq. (30), this yields

$$\gamma V = T([1-ik\pi|k\rangle\langle k|T] + \tilde{G}_{0k}\tilde{T}[1-ik\pi|k\rangle\langle k|T])^{-1} \tag{62}$$

$$= T[1-ik\pi|k\rangle\langle k|T]^{-1}(1+\tilde{G}_{0k}\tilde{T})^{-1}, \tag{63}$$

so that finally,

$$\gamma V = \tilde{T}(1+\tilde{G}_{0k}\tilde{T})^{-1}. \tag{64}$$

It follows that, provided they converge, γV can be obtained from *either* of the following Born-Neumann series expansions:

$$\gamma V=\sum_{n=0}^{\infty} T(-G_{ok}^{+}T)^{n} \tag{65}$$

or

$$\gamma V=\sum_{n=0}^{\infty} \tilde{T}(-\tilde{G}_{0k}\tilde{T})^{n}. \tag{66}$$

The convergence properties of Eq. (65) depend, of course, on the spectral radius of the kernel $G_{0k}^{+}T$, which in turn depends crucially on both the on- and off-shell elements of the T matrix. For this reason, general conclusions regarding the convergence of Eq. (65) have been extremely difficult to obtain despite heroic efforts (Prosser [8]). We shall see that this is not the case for Eq. (66).

The convergence properties of Eq. (66) will be studied using the Fredholm method of solving Eq. (64). To do so requires knowledge of the properties of the kernel $\tilde{G}_{0k}\tilde{T}$, which are yet to be established. It is clear, however, that when both expansions converge, they *must* agree since convergent power series yield a unique result (Kaplan [15]). In order to investigate the convergence of Eq. (66), we now consider the nonlocal character of $\tilde{T}$ in the coordinate representation.

B. Nonlocal character of $\tilde{T}(z,z')$

The aim of this section is to establish that $\tilde{T}(z,z')=0$, when $z>z'$. It is not difficult to show that Eq. (31) has the solution

$$\tilde{T}=\gamma V+\gamma V\tilde{G}\gamma V, \tag{67}$$

where

$$\tilde{G}=\tilde{G}_0+\tilde{G}_0\gamma V\tilde{G} \tag{68}$$

[see work of Newton [6], pp. 343–344; especially Eq. (12.42) and the following unnumbered equation]. From Eq. (12.40a) in Ref. [6], we see that

$$G^{+}(k;z,z')=-k\psi^{+}(k,z_{<})f(k,z_{>}), \tag{69}$$

where $\psi^{+}(k,z)$ is the regular (physical or causal) scattering solution of the interacting Schrödinger equation and $f(k,z)$ is an irregular solution of the same equation, introduced by Jost [6]. Then by defining an interacting Green's function $\tilde{G}(k;z,z')$, which vanishes for $z\geqslant z'$, it is easy to see that

$$\tilde{G}(k;z,z')=G^{+}(k;z,z')+k\psi^{+}(k,z')f(k,z). \tag{70}$$

Obviously, $k\psi^{+}(k,z')f(k,z)$ satisfies the *homogeneous* interacting Green's function Schrödinger equation. Then

$$\tilde{G}(z,z')=0, \quad z\geqslant z', \tag{71}$$

and for local potentials

$$\tilde{T}(z,z')=\gamma V(z)\,\delta(z-z')+\gamma^{2}\int_{-\infty}^{\infty}dz''\int_{-\infty}^{\infty}dz'''V(z)\,\delta(z-z'')\tilde{G}(z'',z''')V(z''')\,\delta(z'''-z') \tag{72}$$

or

$$\tilde{T}(z,z')=\gamma V(z)\,\delta(z-z')+\gamma^{2}V(z)\tilde{G}(z,z')V(z'). \tag{73}$$

It is therefore clear that as a function of either z or z', $\tilde{T}(z,z')$ has support determined by $V(z)$ or $V(z')$. Also, for $z>z'$, the first term on the right-hand side (RHS) of Eq. (73) is zero due to the Dirac δ function, and the second term is zero due to $\tilde{G}(z,z')$. Therefore, we have proved that

$$\tilde{T}(z,z')=0, \quad z>z'. \tag{74}$$

Finally, $\tilde{T}(z,z')$ has an integrable singularity at $z=z'$.

C. Convergence of the inverse series for γV

We note next that the kernel of the Volterra-based Born-Neumann series for γV, [Eq. (64)], is given by

$$\tilde{K}(z,z')=\langle z|\tilde{G}_{0k}\tilde{T}|z'\rangle. \tag{75}$$

It is necessary to compute $\mathrm{Tr}(\tilde{K}^{n})$, but it is sufficient to examine $\mathrm{Tr}(\tilde{K}^{2})$ to see how the general case behaves,

$$\mathrm{Tr}(\tilde{K}^{2})=\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dz'\tilde{K}(z,z')\tilde{K}(z',z). \tag{76}$$

This can be written as

$$\mathrm{Tr}(\tilde{K}^{2})=\int_{-\infty}^{\infty}dz_{1}\cdots\int_{-\infty}^{\infty}dz_{4}\tilde{G}_{0k}(z_{1},z_{2})\times\tilde{T}(z_{2},z_{3})\tilde{G}_{0k}(z_{3},z_{4})\tilde{T}(z_{4},z_{1}). \tag{77}$$

However, by the Volterra property of $\tilde{G}_{0k}$ and $\tilde{T}$, a nonzero contribution can only occur if

$$z_{1}>z_{4}>z_{3}>z_{2}>z_{1}, \tag{78}$$

which obviously is *never* satisfied. We conclude that the Volterra property is satisfied for the product of two (or more) Volterra kernels and

$$\mathrm{Tr}(\tilde{K}^{2})=\mathrm{Tr}(\tilde{K}^{n})\equiv 0, \quad n\geqslant 2. \tag{79}$$

It is similarly easy to prove that $\mathrm{Tr}(\tilde{K})=0$.

We therefore conclude that the Fredholm determinant for Eq. (64) equals 1. *This guarantees that, for a not-too-singular, local interaction having compact support, the Volterra-based inverse scattering series converges absolutely and uniformly independent of the strength of the interaction*. This is an amazing result since it ensures that this inverse scattering series *always* converges for any magnitude (complex) coupling constant.

D. Utilization of the Volterra inverse series for γV in orders of $\tilde{T}$ and the relation to data requirements

In order to use the new Volterra inverse series to determine γV, the final step is to develop explicit expressions for it in terms of "far-field" measured quantities. The standard Born-Neumann inversion of the Lippmann-Schwinger based approach, to obtain a local potential, requires knowledge *only* of the reflection amplitude $r(k)$ as a function of k. We shall see that the additional data are required in order to use the Volterra inverse series. Recall that by Eq. (66),

$$\gamma V = \sum_{n=0}^{\infty} \tilde{T}(-\tilde{G}_{0k}\tilde{T})^n, \tag{80}$$

where

$$T = \tilde{T}[1 - ik\pi|k\rangle\langle k|T]. \tag{81}$$

We shall express γV as a power series in orders of $\tilde{T}$,

$$\gamma V = \sum_{j=1}^{\infty} \tilde{\lambda}^j \tilde{V}_j, \tag{82}$$

where obviously

$$\tilde{V}_j = \tilde{T}(-\tilde{G}_{0k}\tilde{T})^{j-1}, \quad j = 1,2, \ldots. \tag{83}$$

Next, recall that by Eq. (31),

$$\tilde{T} = \sum_{n=0}^{\infty} (\gamma V \tilde{G}_{0k})^n \gamma V, \tag{84}$$

$$\tilde{\lambda}\tilde{T} = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \tilde{\lambda}^j \tilde{V}_j \tilde{G}_{0k} \right)^n \sum_{j'=1}^{\infty} \tilde{\lambda}^{j'} \tilde{V}_{j'}, \tag{85}$$

since $\tilde{T}$ is obviously first order in $\tilde{\lambda}$. We then collect coefficients of each power of $\tilde{\lambda}^j$,

$$\tilde{\lambda}^1: \quad \tilde{T} = \tilde{V}_1, \tag{86}$$

$$\tilde{\lambda}^2: \quad 0 = \tilde{V}_2 + \tilde{V}_1 \tilde{G}_{0k} \tilde{V}_1, \tag{87}$$

$$\tilde{\lambda}^3: \quad 0 = \tilde{V}_3 + \tilde{V}_2 \tilde{G}_{0k} \tilde{V}_1 + \tilde{V}_1 \tilde{G}_{0k} \tilde{V}_2 + \tilde{V}_1 \tilde{G}_{0k} \tilde{V}_1 \tilde{G}_{0k} \tilde{V}_1, \tag{88}$$

etc. Matrix elements of these expressions are first evaluated in the k representation and the results are subsequently transformed to the z representation. This is because the starting expression involves the k-representation matrix elements of $\tilde{T}$. However, using the lower-order operators $\tilde{V}_l$ to express $\tilde{V}_j$ solely in terms of $\tilde{V}_1$ and $\tilde{G}_{0k}$, one easily finds that, in general,

$$\tilde{V}_j = -\tilde{V}_{j-1} \tilde{G}_{0k} \tilde{V}_1. \tag{89}$$

This is the most convenient form with which we can evaluate the higher-order corrections. The Volterra-based expressions can be compared to the Born-Neumann inverse series based on the usual Lippmann-Schwinger equation [1–4],

$$\gamma V = \sum_{n=0}^{\infty} T(-G_{0k}^{+} T)^n = \sum_{j=1}^{\infty} \lambda^j V_j, \tag{90}$$

$$T = \sum_{n=0}^{\infty} (\gamma V G_{0k}^{+})^n \gamma V, \tag{91}$$

and this leads to

$$\lambda T = \sum_{n=0}^{\infty} \left(\sum_{j=1}^{\infty} \lambda^j V_j G_{0k}^{+} \right)^n \sum_{j'=1}^{\infty} \lambda^{j'} V_{j'}, \tag{92}$$

implying then

$$\lambda^1: \quad T = V_1, \tag{93}$$

$$\lambda^2: \quad 0 = V_2 + V_1 G_{0k}^{+} V_1, \tag{94}$$

$$\lambda^3: \quad 0 = V_3 + V_2 G_{0k}^{+} V_1 + V_1 G_{0k}^{+} V_2 + V_1 G_{0k}^{+} V_1 G_{0k}^{+} V_1, \tag{95}$$

etc. Again, one easily shows that in general,

$$V_j = -V_{j-1} G_{0k}^{+} V_1. \tag{96}$$

However, it is crucial to recognize that

$$\tilde{V}_j \neq V_j, \tag{97}$$

because they correspond to orders of completely different parameters ($\tilde{V}_j$ is jth order in $\tilde{T}$ while V_j is jth order in T). By Eq. (81) above, it is clear that each separate factor of T involves *all orders* of $\tilde{T}$ and vice versa,

$$[1 + ik\pi\tilde{T}|k\rangle\langle k|]T = \tilde{T}, \tag{98}$$

so that

$$T = [1 + ik\pi\tilde{T}|k\rangle\langle k|]^{-1}\tilde{T} \tag{99}$$

$$= \sum_{n=0}^{\infty} (-ik\pi\tilde{T}|k\rangle\langle k|)^n \tilde{T}. \tag{100}$$

Thus it is clear that $\tilde{V}_j$ and V_j cannot be the same.

Now we ask how can one combine the measured data with the Volterra inverse series? We compute the backscattering matrix element of Eq. (86):

$$\langle -k|\tilde{T}|k\rangle \equiv \tilde{T}(-k,k) = \langle -k|\tilde{V}_1|k\rangle \equiv \tilde{V}_1(-k,k). \tag{101}$$

But $\tilde{T}(-k,k)$ is *not* directly measured. The far-field quantities typically measured are the reflection amplitude $r(k) = -ik\pi T(-k,k)$ and the transmission amplitude $t(k) = 1 - ik\pi T(k,k)$. By Eq. (81), we write

$$T(-k,k) = \tilde{T}(-k,k) - ik\pi\tilde{T}(-k,k)T(k,k), \tag{102}$$

so

$$\tilde{V}_1(-k,k)=\frac{ir(k)}{k\pi t(k)}. \tag{103}$$

This expression is inverse Fourier transformed to the space domain, yielding $\tilde{V}_1(z)$. The result is

$$\tilde{V}_1(z)=\frac{2i}{\pi}\int_{-\infty}^{\infty}dk\frac{e^{-2ikz}r(k)}{kt(k)}=-\frac{4}{\pi}\int_0^{\infty}dk\frac{1}{k}\mathrm{Im}\frac{e^{-2ikz}r(k)}{t(k)}. \tag{104}$$

One obtains the higher order $\tilde{V}_2(z)$ according to

$$\tilde{V}_j(z)=\int_{-\infty}^{\infty}d(2k)e^{-2ikz}\langle -k|\tilde{V}_{j-1}\tilde{G}_{0k}\tilde{V}_1|k\rangle \tag{105}$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d(2k)e^{-2ikz}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}\times dz''e^{ik(z'+z'')}\tilde{V}_{j-1}(z')\tilde{G}_{0k}(z',z'')\tilde{V}_1(z''). \tag{106}$$

Again, it is instructive to carry out the application to scattering by the Dirac δ-function interaction discussed earlier. In that case, we have

$$r(k)=\frac{-ik\gamma}{2+ik\gamma}e^{2ikz_0}, \tag{107}$$

$$t(k)=\frac{2}{(2+ik\gamma)}. \tag{108}$$

It is then easily shown that

$$\tilde{V}_1(z)=\gamma\delta(z-z_0). \tag{109}$$

The second-order correction is given by

$$\tilde{V}_2(z)=\gamma^2\int_{-\infty}^{\infty}d(2k)e^{-2ikz}\times\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dz''e^{ik(z'+z'')}\delta(z'-z_0)\times\tilde{G}_{0k}(z',z'')\delta(z''-z_0) \tag{110}$$

$$=\gamma^2\int_{-\infty}^{\infty}d(2k)e^{-2ikz}\tilde{G}_{0k}(z_0,z_0)\equiv 0. \tag{111}$$

It should be clear that all $\tilde{V}_j$ vanish for $j\geqslant 2$. We conclude that the Volterra inverse scattering series converges to the exact answer in a single term, in the same manner as the forward Volterra series for the Dirac δ function interaction. *We note that a crucial change from the Born-Neumann approach to the Lippmann-Schwinger-based inversion is that now we require both $r(k)$ and $t(k)$ to use the Volterra inverse series.*

Before leaving this example, we point out that Razavy [4] has considered the interaction $2\lambda\delta(x)$ within the Lippmann-Schwinger-based inverse scattering series. Again, analytical results are obtained and the exact result is equal to the sum of the V_1 and V_2 terms. However, the higher-order terms were not evaluated they do not vanish and involve alternating signs. Thus, it appears that the series is only conditionally convergent, depending on the order in which the terms are grouped and summed.

E. The Volterra series for nonlocal potentials

Up to this point, we have established that the Volterra property is shared by the coordinate representation matrix element $\langle z|\tilde{T}|z'\rangle$, if the interaction is local. *In fact, we now show that this result is true for all nonlocal interactions as long as they also possess the Volterra kernel property.* We recall that the exact solution for $\tilde{T}$ is

$$\tilde{T}=\gamma V+\gamma^2 V\tilde{G}V, \tag{112}$$

where

$$\tilde{G}=\tilde{G}_0+\tilde{G}_0\gamma V\tilde{G} \tag{113}$$

and

$$\tilde{G}(z,z')=0,\quad z\geqslant z'. \tag{114}$$

Now we do not restrict V to be local but we require that

$$V(z,z')=0,\quad z>z'. \tag{115}$$

We want to prove that it remains true that $\tilde{T}(z,z')=0, z>z'$. Our equation now is

$$\tilde{T}(z,z')=\gamma V(z,z')+\gamma^2\int_z^{\infty}dz''\times\int_{-\infty}^{z'}dz'''V(z,z'')\tilde{G}(z'',z''')V(z''',z'), \tag{116}$$

where we use the facts that z'' must be greater than z and z' must be greater than z''', due to the presence of the factors $V(z,z'')$ and $V(z''',z')$. Now suppose that $z>z'$. The first term on the RHS of Eq. (116) vanishes for this condition. But the second term only has nonzero contributions for $z'>z'''>z''>z$ since all terms involving $z''>z'''$ vanish due to the factor $\tilde{G}(z'',z''')$. Therefore the only nonzero terms contradict the condition $z>z'$. We conclude that $\tilde{T}(z,z')=0$ if $z>z'$, if the nonlocal potential has the same property.

The analysis in which the kernel of the inverse scattering equation has the Volterra property and therefore a Fredholm determinant that equals 1 is easily carried out and we do not write it explicitly here. If one has the most general form of nonlocal potential, $V(z,z')$, for 1D scattering, then it turns out that the inversion requires measuring both the far and the near fields. Clearly, the operator equations obtained from the inversion of $k^2\gamma V$ in terms of the $\tilde{V}_j$'s hold regardless of whether the potential is local or nonlocal. This is also true of the absolute convergence of the series for $k^2\gamma V$ in terms of the $\tilde{V}_j$ (provided V itself has a Volterra-kernel structure). However, if the nonlocal potential has the general form $V(z,z')$, then Eq. (86) becomes

$$\tilde{T}(k',k)=\tilde{V}_1(k',k), \tag{117}$$

with k' and k independent of one another. Then one uses Eq. (81) to determine the off-shell elements of $\tilde{T}$ in terms of the physical T matrix elements:

$$T(k',k)=\tilde{T}(k',k)-ik\pi\tilde{T}(k',k)T(k,k) \tag{118}$$

so that

$$\tilde{T}(k',k)=\frac{T(k',k)}{1-ik\pi T(k,k)}. \tag{119}$$

Thus, the knowledge of the on-shell and half-off-shell T-matrix elements enables one to determine the corresponding elements of the $\tilde{T}$ matrix. Then inverse Fourier transforming on *both* k' and k independently yields $\tilde{V}_1(z,z')$, which enables one to determine all higher $\tilde{V}_j(z,z'),j>1$.

It should be clear that scattering interactions that can be expressed in the form $V(z)(d^n/dz^n)\delta(z-z')$ will also produce Volterra kernels (that is, interactions involving derivatives of the field). Thus, the range of systems for which our results hold is very broad.

IV. APPLICATION TO THE SQUARE WELL OR BARRIER

As a second example, we present the results of the Volterra-based inverse series for acoustic scattering by a finite width well or barrier. Again the reflection and transmission amplitudes can be obtained analytically, as can the various $\tilde{V}_j$ terms in the power series for the potential. One easily can show that

$$r(k)=\frac{V_0}{(2-V_0)+2i\sqrt{1-V_0}\cot(ak\sqrt{1-V_0})}, \tag{120}$$

,

$$t(k)=\frac{2\sqrt{1-V_0}ie^{-ika}}{V_0\sin(ka\sqrt{1-V_0})}r(k). \tag{121}$$

In the case of $V_0<0$, one can have any finite value for the magnitude of the interaction (corresponding to any finite increase in the velocity of sound in the medium). In the case of a barrier, $0<V_0<1$; otherwise one encounters an infinite $(V_0=1)$ or pure imaginary $(V_0>1)$ velocity of sound. It follows that

$$\tilde{V}_1(2k)=\frac{V_0}{2\pi k\sqrt{1-V_0}}\sin(ka\sqrt{1-V_0})e^{ika}. \tag{122}$$

The $\tilde{V}_1(z)$ is then

$$\tilde{V}_1(z)=\frac{V_0}{\pi\sqrt{1-V_0}}\int_{-\infty}^{\infty}dk\frac{\sin(ka\sqrt{1-V_0})}{k}e^{ik(a-2z)}, \tag{123}$$

which is recognized as the Fourier transform of the sinc function. This is well known to be a square well or barrier:

$$\tilde{V}_1=\frac{V_0}{\sqrt{1-V_0}},\quad |a-2z|<a\sqrt{1-V_0} \tag{124}$$

$$=0,\ \text{all other } z. \tag{125}$$

Rearranging, we find that the region where $\tilde{V}_1$ is nonzero is

$$z_{min}<z<z_{max}, \tag{126}$$

$$z_{min}=\frac{a}{2}(1-\sqrt{1-V_0}), \tag{127}$$

$$z_{max}=\frac{a}{2}(1+\sqrt{1-V_0}). \tag{128}$$

For a barrier, $0<V_0<1$ and the first-order result has a *higher* barrier than the true one. For a well, $V_0<0$ and the first-order result is shallower than the true one. Thus, although the first-order result has the correct analytical form of a square well or barrier, it has incorrect width and height (or depth). However, the explicit form of the result is such that it is trivial to obtain the *exact* potential from z_{min} and z_{max}. It is easily seen that

$$V_0=1-\left(\frac{z_{max}-z_{min}}{z_{max}+z_{min}}\right)^2 \tag{129}$$

and

$$a=z_{max}+z_{min}, \tag{130}$$

$$a=\frac{2z_{min}}{1-\sqrt{1-V_0}}=\frac{2z_{max}}{1+\sqrt{1-V_0}}. \tag{131}$$

These exact, analytical expressions are found to work very well in computational studies as well. Thus, it is *not* necessary to evaluate the $\tilde{V}_j$ beyond $j=1$ in order to obtain the exact parameters for a square well or barrier interaction. Even so, these higher-order terms can also be evaluated analytically.

These results can again be compared to those obtained using the Fredholm-based Born-Neumann inverse scattering series. Razavy [4] has also obtained an expression for the V_1 term. In fact, the result is of the form of an infinite series, so a closed expression has not been possible. This also prevented him from obtaining higher-order corrections. However, the structure manifested at the first order is *not* a simple square well but rather an infinite sequence of steps of decreasing magnitude. Razavy does not consider the convergence of the series. Despite these qualitatively incorrect features, it is nevertheless possible to use Razavy's result to determine the square interaction parameters *exactly*. This is because the terms in the infinite series permit one to obtain the correct V_0 and the a parameter from the first of the infinite series of steps. However, because the Fredholm-based inversion produces unphysical artifacts that are absent from the Volterra-based results, the latter provides a more robust framework for an inversion when one has an interaction that does not yield an explicit formula for the various terms in the series.

It is remarkable that the Volterra-based inverse scattering series for both of these simple potentials is able to provide either the exact answer or the exact functional form of the interaction with only the first-order term. Furthermore, the fact that all higher-order terms can be evaluated analytically is very useful. *We stress that these results are consequences of the fact that the Volterra-based inversion makes use of both the reflection and transmission information.*

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have used the fact that the acoustic scattering Lippmann-Schwinger integral equation (in 1D) involving the causal (or anticausal) Green's function can be renormalized to write it as a Volterra integral equation. Such equations possess the best possible convergence behavior under Born-Neumann iteration. Furthermore, for a wide class of interactions (local, differential, or nonlocal but with the Volterra property), the auxilliary transition operator also possesses the Volterra property. Consequently, the inverse acoustic scattering series obtained by reverting the Volterra-based series in terms of $\tilde{V}_j$ also converges absolutely and uniformly for all $|\gamma|<\infty$. This does not, of course, ensure that the rate of convergence is conveniently rapid. It is well known that an absolutely convergent series can be rearranged or grouped in any manner without affecting its convergence (Kaplan [15]). Of course, this is not true for divergent or conditionally convergent series. In the case of seismic scattering, one may expect the changes in the velocity of sound to be modeled reasonably by piecewise constant interactions since the distance over which there can be large changes should be small compared to the distances over which the sonic speed changes less rapidly.

Our results show that a Volterra-based inversion can be done as a single comprehensive task, *provided one has both the reflection and transmission amplitudes as functions of k*. Indeed, all 1D scattering problems that can be formulated in a Lippmann-Schwinger framework have now been shown to be invertible, given $r(k)$ and $t(k)$. In subsequent work, we shall consider this approach for scattering in higher dimensions as well. The implications for various applications such as medical imaging, seismic exploration, nondestructive testing, etc. are under current study and results will be reported as they are obtained.

Note added. By appropriate use of Eq. (30), we have been able to express the Volterra-based inversion in a form that requires *only* $r(k)$ as input, rather than both $r(k)$ and $t(k)$. This is an important reduction in the experimental data required to apply our approach. It has been pointed out to us that for the Dirac δ interaction, an approach based on the Heitler damping relation also yields the exact result [16] This approach is for evaluating a first-order approximation only. A complete discussion of the relation to the present approach will be given elsewhere.

ACKNOWLEDGMENTS

D.J.K. is indebted to Professor A. B. Weglein for introducing him to the acoustic inverse scattering problem, and for sharing his deep knowledge of the Born-Neumann inversion of the Lippmann-Schwinger equation. He acknowledges Dr. Jon Sheiman and Dr. Jacques Leveille for very helpful discussions. D.J.K. was supported in part by the National Science Foundation under Grant No. CHE-0074311, the R.A. Welch Foundation under Grant No. E 0608, and the Texas Learning and Computing Center through the UH Institute for Digital Informatics and Analysis. A.V. was supported by Texas Higher Education Coordinating Board under Grant No. ARP 003652-0071-2001.

APPENDIX

In this Appendix, we give a few more details regarding the Fredholm solution of Eq. (28),

$$p_k(z)=e^{ikz}+\int_z^\infty dz'' k^2\gamma K(z,z'')p_k(z''), \tag{A1}$$

where $K(z,z'')$ is defined in Eqs. (37) and (38). The solution may be written as

$$p_k=e^{ikz}+\int_0^\infty dz'\frac{D(z,z')}{D}e^{ikz'}, \tag{A2}$$

where

$$D=1-k^2\gamma\int_0^\infty dz K(z,z) + \frac{(k^2\gamma)^2}{2!}\int_0^\infty dz\int_0^\infty dz' \det\begin{pmatrix} K(z,z) & K(z,z') \\ K(z',z) & K(z',z')\end{pmatrix} - \cdots \tag{A3}$$

and

$$D(z,z')=k^2\gamma K(z,z') - (k^2\gamma)^2\int_0^\infty dz'' \det\begin{pmatrix} K(z,z') & K(z,z'') \\ K(z'',z') & K(z'',z'')\end{pmatrix} + \cdots. \tag{A4}$$

Note that $K(z,z)$ vanishes as long as $V(z)$ is not too singular. Therefore, all diagonal terms in the determinants appearing in Eq. (A3) for D vanish. All other terms vanish, as discussed in the text above, since they are of the form $\mathrm{Tr}(K^n)$. Consequently, $D=1$. Now consider the integral $\int_0^\infty dz' D(z,z')\exp(ikz')$. We assume, for simplicity and convenience, that the potential has compact support on the domain $[0,Z]$, and that it is bounded. For *any* value of k and γ, we conclude that $k^2\gamma K(z,z')<|a|$, where a is some finite number. By Hadamard's theorem [6,13], the value of an nth order determinant formed from such elements is bounded by $|a|^n n^{n/2}$. Then the nth term, say t_n, in Eq. (A4) is bounded by

$$t_n<\frac{1}{n!}Z^n|a|^n n^{n/2}. \tag{A5}$$

Using Stirling's approximation, one has

$$t_n<\frac{Z^n|a|^n}{e^{-n}n^{n/2}n^{1/2}}. \tag{A6}$$

By the root test (Kaplan [15]), we see that

$$\lim_{n\to\infty}(t_n)^{1/n}=\lim_{n\to\infty}\left(\frac{Z|a|e}{n^{1/2}n^{1/2n}}\right)=0. \tag{A7}$$

The radius of convergence is one divided by this limit so we conclude that the series for $D(z,z')$ converges absolutely independent of the strength of the coupling parameter γ or the value of k.

[1] A.B. Weglein, K.H. Matson, D.J. Foster, P.M. Carvalho, D. Corrigan, and S.A. Shaw, Soc. Exploration Geophys. (2000); A.B. Weglein and R.H. Stolt, in The Leading Edge (1999), p. 950; see also the subseries approach to removing multiples from seismic data in A.B. Weglein, F.A. Gasparotto, P.M. Carvalho, and R.H. Stolt, Geophysics **62**, 1775 (1997).

[2] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).

[3] H.E. Moses, Phys. Rev. **102**, 559 (1956).

[4] M. Razavy, J. Acoust. Soc. Am. **58**, 956 (1975).

[5] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[6] R.G. Newton, *Scattering Theory of Waves and Particles* (Springer-Verlag, New York, 1982).

[7] M.L. Goldberger and K.M. Watson, *Collision Theory* (Wiley, New York, 1964).

[8] R.T. Prosser, J. Math. Phys. **10**, 1819 (1969); **17**, 1775 (1976); **21**, 2648 (1980).

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience, New York, 1953).

[10] W.N. Sams and D.J. Kouri, J. Chem. Phys. **51**, 4809 (1969).

[11] D.J. Kouri, J. Math. Phys. **14**, 1116 (1973).

[12] A.B. Weglein (private communication).

[13] L.S. Rodberg and R.M. Thaler, *The Quantum Theory of Scattering* (Academic Press, New York, 1967).

[14] J. Mathews and R.L. Walker, *Mathematical Methods of Physics* (Benjamin, New York, 1965).

[15] W. Kaplan, *Advanced Calculus* (Addison-Wesley, Reading, MA, 1952).

[16] A.J. Devaney and A.B. Weglein, J. Inverse Problems **5**, 49 (1989).