Optimal control of the transient behavior of coupled solid-state lasers

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We apply optimal control theory to substantially reduce transient times for transitions between in-phase and out-of-phase states in coupled solid-state lasers. The control is a time-varying optical field that is injected into the cavities of each laser. We have analytically derived the optimal control and numerically solved the optimality system. Numerical simulations indicate that transient times can be significantly reduced upon increasing the injection strength very briefly.

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Laser arrays hold great promise for space communication applications, which require compact sources with high optical intensities and fast switching times [1]. Recently, both solid-state [2,3] and semiconductor [4,5] laser arrays have been investigated to that effect, and various relevant aspects of their dynamical behavior such as chaotic synchronization [6], chaotic communication [7], and amplitude dropout [8] have already been reported.

The most efficient mode of operation for space communication is realized when the array elements are synchronized to an in-phase (IP) state, such that the output interferes constructively and the light intensity at the central lobe scales as N^2, where N is the number of lasers in the array. Unfortunately, the IP state is typically unstable; instead, the system is driven to the stable out-of-phase (OP) state, whose destructive interference pattern results in low output intensities at the central lobe [3,5]. The IP behavior can be stabilized by injecting a common driving laser field into the laser array elements [3,5,9]. Then, for sufficiently high driving amplitude, the elements are entrained and the output intensities interfere constructively; full entrainment of the array is realized above a certain threshold, determined by the coupling of the array elements [3,5].

In addition to synchronization, an equally important issue in applications is the time required to reach the IP behavior from an arbitrary state. In particular, it is desirable to minimize the transient time between the IP and OP states, upon removal of the injected entrainment field. This aspect is important, for example, in fast switching and communications applications.

Despite the obvious practical relevance and potential of the topics, transient behavior, switching, and control thereof in (arrays of) lasers have not been widely studied. We mention though the related work of Porta *et al.* [10], who applied a two-step steering function to the pumping of a single CO_2 laser and reduced the turn-on time by a factor of 3. Using chaos control methods, Uchida *et al.* [11] reported statistical properties of the transient response times between periodic attractors. Switching was realized via high-frequency injection in a laser diode subject to optical feedback. Lippi *et al.* [12] developed a global steering (targeting) technique to induce transitions between states of generic two-dimensional separable nonlinear systems described by a Lotka-Volterra model.

Here we apply optimal control (OC) techniques to reduce the transient switching times between OP and IP states in an array of two coupled solid-state lasers. The OC method is completely general and systematic and does not depend essentially on the internal features of the system. Since it tailors the effort precisely to the desired task, the OC method keeps the cost at its minimum possible and yields significant reductions of the transient time, without resulting in overshoots.

We illustrate the approach on a system of coupled solid-state lasers and demonstrate its efficiency. We start with a complete description of the dynamics of coupled lasers and demonstrate the OC on the dynamics of the phase model. The phase model adequately describes the dynamics of solid-state lasers provided intensity and gain oscillations are small (we will discuss the applicability of the phase model to laser dynamics in the paper later on). The use of the phase model also enhances the generality of the OC approach to other applications, since coupled phase oscillators provide a realistic description of a wide variety of dynamical systems, such as Josephson junctions [13], neural oscillators [14], frictional dynamics [15], and others [16].

OC of systems of ordinary differential equations was developed by Pontryagin and his co-workers in the 1950s [17]. The basic idea is to adjust a coefficient or a source term, viewed as the control, in the differential system to maximize (or minimize) a goal that is represented in terms of the control and corresponding solution (state) of the differential system. Pontryagin's maximum principle for OC of ordinary differential equations does not carry over directly to partial differential equations, but some of the associated techniques do. The corresponding theory was developed by Lions [20] and applied to a wide variety of distributed systems, ranging from economics and management to physical and biological models [18,19,21]. For the sake of simplicity, we present the application of OC to a system of two coupled lasers, but the procedure extends canonically to arbitrary arrays.

We start from the dimensionless system of equations describing the dynamics of two evanescently coupled solid-state lasers, where the polarization is adiabatically eliminated [3,8]:

$$\dot{E}_j(t)=(G_j-\alpha_j+i\delta_j)E_j+\kappa(E_{j+1}+E_{j-1})+E_e(t),$$

$$\dot{G}_j(t)=\frac{\tau_c}{\tau_f}[p_j-(1+|E_j|^2)G_j], \quad j=1,2. \tag{1}$$

In Eqs. (1) free end boundary conditions $[E_0(t)=E_3(t)=0]$ are imposed. The variables E_j and G_j are the dimensionless complex electric field and gain, respectively, for the jth laser. All times and frequencies are scaled relative to the cavity round trip time, τ_c, and τ_f is the fluorescence time of the laser medium; α_j and p_j are the dimensionless cavity decay and pump rates, respectively, for the jth laser, κ is the evanescent coupling constant between the two lasers, and $E_e(t)$ is the slowly varying amplitude of the external field that drives each laser. System (1) is written in a frame rotating with frequency ω_e, at which the external field has a nonzero Fourier component. This frequency is tuned to minimize the detuning from the cavity resonances. In practice, the output power emitted by the array depends on the tuning of external field to the cavities [9]. The detuning $\delta_j=\omega_e-\omega_{cj}-G_j\Delta\omega_j\approx\omega_e-\omega_{cj}$, where ω_{cj} is the cavity resonance frequency for laser j and $\Delta\omega$ is the atomic detuning from ω_e in units of the polarization decay rate. We allow for a small spread in detunings as a way to test the robustness of the entrainment mechanism to a physically reasonable parameter spread.

We assume $\alpha_j=\alpha$, $p_j=p$, $p>\alpha$ [3]. Substituting $E_j(t)=\sqrt{I_j(t)}\exp[i\phi_j(t)]$, where $I_j(t)$ and $\phi_j(t)$ are the intensity and the phase of laser j, respectively, and assuming $E_e(t)=E_e\equiv\sqrt{I_e}$ to be a constant field, the model equations for the two lasers read

$$\dot{I}_j=2(G_j-\alpha)I_j+2\kappa\sqrt{I_1I_2}\cos(\phi_2-\phi_1)+2\sqrt{I_eI_j}\cos\phi_j,$$

$$\dot{\phi}_j=\delta_j+(-1)^j\kappa\frac{\sqrt{I_1I_2}}{I_j}\sin(\phi_1-\phi_2)-\sqrt{I_e/I_j}\sin\phi_j,$$

$$\dot{G}_j=(p-G_j-G_jI_j)\omega_0, \tag{2}$$

where $\omega_0=\tau_c/\tau_f$. System (2) has been studied theoretically for N coupled lasers [3] and the condition for full entrainment was derived. This condition assumes small deviations in detunings and small couplings between the lasers in the array. We denote the dimensionless amplitude of the injected field by $A_e=\sqrt{I_e/I}$, where $I=p/\alpha-1$. Ideally, to entrain an array of N identical lasers requires an injected field amplitude $A_e=4|\kappa|$ or $E_e=4|\kappa|\sqrt{I}$. The functional form of the total output intensity may significantly depend on the parameters of the array, such as detunings and the coupling constant.

For certain ranges of parameters of the laser array, the intensities and gain oscillations are not large and the complete description of the full system [Eqs. (1) and (2)] can be reduced to the ''phase model''

$$\dot{\phi}_1(t)=\delta_1+\kappa\sin(\phi_2-\phi_1)-A_0(t)\sin\phi_1,$$

$$\dot{\phi}_2(t)=\delta_2+\kappa\sin(\phi_1-\phi_2)-A_0(t)\sin\phi_2, \tag{3}$$

where δ_i, $i=1,2$, are the detunings, κ is the coupling constant, and $A_0(t)$ is the amplitude of the injected field.

We have extensively discussed the conditions for which the phase model adequately describes the dynamics of the ''full'' relaxation oscillations model [3,8]. Here we reiterate these conditions only to verify that we are indeed in the parameter range where phase description of laser dynamics is valid. The conditions ask [8] that $\omega\gg\max(1,\varepsilon)$, where $\omega=\alpha/\sqrt{(p-\alpha)\omega_0}$ and $\varepsilon=-\kappa\alpha/(p-\alpha)\omega_0$. For the set of parameters chosen above (which correspond to experimentally measurable parameters for the neodymium-doped yttrium aluminum garnet laser [22]), $\omega=200$ and $\varepsilon=40$. Therefore, we meet this assumption if $|\kappa|\ll 5\times10^{-5}$, which, indeed, is satisfied in our simulations since we use $|\kappa|=1.3\times10^{-5}$.

We consider the physically relevant transitions between OP and IP states. For the case of initial OP lasers, taking the injected field amplitude to be a function of time $A_0(t)$, the initial conditions read

$$0<\phi_1(0)<\pi \quad \text{and} \quad \phi_2(0)=\phi_1(0)-\pi. \tag{4}$$

The control function $A_0(t)$ is a Lebesgue integrable function such that

$$4|\kappa|\leqslant A_0(t)\leqslant M_0. \tag{5}$$

The lower bound is set at $4|\kappa|$ since $4|\kappa|$ is the constant amplitude external field input that will drive the phases to the final IP state, and M_0 is an arbitrary upper bound.

Since the aim is to drive the phases ϕ_1 and ϕ_2 close together quickly, we consider the following objective functional, which is to be minimized as a function of A_0:

$$J_0(A_0)=\frac{1}{2}\int_0^T[(\phi_1-\phi_2)^2(t)+\varepsilon A_0^2(t)]dt. \tag{6}$$

The term εA_0^2 is a stabilizing term that represents ''the cost of the control,'' the positive parameter ε is chosen small to make the $(\phi_1-\phi_2)^2$ term dominant, and the total time interval T is chosen to be shorter than the transient time experienced by the system in the absence of an optimal control. We seek the optimal control $A_0^*(t)$ such that

$$J_0(A_0^*)=\min_{4|\kappa|\leqslant A_0\leqslant M_0} J_0(A_0).$$

Here we present in detail the transition from OP to IP with initial conditions (4). For the transition from OP to IP with initial conditions $\pi\leqslant\phi_1(0)\leqslant2\pi$, $\phi_2(0)=\phi_1(0)-\pi$, the first term in the objective functional is to be replaced by $(\phi_1-\phi_2-2\pi)^2$. Transitions from IP to OP are treated similarly.

The existence of an OC A_0^* and corresponding optimal state pair ϕ_1^*,ϕ_2^* is guaranteed by the convexity of the problem as a function of the control and by the Lipschitz property of the right hand side of Eq. (1) in ϕ_1 and ϕ_2 [23].

The optimization problem stated above is solved using Pontryagin's maximum principle [17,19], which converts the problem (3)–(6) into a problem of minimizing pointwise a Hamiltonian H:

$$H=\frac{1}{2}(\phi_1-\phi_2)^2+\frac{\varepsilon}{2}A_0^2+\lambda_{01}[\delta_1+\kappa\sin(\phi_2-\phi_1)-A_0(t)\sin\phi_1]+\lambda_{02}(\delta_2+\kappa\sin(\phi_1-\phi_2)-A_0(t)\sin\phi_2) \quad (7)$$

with respect to A_0. The adjoint functions λ_{01} and λ_{02} act like "Lagrange multipliers" and couple the differential equations [system (3)] to the minimization problem. Pontryagin's principle yields the following system of differential equations and boundary conditions for the adjoint functions λ_{01} and λ_{02}:

$$\dot{\lambda}_{01}=-\partial H/\partial\phi_1 = -\phi_1+\phi_2+\lambda_{01}[\kappa\cos(\phi_2-\phi_1)+A_0(t)\cos\phi_1]-\lambda_{02}\kappa\cos(\phi_1-\phi_2), \quad (8)$$

$$\dot{\lambda}_{02}=-\partial H/\partial\phi_2 = \phi_1-\phi_2-\lambda_{01}\kappa\cos(\phi_2-\phi_1)+\lambda_{02}[\kappa\cos(\phi_1-\phi_2)+A_0(t)\cos\phi_2],$$

$$\lambda_{01}(T)=0,\quad \lambda_{02}(T)=0 \quad \text{(transversality conditions)}. \quad (9)$$

To minimize the Hamiltonian with respect to A_0, we use the necessary condition $\partial H/\partial A_0=0$ at A_0^*, and solve for A_0^*, taking the bounds into account. As a result, we obtain an explicit characterization of the OC as a function of the state and adjoint functions:

$$A_0^*(t)=\min\left(\max\left[4|\kappa|,\frac{1}{\varepsilon}[\lambda_{01}(t)\sin\phi_1(t)+\lambda_{02}(t)\sin\phi_2(t)]\right],M_0\right). \quad (10)$$

Note that concavity of H with respect to A_0 yields $\partial^2H/\partial A_0^2=\varepsilon>0$, which ensures that we are indeed finding a minimizer.

The OC A_0^* is found by solving the optimality system (OS), i.e., the phase equations (3),(4) and the adjoint equations (8),(9) together with the explicit characterization of the OC, Eq. (10). We discuss the numerical solutions of the OS for various choices of M_0. Since the state system has initial conditions and the adjoint system has final time conditions, the OS cannot be solved by an ordinary forward marching scheme. Instead, an iterative method with a fourth order Runge-Kutta scheme is used, whereby the iterative method consists of the following steps for this two-point boundary value problem. (i) Guess the value of the OG (A_0^*) over the prescribed time *T*. (ii) Solve the state system *forward* in time (for a time period $0\leqslant t\leqslant T$) using the Runge-Kutta scheme. (iii) Solve the adjoint system *backward* in time for a time period ($T\geqslant t\geqslant 0$) using the Runge-Kutta scheme and the solution of the state equations from step (ii). (iv) Update the control by using a convex combination of the previous control and the value calculated from the characterization (10) using the new values of the states and adjoint functions. (v) Repeat steps (i)–(iv) until the difference between the values of unknowns at the present iteration and the previous iteration becomes arbitrarily small.

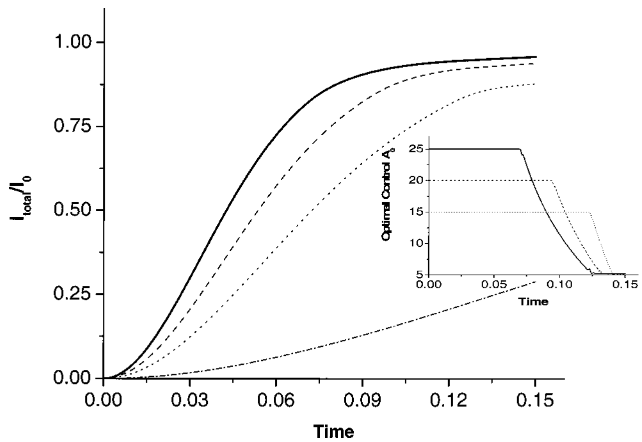

FIG. 1. Normalized total output intensity I_{total}/I_0 as a function of time t in the transition from OP to IP for different values of M_0: 5.2 (dash-dotted curve); 15 (dotted curve); 20 (dashed curve); 25 (solid curve). Inset: the optimal control $A_0(t)$ as a function of time t for different values of M_0: 15 (dotted curve); 20 (dashed curve); 25 (solid curve). The other dimensionless parameters are $T=0.15$, $\kappa=-1.3$, $\delta_1=0.4$, and $\delta_2=-0.4$.

We present numerical results for the case of OP initial conditions $\phi_1(0)=1.7\pi$ and $\phi_2(0)=\phi_1-\pi$. The physical and numerical parameters are given in the figure captions. Since the transient time for the minimal entrainment required to achieve synchronization ($A_0\equiv 4|\kappa|$) is approximately 0.15 time units, we chose $T=0.15$. We then compared the relative effects of the time-varying OCs and the minimal entrainment control on the total output intensity (in units of the intensity of a single uncoupled laser) [8], $I_{\mathrm{total}}/I_0=4\cos^2[(\phi_1-\phi_2)/2]$, where I_0 is the single laser intensity. Various upper bounds M_0 have been used.

In Fig. 1, we present the results for the transition from OP to IP for a few typical cases. The value of the injection amplitude A_0 to obtain the IP behavior (for the parameters of our choice) is given by $A_0=4|\kappa|=5.2$ (since in our simulations $\kappa=-1.3$). We use OC to decrease the transient time to reach the IP behavior. We have chosen three values for the upper bounds between 15 and 25, for which the total output intensities are plotted as functions of time. The total intensity without the OC (i.e., using a constant input $A_0=5.2$) is the reference curve (dash-dotted curve). As expected, the transient time shortens significantly as controls are allowed to take higher values. In the inset, we show three optimal controls during the simulated time, corresponding to the three different upper bounds. In Fig. 2, we show the time that is required to reach 85% (solid curve), 75% (dashed curve), 50% (dotted curve), and 25% (dash-dotted curve) of the maximum intensity as a function of the upper bound imposed on the optimal control.

Several comments are in order.

(1) The OC displayed in the Fig. 1 inset shows two con-

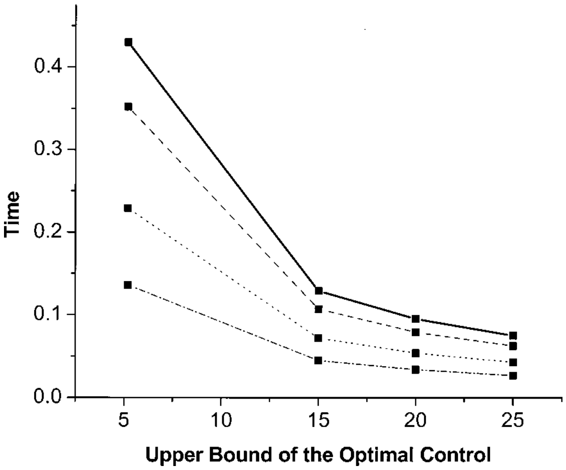

FIG. 2. Time needed to reach 85% (solid curve), 75% (dashed curve), 50% (dotted curve), and 25% (dash-dotted curve) of the total output intensity as a function of the upper bound M_0.

stant regions. The higher one is imposed by the upper limit of the OC, M_0. The lower one is, to a certain extent, the combined result of the OC method and the size of T. By choosing smaller values for T, this region can be reduced or even eliminated completely.

(2) We studied the asymptotic behavior of the total output intensity when the injection amplitude A_0 increases to such levels that the coupling contribution becomes insignificant. Neglecting the coupling one can solve Eqs. (3) and (4) explicitly and show that, in this limit, the total output intensity behaves as

$$I_{\text{total}}/I_0 = 4\cos^2[(\phi_1 - \phi_2)/2]$$
$$= 4\cos^2[\tan^{-1}C_1 e^{-A_0 t} - \tan^{-1}C_2 e^{-A_0 t}],$$

which means that the transient time decreases as $t \sim 1/A_0$. The coefficients C_1 and C_2 can be determined from the initial conditions $C_i = \tan\phi_i(0)/2$, $i = 1,2$.

(3) Finally, we briefly discuss the relevance of these results for the full system Eqs. (2). For small values of the coupling constant κ and detunings δ_i, only very small values of injection amplitude are required for full entrainment of coupled lasers. In that case, the intensities will not change in time and the conditions of using the phase model are satisfied [3,8]; therefore the phase model [Eqs. (3)] yields the same results as the full model [Eqs. (2)]. However, if the injection amplitude is large, relaxation oscillations of the intensity have to be considered and the phase model may not be applicable. We tested the control function obtained from the phase model [Eqs. (3)] on the full model [Eqs. (2)] for small values of the coupling constant ($\kappa = 10^{-6}$) and detunings ($\delta_1 = -\delta_2 = 0.4\times 10^{-6}$) and obtained perfect agreement for small injection field amplitudes. For the higher values of the coupling constants, detunings, and control amplitudes used here, optimal controls computed from the phase model alone still provide a fairly efficient control tool for the whole system, even though we observed small discrepancies between the intensities, particular at short times.

The results presented here demonstrate that OC theory is a very efficient systematic tool for significantly reducing the transient times for the array of coupled lasers when switched between IP and OP states. It is important to mention that a significant reduction of transient times is obtained at the price of briefly increasing the injection strength. In our example, a brief increase by a factor of 3 in injection may lead to a decrease of the transient time by an order of magnitude. Since the typical injection power required to mode-lock a single laser is less than 1% of the total output power of the injected laser, the ''price'' for such transient reduction in switching application is reasonable and worthwhile.

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