Performance analysis of an irreversible quantum heat engine working with harmonic oscillators

Bihong Lin
Department of Physics, Xiamen University, Xiamen 361005, People's Republic of China
and Department of Physics, Quanzhou Normal College, Quanzhou 362000, People's Republic of China

Jincan Chen*
CCAST (World Laboratory), P.O. 8730, Beijing 100080, People's Republic of China
and Department of Physics, Xiamen University, Xiamen 361005, People's Republic of China

(Received 23 August 2002; published 11 April 2003)

The cycle model of a regenerative quantum heat engine working with many noninteracting harmonic oscillators is established. The cycle consists of two isothermal and two constant-frequency processes. The performance of the cycle is investigated, based on the quantum master equation and semigroup approach. The inherent regenerative losses in the two constant-frequency processes are calculated. The expressions of several important performance parameters such as the efficiency, power output, and rate of the entropy production are derived for several interesting cases. Especially, the optimal performance of the cycle in high-temperature limit is discussed in detail. The maximum power output and the corresponding parameters are calculated. The optimal region of the efficiency and the optimal ranges of the temperatures of the working substance in the two isothermal processes are determined.

DOI: 10.1103/PhysRevE.67.046105 PACS number(s): 05.70.−a, 05.90.+m

I. INTRODUCTION

In recent years, the optimal analyses relative to the performance characteristics of thermodynamic cycles have been extended from classical to quantum cycles [1–14]. The quantum thermodynamic cycles working with the spin systems [2,7,13], harmonic oscillator systems [11,13,14], and ideal quantum gases [4,5,10] have become one of the interesting research subjects. The influence of several factors on the performance of quantum heat engines has been investigated and many meaningful conclusions have been obtained. However, these investigations rarely dealt with the performance of a regenerative quantum heat engine working with harmonic oscillators.

For a class of either classical or quantum heat engines with regenerative processes [10,15–17], their performances are, in general, closely dependent on the properties of the working substance. For different working substances, there exist different regenerative losses, so that the performances of heat engines are different from each other. Thus, it is of great significance to study the performance of a regenerative quantum heat engine using harmonic oscillators as the working substance.

The paper is organized in the following manner. In Sec. II, the properties of a harmonic oscillator system are discussed simply and the expression of the first law of thermodynamics of the system is obtained. In Sec. III, a cycle model of a harmonic quantum heat engine consisting of two isothermal and two constant-frequency processes is established and the expressions of the amounts of heat exchange in the various processes of the cycle are derived. In Sec. IV, the regenerative characteristics of the cycle are analyzed and the inherent regenerative losses are determined. In Sec. V, the time evolutions of the harmonic populations in the various processes are calculated, based on the quantum master equation and semigroup approach. In Sec. VI, the general expressions of several important parameters such as the efficiency, power output, and rate of entropy production are given. The performance characteristics of the cycle are investigated for several interesting cases. Especially, the performance of the quantum heat engine in the high-temperature limit is optimized. The maximum power output and the corresponding parameters of the cycle are calculated. The optimally operating regions of the cycle are determined. Finally, some conclusions are given in Sec. VII.

II. A HARMONIC OSCILLATOR SYSTEM

We first consider a quantum system consisting of many noninteracting harmonic oscillators. The Hamiltonian of the system is given by [14,18]

$$\hat{\mathbf{H}}(t)=\omega(t)\hat{\mathbf{N}}=\omega(t)\hat{\mathbf{a}}^{\dagger}\hat{\mathbf{a}}, \tag{1}$$

where $\hat{\mathbf{a}}^{\dagger}$, $\hat{\mathbf{a}}$ are the Bosonic creation and annihilation operators, $\hat{\mathbf{N}}=\hat{\mathbf{a}}^{\dagger}\hat{\mathbf{a}}$ is the number operator, and $\omega>0$ is the oscillator's frequency. The internal energy of the harmonic oscillator system is of the expectation value of the Hamiltonian, i.e.,

$$E=\langle\hat{\mathbf{H}}\rangle=\omega(t)\langle\hat{\mathbf{N}}\rangle=\omega(t)n, \tag{2}$$

where $n=\langle\hat{\mathbf{N}}\rangle$ is the population of the oscillators. Based on the statistical mechanics, the population of the oscillators can be obtained from the Bose-Einstein distribution [19]

*Author to whom correspondence should be addressed. Mailing address: Department of Physics, Xiamen University, Xiamen 361005, People's Republic of China. Email address: jcchen@xmu.edu.cn

 ©2003 The American Physical Society

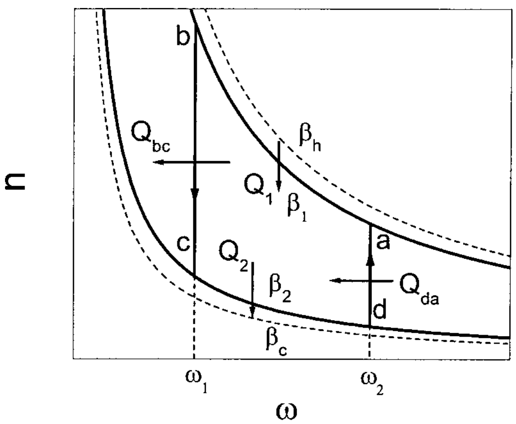

FIG. 1. The n-ω diagram of an irreversible quantum heat engine using harmonic oscillators as the working substance, where the unit of ω is joule.

$$n=\frac{1}{\exp(\beta'\omega)-1}, \tag{3}$$

where $\beta'=1/T$ and T is the absolute temperature in energy units.

When the harmonic oscillators mentioned above are used as the working substance of a quantum heat engine, the internal energy of the working substance may change by changing either the frequency or the population of the oscillators. From Eq. (2), one can obtain

$$dE=n\,d\omega+\omega\,dn. \tag{4}$$

Comparing Eq. (4) with the differential form of the first law of thermodynamics,

$$dE=dW+dQ, \tag{5}$$

one can find that the first term in the right-hand side of Eq. (4) is associated with work and the second term with heat:

$$dW=n\,d\omega \tag{6}$$

and

$$dQ=\omega\,dn. \tag{7}$$

It is thus clear that for a harmonic oscillator system, Eq. (4) gives the differential form of the first law of thermodynamics.

III. A QUANTUM HEAT ENGINE

Using Eq. (3), one can plot the cycle diagram of a harmonic quantum heat engine consisting of two isothermal and two constant-frequency processes and operating between two heat reservoirs at constant temperatures T_h and T_c, as shown in Fig. 1, where the temperature T_c of the heat sink is restricted to be higher than the temperature of Bose-Einstein condensation of the harmonic oscillators. For the convenience of writing, "temperature" will refer to β rather than T. In the cycle, the two isothermal processes with the "temperatures" $\beta'=\beta_1$ and $\beta'=\beta_2$ of the working substance are connected by the two constant-frequency processes $\omega=\omega_1$ and $\omega=\omega_2$ with $\omega_2>\omega_1$. In the two isothermal processes, the oscillators are, respectively, coupled to the heat reservoirs at constant temperatures $\beta=\beta_h$ and $\beta=\beta_c$, and the amounts of heat exchange between the working substance and the heat reservoirs are represented by Q_1 and Q_2. Due to finite-rate heat transfer between the working substance and the heat reservoir, the "temperatures" β_1 and β_2 of the working substance in the two isothermal processes are different from those of the heat reservoirs and there is a relation: $\beta_c>\beta_2>\beta_1>\beta_h$. In order to improve the performance of the cycle, a regenerator is often used in the two constant-frequency processes. The amounts of heat exchange between the working substance and the regenerator during the two constant-frequency processes are represented by Q_{bc} and Q_{da}, respectively,

Using Eqs. (3), (6), and (7), one can find that the amounts of heat exchange in the four processes mentioned above are, respectively, given by

$$Q_1=\int_a^b \omega\,dn=\frac{\omega_1}{e^{\beta_1\omega_1}-1}-\frac{\omega_2}{e^{\beta_1\omega_2}-1}+\frac{1}{\beta_1}\ln\left(\frac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right), \tag{8}$$

$$Q_2=\int_c^d \omega\,dn=\frac{\omega_2}{e^{\beta_2\omega_2}-1}-\frac{\omega_1}{e^{\beta_2\omega_1}-1}+\frac{1}{\beta_2}\ln\left(\frac{1-e^{-\beta_2\omega_1}}{1-e^{-\beta_2\omega_2}}\right), \tag{9}$$

$$Q_{bc}=\int_b^c \omega\,dn=\omega_1(n_c-n_b)=\omega_1\left(\frac{1}{e^{\beta_2\omega_1}-1}-\frac{1}{e^{\beta_1\omega_1}-1}\right), \tag{10}$$

and

$$Q_{da}=\int_d^a \omega\,dn=\omega_2(n_c-n_d)=\omega_2\left(\frac{1}{e^{\beta_1\omega_2}-1}-\frac{1}{e^{\beta_2\omega_2}-1}\right), \tag{11}$$

where n_a, n_b, n_c, and n_d are the mean values of the harmonic oscillator population in a, b, c, and d states in Fig. 1, respectively. Using Eqs. (8)–(11), we can calculate the work output per cycle as

$$\begin{aligned}W&=Q_1+Q_2+Q_{bc}+Q_{da}\\&=\frac{1}{\beta_1}\ln\left(\frac{e^{\beta_1\omega_2}-1}{e^{\beta_1\omega_1}-1}\right)-\frac{1}{\beta_2}\ln\left(\frac{e^{\beta_2\omega_2}-1}{e^{\beta_2\omega_1}-1}\right).\end{aligned} \tag{12}$$

Using above equations, we can discuss the optimal performance of a quantum heat engine using harmonic oscillators as the working substance.

IV. REGENERATIVE CHARACTERISTICS

From Eqs. (10) and (11), one can calculate the net amount of heat transfer between the working substance and the regenerator during the two regenerative processes as

$$\Delta Q = Q_{bc} + Q_{da}$$
$$= \omega_2 \left(\frac{1}{e^{\beta_1 \omega_2} - 1} - \frac{1}{e^{\beta_2 \omega_2} - 1} \right)$$
$$- \omega_1 \left(\frac{1}{e^{\beta_1 \omega_1} - 1} - \frac{1}{e^{\beta_2 \omega_1} - 1} \right). \quad (13)$$

It is seen from Eq. (13) that ΔQ is smaller than zero, because the function $f(\omega) = \omega/(e^{\beta_1 \omega} - 1) - \omega/(e^{\beta_2 \omega} - 1)$ is a monotonically decreasing function of ω. This implies the fact that the amount of heat Q_{bc} flowing from the working substance into the regenerator in one regenerative process is larger than that of heat Q_{da} flowing from the regenerator into the working substance in the other regenerative process. The redundant heat in the regenerator per cycle must be released to the heat sink at ''temperature'' β_c in a timely manner [20,21]. This results in the increase of the amount of heat rejected to the heat sink per cycle from Q_2 to $Q_c = Q_2 - \Delta Q$, while the amount of heat Q_1 supplied by the heat reservoir per cycle is unvarying. If not, the temperature of the regenerator would be changed such that the regenerator would not operate normally. It is thus obvious that a harmonic quantum heat engine consisting of two isothermal and two constant-frequency processes does not possess, in principle, the condition of prefect regeneration.

V. CYCLE TIME

In order to discuss further the performance of a harmonic quantum heat engine, we have to solve the equation of motion that determines the time evolution of the harmonic populations. For a harmonic quantum heat engine, the working substance is coupled thermally to a heat reservoir at temperature T. Using the Heisenberg picture for the rate of change of an operator, one obtains [14,22–25] (throughout this paper we adopt $\hbar = I$ for simplicity)

$$\frac{d\hat{\mathbf{X}}}{dt} = i[\hat{\mathbf{H}}, \hat{\mathbf{X}}] + \frac{\partial \hat{\mathbf{X}}}{\partial t} + L_D(\hat{\mathbf{X}}), \quad (14)$$

where $L_D(\hat{\mathbf{X}}) = \Sigma \gamma_\alpha (\hat{\mathbf{V}}_\alpha^\dagger [\hat{\mathbf{X}}, \hat{\mathbf{V}}_\alpha] + [\hat{\mathbf{V}}_\alpha^\dagger, \hat{\mathbf{X}}] \hat{\mathbf{V}}_\alpha)$ is a dissipation term and originates from a thermal coupling of the working substance to a heat reservoir, $\hat{\mathbf{V}}_\alpha$ and $\hat{\mathbf{V}}_\alpha^\dagger$ are operators in the Hilbert space of the system and are Hermitian conjugates, and γ_α are phenomenological positive coefficients. For a harmonic oscillator system, $\hat{\mathbf{V}}_\alpha$ are chosen to be the Bosonic creation and annihilation operators: $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$, and $\hat{\mathbf{H}} = \omega \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}}$. Substituting $\hat{\mathbf{a}}^\dagger$, $\hat{\mathbf{a}}$, $\hat{\mathbf{H}}$, and $\hat{\mathbf{X}} = \hat{\mathbf{N}}$ into Eq. (14), one can prove [14] that

$$\frac{dn}{dt} = -2a e^{q\beta\omega} [(e^{\beta\omega} - 1)n - 1], \quad (15)$$

where $a > 0$, $-1 < q < 0$, and β, ω, and n are, in general, dependent on time [14]. The explicit quantum mechanical nature of a heat engine working with harmonic oscillators is manifested by the dual characters of ω, i.e., $\hbar\omega (\hbar = 1)$ defines the energy level structure of the heat engine and ω is a frequency of the oscillators so that ω^{-1} defines an intrinsic time scale. This implicitly assumes an instantaneous response of the heat reservoir to changes in the frequency ω and the time duration of a process should be long enough so that resonance conditions are established instantaneously. This means that the time duration at each process has to be much larger than the intrinsic time scale [14]. Thus, the change of ω with time is small. This point can also be directly expounded from Eq. (3).

The solution of Eq. (15) gives the expression of time evolution as

$$t = -\frac{1}{2a} \int_{n_i}^{n_f} \frac{dn}{e^{q\beta\omega} [(e^{\beta\omega} - 1)n - 1]}, \quad (16)$$

where n_i and n_f are the initial and final values of n along a given path $n(\beta', \omega)$. Equation (16) is a general expression of time evolution for a harmonic oscillator system coupled with the heat reservoir.

Based on Eqs. (3) and (16), we can calculate the times spent on the four processes in the cycle. Substituting $n(\omega) = 1/(e^{\beta_1 \omega} - 1)$, $\beta = \beta_h$, $n_i = n_i(\beta_1, \omega_2)$, and $n_f = n_f(\beta_1, \omega_1)$ into Eq. (16), one can obtain the time of the high-temperature isothermal process as

$$t_1 = \frac{\beta_1}{2a} \int_{\omega_1}^{\omega_2} [e^{q\beta_h \omega} (e^{\beta_1 \omega} - e^{\beta_h \omega})(1 - e^{-\beta_1 \omega})]^{-1} d\omega. \quad (17)$$

Similarly, substituting $n(\omega) = 1/(e^{\beta_2 \omega} - 1)$, $\beta = \beta_c$, $n_i = n_i(\beta_2, \omega_1)$, and $n_f = n_f(\beta_2, \omega_2)$ into Eq. (16), one can obtain the time of the low-temperature isothermal process as

$$t_3 = \frac{\beta_2}{2a} \int_{\omega_1}^{\omega_2} [e^{q\beta_c \omega} (e^{\beta_c \omega} - e^{\beta_2 \omega})(1 - e^{-\beta_2 \omega})]^{-1} d\omega. \quad (18)$$

In the two constant-frequency processes, the ''temperature'' of the working substance changes from β_1 to β_2 or from β_2 to β_1, so they need a non-negligible time compared with the time of the isothermal processes. Substituting $n(\beta') = 1/(e^{\beta' \omega_1} - 1)$, $\beta = \beta_{1r}$, $n_i = n_i(\beta_1, \omega_1)$, and $n_f = n_f(\beta_2, \omega_1)$ into Eq. (16), one can obtain the time of the constant-frequency process with $\omega = \omega_1$ as

$$t_2 = \frac{\omega_1}{2a} \int_{\beta_1}^{\beta_2} [e^{q\beta_{1r} \omega_1} (e^{\beta_{1r} \omega_1} - e^{\beta' \omega_1})(1 - e^{-\beta' \omega_1})]^{-1} d\beta', \quad (19)$$

where β_{1r} is the ''temperature'' of the regenerator in the regenerative processes with $\omega = \omega_1$ and $\beta_{1r} > \beta'$ because heat is transferred from the working substance to the regenerator. Similarly, substituting $n(\beta') = 1/(e^{\beta' \omega_2} - 1)$, $\beta = \beta_{2r}$, $n_i = n_i(\beta_2, \omega_2)$, and $n_f = n_f(\beta_1, \omega_2)$ into Eq. (16), one can obtain the time of the constant-frequency process with $\omega = \omega_2$ as

$$t_4=\frac{\omega_2}{2a}\int_{\beta_1}^{\beta_2}[e^{q\beta_{2r}\omega_2}(e^{\beta'\omega_2}-e^{\beta_{2r}\omega_2})(1-e^{-\beta'\omega_2})]^{-1}d\beta', \tag{20}$$

where β_{2r} is the ''temperature'' of the regenerator in the regenerative process with $\omega=\omega_2$ and $\beta_{2r}<\beta'$ because heat is transferred from the regenerator to the working substance.

So far we have obtained the times spent on two isothermal and two regenerative processes. Consequently, the cycle time is determined by

$$\tau=t_1+t_2+t_3+t_4. \tag{21}$$

VI. ANALYSIS ON SEVERAL IMPORTANT PARAMETERS

The efficiency and power output are two of the important performance parameters, which are often considered in the optimal design and theoretical analysis of heat engines. Using Eqs. (8), (12), and (21), we find that the efficiency and power output may be, respectively, expressed as

$$\eta=\frac{W}{Q_h}=\frac{\dfrac{1}{\beta_1}\ln\left(\dfrac{e^{\beta_1\omega_2}-1}{e^{\beta_1\omega_1}-1}\right)-\dfrac{1}{\beta_2}\ln\left(\dfrac{e^{\beta_2\omega_2}-1}{e^{\beta_2\omega_1}-1}\right)}{\dfrac{\omega_1}{e^{\beta_1\omega_1}-1}-\dfrac{\omega_2}{e^{\beta_1\omega_2}-1}+\dfrac{1}{\beta_1}\ln\left(\dfrac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right)} \tag{22}$$

and

$$P=\frac{W}{\tau}=\frac{\dfrac{1}{\beta_1}\ln\left(\dfrac{e^{\beta_1\omega_2}-1}{e^{\beta_1\omega_1}-1}\right)-\dfrac{1}{\beta_2}\ln\left(\dfrac{e^{\beta_2\omega_2}-1}{e^{\beta_2\omega_1}-1}\right)}{t_1+t_2+t_3+t_4}. \tag{23}$$

In addition, using Eqs. (8), (9), (13), and (21), one can obtain the expression of the rate of the entropy production as

$$\sigma=\frac{\Delta S}{\tau}=\frac{\beta_hQ_h+\beta_cQ_c}{\tau}=\frac{\left[(\beta_h+\beta_c)\left(\dfrac{\omega_2}{e^{\beta_1\omega_2}-1}-\dfrac{\omega_1}{e^{\beta_1\omega_1}-1}\right)-\dfrac{\beta_h}{\beta_1}\ln\left(\dfrac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right)+\dfrac{\beta_c}{\beta_2}\ln\left(\dfrac{1-e^{-\beta_2\omega_2}}{1-e^{-\beta_2\omega_1}}\right)\right]}{t_1+t_2+t_3+t_4}. \tag{24}$$

Using Eqs. (22)–(24), one can, in principle, optimize these important performance parameters of the quantum heat engine.

(a) Only if the temperature of the heat sink is low enough, i.e., $\beta_2\omega_i\gg 1$ $(i=1,2)$, Eqs. (22)–(24) can be, respectively, given by

$$\eta=1-\frac{\omega_1/(e^{\beta_1\omega_1}-1)-\omega_2/(e^{\beta_1\omega_2}-1)+(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})/\beta_2}{\omega_1/(e^{\beta_1\omega_1}-1)-\omega_2/(e^{\beta_1\omega_2}-1)+\dfrac{1}{\beta_1}\ln\left(\dfrac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right)}, \tag{25}$$

$$P=\frac{\dfrac{1}{\beta_1}\ln\left(\dfrac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right)-\dfrac{1}{\beta_2}(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})}{t_1+t_2+t_3+t_4}, \tag{26}$$

and

$$\sigma=\frac{(\beta_h+\beta_c)\left(\dfrac{\omega_2}{e^{\beta_1\omega_2}-1}-\dfrac{\omega_1}{e^{\beta_1\omega_1}-1}\right)-\dfrac{\beta_h}{\beta_1}\ln\left(\dfrac{1-e^{-\beta_1\omega_2}}{1-e^{-\beta_1\omega_1}}\right)+\dfrac{\beta_c}{\beta_2}(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})}{t_1+t_2+t_3+t_4}. \tag{27}$$

(b) When the temperature of two heat reservoirs are low enough, i.e., $\beta\omega\gg 1$, Eqs. (25)–(27) can be further simplified as

$$\eta=1-\frac{\beta_1\omega_1e^{-\beta_1\omega_1}-\beta_1\omega_2e^{-\beta_1\omega_2}+(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})\beta_1/\beta_2}{(1+\beta_1\omega_1)e^{-\beta_1\omega_1}-(1+\beta_1\omega_2)e^{-\beta_1\omega_2}}, \tag{28}$$

$$P=\frac{(e^{-\beta_1\omega_1}-e^{-\beta_1\omega_2})/\beta_1-(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})/\beta_2}{t_1+t_2+t_3+t_4}, \tag{29}$$

and

$$\sigma=\frac{(\beta_h+\beta_1)(\omega_2e^{-\beta_1\omega_2}-\omega_1e^{-\beta_1\omega_1})-(e^{-\beta_1\omega_1}-e^{-\beta_1\omega_2})\beta_h/\beta_1+(e^{-\beta_2\omega_1}-e^{-\beta_2\omega_2})\beta_c/\beta_2}{t_1+t_2+t_3+t_4}, \tag{30}$$

respectively.

(c) When the temperature of the heat reservoir is high enough and the temperature of the heat sink is low enough, i.e., $\beta_1\omega_i \ll 1$, $\beta_2\omega_i \gg 1$ ($i=1,2$), Eqs. (22)–(24) can be, respectively, expressed as

$$\eta = 1 - \frac{(e^{-\beta_2\omega_1} - e^{-\beta_2\omega_2})/\beta_2}{\ln(\omega_2/\omega_1)/\beta_1 - (\omega_2-\omega_1)}, \tag{31}$$

$$P = \frac{\ln(\omega_2/\omega_1)/\beta_1 - (\omega_2-\omega_1) - (e^{-\beta_2\omega_1} - e^{-\beta_2\omega_2})/\beta_2}{t_1+t_2+t_3+t_4}, \tag{32}$$

and

$$\sigma = \frac{\beta_h(\omega_2-\omega_1) - \ln(\omega_2/\omega_1)\beta_h/\beta_1 + (e^{-\beta_2\omega_1} - e^{-\beta_2\omega_2})\beta_c/\beta_2}{t_1+t_2+t_3+t_4}. \tag{33}$$

(d) Only if the temperature of the heat reservoir is high enough, i.e., $\beta_1\omega_i \ll 1$ ($i=1,2$), Eqs. (22)–(24) can be, respectively, expressed as

$$\eta = 1 - \beta_1 \ln\left(\frac{1-e^{-\beta_2\omega_2}}{1-e^{-\beta_2\omega_1}}\right) \Bigg/ \beta_2 \ln\left(\frac{\omega_2}{\omega_1}\right), \tag{34}$$

$$P = \frac{\dfrac{1}{\beta_1}\ln\left(\dfrac{\omega_2}{\omega_1}\right) - \dfrac{1}{\beta_2}\ln\left(\dfrac{e^{\beta_2\omega_1}-1}{e^{\beta_2\omega_2}-1}\right)}{t_1+t_2+t_3+t_4}, \tag{35}$$

and

$$\sigma = \left[\frac{\beta_c}{\beta_2}\ln\left(\frac{1-e^{-\beta_2\omega_2}}{1-e^{-\beta_2\omega_1}}\right) - (\beta_c-\beta_h)(\omega_2-\omega_1) - \frac{\beta_h}{\beta_1}\ln\left(\frac{\omega_2}{\omega_1}\right)\right] \Bigg/ (t_1+t_2+t_3+t_4). \tag{36}$$

(e) When the temperatures of two heat reservoirs are high enough, i.e., $\beta\omega \ll 1$, the results obtained above can be simplified. For example, Eqs. (8)–(13), (17)–(20), and (22) can be, respectively, simplified as

$$Q_1 = \frac{1}{\beta_1}\ln\left(\frac{\omega_2}{\omega_1}\right), \tag{37}$$

$$Q_2 = \frac{1}{\beta_2}\ln\left(\frac{\omega_1}{\omega_2}\right), \tag{38}$$

$$Q_{bc} = \frac{1}{\beta_2} - \frac{1}{\beta_1}, \tag{39}$$

$$Q_{da} = \frac{1}{\beta_1} - \frac{1}{\beta_2}, \tag{40}$$

$$W = \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\ln\left(\frac{\omega_2}{\omega_1}\right), \tag{41}$$

$$\Delta Q = Q_{bc} + Q_{da} = 0, \tag{42}$$

$$t_1 = \frac{\omega_2-\omega_1}{2a\omega_1\omega_2(\beta_1-\beta_h)}, \tag{43}$$

$$t_3 = \frac{\omega_2-\omega_1}{2a\omega_1\omega_2(\beta_c-\beta_2)}, \tag{44}$$

$$t_2 = \frac{1}{2a\omega_1}\int_{\beta_1}^{\beta_2} \frac{d\beta'}{\beta'(\beta_{1r}-\beta')}, \tag{45}$$

$$t_4 = \frac{1}{2a\omega_2}\int_{\beta_1}^{\beta_2} \frac{d\beta'}{\beta'(\beta'-\beta_{2r})}, \tag{46}$$

and

$$\eta = 1 - \beta_1/\beta_2. \tag{47}$$

It should be noted that the "temperatures" β_{1r} and β_{2r} of the regenerator in two constant-frequency processes are not constant and vary with time. If there is not any additional assumption, Eqs. (45) and (46) cannot be calculated further. One of the simplest assumptions is that β_{1r} and β' are linear dependent and so are β_{2r} and β', i.e., $\beta_{1r} \propto \beta'$ and $\beta_{2r} \propto \beta'$. Then, the times spent on the two regenerative processes can be simply given by

$$t_2 + t_4 = \gamma(1/\beta_1 - 1/\beta_2), \tag{48}$$

where γ is a proportional constant independent of temperature. It will be seen from other assumptions given below that this simple assumption is reasonable.

In general, the larger the temperature difference of the working substance in the two isothermal processes is, the larger is the amount of regeneration and the longer is the time of the regenerative processes. When the times spent on the two regenerative processes are assumed to be directly proportional to the amount of regeneration [21], the times spent on two regenerative processes can be expressed as

$$t_2 + t_4 = \alpha(|Q_{bc}| + Q_{da}) = 2\alpha\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) = \gamma\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right), \tag{49}$$

where α is also a proportional constant independent of temperature. Equations (48) and (49) just indicate that the two assumptions mentioned above are equivalent to each other.

It is of significance to note another assumption adopted in many papers: the temperature of the working substance in the regenerative processes varies with time linearly [15,20,26,27], i.e.,

$$\frac{dT}{dt} = \pm\kappa, \tag{50}$$

where κ is a constant independent of temperature but dependent on the properties of the working substance, and the positive and negative signs correspond to the regenerative heating and cooling processes, respectively. As long as $2/\kappa = \gamma$ is chosen, one easily derives Eq. (48) from Eq. (50). It tells us once again that the assumption given in this paper is reasonable.

Using Eqs. (43), (44), and (48), one can obtain the cycle time as

$$\tau = d\left(\frac{1}{\beta_1 - \beta_h} + \frac{1}{\beta_c - \beta_2}\right) + \gamma\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right), \tag{51}$$

where $d = (\omega_2 - \omega_1)/2a\omega_1\omega_2$. Substituting Eqs. (41) and (51) into Eqs. (23), one obtains the power output as

$$P = \frac{b(y-1)}{d\beta_1 y[1/(\beta_1 - \beta_h) + 1/(\beta_c - \beta_1 y)] + \gamma(y-1)}, \tag{52}$$

where $b = \ln(\omega_2/\omega_1)$ and $y = \beta_2/\beta_1$. Similarly, substituting Eqs. (37), (38), and (51) into Eq. (24) gives

$$\sigma = \frac{b(\beta_c - \beta_h y)}{d\beta_1 y[1/(\beta_1 - \beta_h) + 1/(\beta_c - \beta_1 y)] + \gamma(y-1)}. \tag{53}$$

Using Eq. (52) and the extremal condition $\partial P/\partial \beta_1 = 0$, one can obtain an optimal relation

$$\beta_1 = \frac{\beta_c + \theta\beta_h}{y + \theta}, \tag{54}$$

where $\theta = \sqrt{\beta_c/\beta_h}$. Solving Eqs. (47), (52), (53), and (54), one can prove that the fundamental optimal relations between some important parameters and the efficiency are, respectively, given by

$$\beta_1 = \beta_h \frac{(1-\eta)[1+\theta(1-\eta_c)]}{(1-\eta_c)[1+\theta(1-\eta)]}, \tag{55}$$

$$\beta_2 = \beta_c \frac{1+\theta(1-\eta_c)}{1+\theta(1-\eta)}, \tag{56}$$

$$P = \frac{\theta\eta(\eta_c - \eta)}{B(1+\theta)(1-\eta)[1+\theta(1-\eta_c)] + D\theta\eta(\eta_c - \eta)}, \tag{57}$$

and

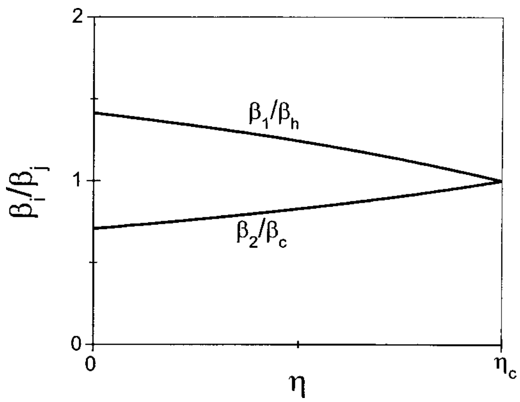

FIG. 2. The (β_1/β_h)-η and (β_2/β_c)-η characteristic curves for $\beta_c/\beta_h = 2$.

$$\sigma = \frac{\beta_c\theta(\eta_c - \eta)^2}{B(1+\theta)(1-\eta)[1+\theta(1-\eta_c)] + D\theta\eta(\eta_c - \eta)}, \tag{58}$$

where $B = d/b$, $D = \gamma/b$, and $\eta_c = 1 - \beta_h/\beta_c$ is the efficiency of a reversible Carnot heat engine.

It is clearly seen from Eq. (57) that the power output is zero when $\eta = 0$ or $\eta = \eta_c$. This implies the fact that when the efficiency is equal to some value, the power output has a maximum. Using Eqs. (55)–(58), we can plot the $\beta_i/\beta_j - \eta$ ($i = 1$, 2 and $j = h, c$), P^*-η and σ^*-η characteristic curves, as shown in Figs. 2, 3, and 4, where $P^* = BP$ and $\sigma^* = T_c B\sigma$ are, respectively, the dimensionless power output and rate of the minimum-average-entropy production. It is also clearly seen from Fig. 3 that there exists a maximum power output. Using Eq. (57), one can prove that when the efficiency

$$\eta_m = 1 - \sqrt{\beta_h/\beta_c} \equiv \eta_{CA} \tag{59}$$

the power output attains its maximum value, i.e.,

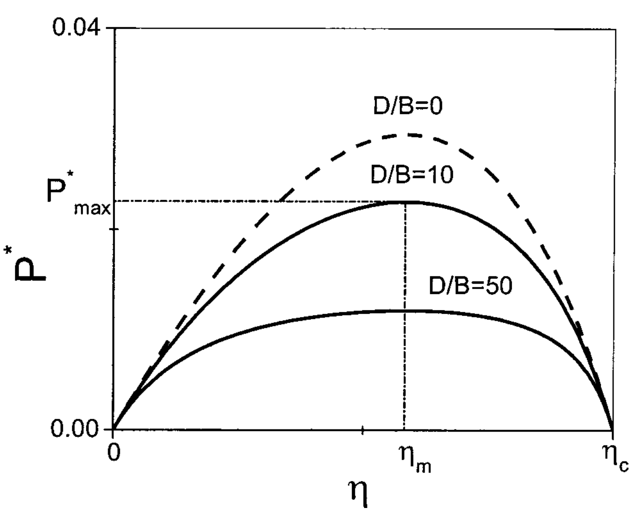

FIG. 3. The dimensionless power output $P^* = BP$ versus the efficiency η. Dashed ($D/B = 0$) and solid ($D/B = 10$ and $D/B = 50$) curves are presented for $\beta_c/\beta_h = 2$.

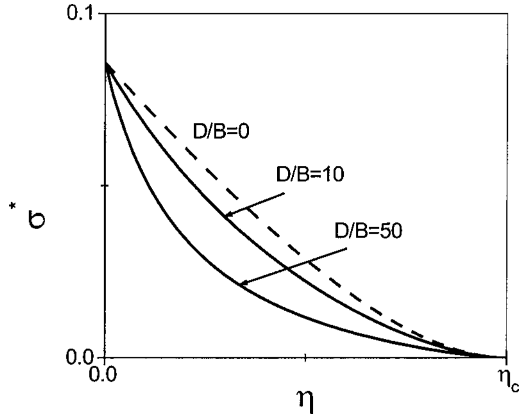

FIG. 4. The dimensionless rate of the entropy production $\sigma^* = T_c B \sigma$ versus the efficiency η. The values of the parameters D/B and β_c/β_h are the same as those used in Fig. 3.

$$P_{\max} = \frac{(\theta-1)^2}{B(1+\theta)^2 + D(\theta-1)^2}, \tag{60}$$

where η_{CA} is the efficiency of an endoreversible Carnot heat engine at the maximum power output [28–30]. Substituting Eq. (59) into Eqs. (55), (56), and (58), one obtains the values of β_1, β_2, and σ at the maximum power output as

$$\beta_{1m} = \frac{\beta_h + \sqrt{\beta_c \beta_h}}{2}, \tag{61}$$

$$\beta_{2m} = \frac{\beta_c + \sqrt{\beta_c \beta_h}}{2}, \tag{62}$$

and

$$\sigma_m = \frac{\beta_c \theta^2 (\theta-1)^2}{B(\theta+1)^2 + D(\theta-1)^2}. \tag{63}$$

It is thus clear that in the high-temperature limit, the efficiency of the harmonic heat engine at the maximum power output is the same as that of a Curzon-Ahlborn engine [28] at the maximum power output. It is because, in the high-temperature limit, the cycle may possess the condition of perfect regeneration through the use of a reversible regenerator and the heat transfer between the harmonic oscillator system and the heat reservoirs can be simply written as

$$Q_i = F(1/\beta_j - 1/\beta_i) \quad (i=1,2; \quad j=h,c), \tag{64}$$

where $F = 2a\omega\beta_j$ is independent of the temperature of the working substance and the temperature difference between the working substance and the heat reservoir. It is obvious that Eq. (64) can be regarded as the Newtonian law.

It is seen from Figs. 3 and 4 that when $\beta_1 = \beta_h$ and $\beta_2 = \beta_c$, $\eta = \eta_c$, $P = 0$, and $\sigma = 0$. This implies the fact that the efficiency of the quantum heat engine mentioned above cannot attain that of a reversible Carnot heat engine. Figure 3 also shows that when $P < P_{\max}$, there are two different efficiencies for a given power output P, where one is smaller than η_m and the other is larger than η_m. When $\eta < \eta_m$, the power output decreases as the efficiency decreases. Obviously, the region of $\eta < \eta_m$ is not optimal for a harmonic quantum heat engine. Consequently, the optimal region of the efficiency should be

$$\eta_m \leqslant \eta < \eta_c. \tag{65}$$

When a quantum heat engine is operated in this region, the power output will increase as the efficiency decreases, and vice versa. It is thus clear that $P_{\max}$ and η_m are two important parameters, because $P_{\max}$ determines the upper bound of the power output, while η_m gives the allowable value of the lower bound of the optimal efficiency.

Analyzing Eq. (65) and Figs. 2–4, we find that the optimal ranges of the "temperature" of the working substance in the two isothermal processes and the cycle time are

$$\beta_{1m} \geqslant \beta_1 > \beta_h, \tag{66}$$

$$\beta_{2m} \leqslant \beta_2 < \beta_c, \tag{67}$$

and

$$\tau \geqslant \tau_m, \tag{68}$$

where

$$\tau_m = \frac{2\eta_c(1-\eta_{\mathrm{CA}})}{\beta_h}\left(\frac{d}{\eta_{\mathrm{CA}}^2} + \frac{\gamma}{(2-\eta_{\mathrm{CA}})^2}\right).$$

When the regenerative time is negligible, $D = 0$. In this case, Eqs. (59), (61), and (62) are still true, while Eqs. (57), (58), (60), and (63) can be, respectively, simplified as

$$P = \frac{\theta\eta(\eta_c - \eta)}{B(1+\theta)(1-\eta)[1+\theta(1-\eta_c)]}, \tag{69}$$

$$\sigma = \frac{\beta_c \theta(\eta_c - \eta)^2}{B(1+\theta)(1-\eta)[1+\theta(1-\eta_c)]}, \tag{70}$$

$$P_{\max} = \frac{(\theta-1)^2}{B(1+\theta)^2}, \tag{71}$$

and

$$\sigma_m = \frac{\beta_c \theta^2 (\theta-1)^2}{B(\theta+1)^2}. \tag{72}$$

The results obtained above indicate clearly that the maximum power output is dependent on the time of the constant-frequency processes, while the efficiency at the maximum power output is not affected by the time of the constant-frequency processes. In this case, the relation curves of the power output, and rate of the minimum-average-entropy production varying with the efficiency are shown by the dashed curves in Figs. 3 and 4, respectively.

VII. CONCLUSIONS

We have established the cyclic model of a typical quantum power cycle working with many noninteracting harmonic oscillators and consisting of two isothermal and two constant-frequency processes. On the basis of the statistical mechanics, motion equation of an operator, and semi-group formalism, we have analyzed the optimal performance characteristics of the harmonic quantum power cycles and derived the general expressions of several important parameters of the quantum heat engine. By using the expressions, the influence of nonperfect regeneration is analyzed. In general, the cycle does not possess the condition of perfect regeneration. However, in the high-temperature limit, it may possess the condition of perfect regeneration. Furthermore, the optimum performance characteristics of the quantum heat engine in high-temperature limit are discussed in detail. For example, the maximum power output and the corresponding parameters are calculated and the optimally operating regions of the quantum heat engine are determined. The results obtained here will be helpful to understanding further the performance of quantum heat engines using harmonic oscillators as the working substance.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation (Grant No. 10275051), People's Republic of China.

[1] E. Geva, J. Mod. Opt. **49**, 635 (2002).
[2] J. He, J. Chen, and B. Hua, Phys. Rev. E **65**, 036145 (2002).
[3] K. Bhattacharyya and S. Mukhopadhyay, J. Phys. A **34**, 1529 (2001).
[4] A. Sisman and H. Saygin, Appl. Energy **68**, 367 (2001).
[5] H. Saygin and A. Sisman, J. Appl. Phys. **90**, 3085 (2001).
[6] R. Kosloff, E. Geva, and J. M. Gordon, J. Appl. Phys. **87**, 8093 (2000).
[7] T. Feldmann and R. Kosloff, Phys. Rev. E **61**, 4774 (2000).
[8] C. M. Bener, D. C. Brody, and B. K. Meister, J. Phys. A **33**, 4427 (2000).
[9] F. Wu, L. Chen, F. Su, and C. Wu, Int. J. Eng. Sci. **38**, 239 (2000).
[10] A. Sisman and H. Saygin, J. Phys. D **32**, 664 (1999).
[11] F. Wu, L. Chen, F. Sun, C. Wu, and Y. Zhu, Energy Convers. Manage. **39**, 733 (1998).
[12] E. Geva and R. Kosloff, J. Chem. Phys. **104**, 7681 (1996).
[13] T. Feldmann, E. Geva, and R. Kosloff, Am. J. Phys. **64**, 485 (1996).
[14] E. Geva and R. Kosloff, J. Chem. Phys. **97**, 4398 (1992).
[15] J. Chen, Int. J. Ambient Energ. **18**, 107 (1997).
[16] S. C. Kaushik and S. Kumar, Energy Convers. Manage. **42**, 295 (2001).
[17] J. Chen, B. Lin, and B. Hua, J. Appl. D **35**, 2051 (2002).
[18] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1990).
[19] P. K. Pathria, *Statistical Mechanics* (Pergamon, London, 1972).
[20] J. Chen and Z. Yan, J. Phys. D **29**, 987 (1996).
[21] J. Chen and Z. Yan, J. Appl. Phys. **84**, 1791 (1998).
[22] R. Alicki, J. Phys. A **12**, L103 (1979).
[23] R. Alicki and K. Leudi, *Quantum Dynamical Semi-groups and Applications* (Springer-Verlag, Berlin, 1987).
[24] E. Geva and R. Kosloff, J. Chem. Phys. **96**, 3054 (1992).
[25] E. Geva and R. Kosloff, Phys. Rev. E **49**, 3903 (1994).
[26] F. A. Brown and G. R. Madrigal, Rev. Mex. Fis. **36**, 363 (1990).
[27] S. C. Kaushik and S. Kumar, Energy Convers. Manage. **42**, 295 (2001).
[28] F. L. Curzon and B. Ahlborn, Am. J. Phys. **43**, 22 (1975).
[29] J. Chen, J. Phys. D **27**, 1144 (1994).
[30] C. Wu, L. Chen, and J. Chen, *Recent Advances in Finite-Time Thermodynamics* (Nova Science, New York, 1999).