Mechanical unfolding of directed polymers in a poor solvent: Critical exponents

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(Received 20 January 2003; revised manuscript received 3 March 2003; published 29 April 2003)

We study the thermodynamics of an exactly solvable model of a self-interacting, partially directed self-avoiding walk in two dimensions when a force is applied on one end of the chain. The critical force for the unfolding is determined exactly, as a function of the temperature, below the Θ transition. The transition is of second order and is characterized by new critical exponents that are determined by a careful numerical analysis. The usual polymer critical index ν on the critical line, and another one which we call ζ, takes a nontrivial value that is numerically close to 2/3.

DOI: 10.1103/PhysRevE.67.041802 PACS number(s): 82.35.Lr, 64.60.Fr, 05.20.Gg, 87.15.−v

I. INTRODUCTION

The nature of the collapsed phase that a polymer attains in poor solvent conditions is still under debate (see, e.g., Refs. [1,2], and references therein). Recent experiments on pulling of polymers and biopolymers (see, e.g., Refs. [3–5]) have enhanced theoretical interest on the unfolding transition a collapsed polymer undergoes when subjected to an external force f applied at its extrema. Until very recently, most of the existing studies on this subject dealt with a refined version of the mean field studies originally proposed in Ref. [6]. A common characteristic of such studies is that, for a self-attracting polymer, they predict a first-order phase transition in any dimension at a critical force $f_c(T)$. At temperatures below the Θ transition, where the self-attraction prevails, and for an applied force less than f_c, the polymer is in a compact phase. For forces greater than f_c, the self-attraction is unable to maintain the polymer in its compact conformation and the polymer chain is stretched along the force direction. However, in $d=2$ extensive Monte Carlo simulations [7], performed on a self-avoiding walk (SAW) model, suggested that the transition is of second order. An exactly solvable model, on a lattice of fractal dimension 2, has been analyzed in Ref. [8] and a second-order transition was found at a critical force $f_c(T)$. In Ref. [9], a rationale was given for the change in order of the transition as the spatial dimension d goes past 2 by means of a renormalization group based argument. Within this framework, it was found that, near criticality, the projection of the end-to-end distance along the force direction per monomer goes like $f-f_c$, near the phase transition, where f is the force and f_c is the critical force. Numerical uncertainties are too large to critically test this prediction in the SAW model of Refs. [7,9]. Another feature of interest of the SAW model is that the transition line $f_c(T)$ shows a reentrance at low temperature, i.e., $f_c(T)$ increases at low T and after reaching a maximum it decreases becoming zero at T_θ, the Θ transition. The reentrant behavior is due to the fact that in the low-T limit, since the entropy does not play any role, the energy dominates the free energy and the open chain is the most favorable configuration. Let us notice that this behavior is similar to the one found in theoretical models of pulling of double stranded DNA [10–12]. Here we will consider a simplified polymer model where the chains are represented by partially directed walks, i.e., steps with negative projection along the x axis, $(1,0)$, are forbidden. This model proved to be helpful in the past in order to find the phase diagram in the (temperature, fugacity) plane for a simplified θ transition [13–18]. We take advantage of previous contributions and generalize the model to the presence of a pulling force along the direction $(1,0)$ (see also Ref. [19]). Surprisingly, in this version the critical force as a function of the temperature T can be found analytically. With transfer matrix techniques we find that the end-to-end distance per monomer goes like $(f-f_c)^{1/\zeta-1}$, with $\zeta<1$. With a sophisticated enumeration technique [16] we show that the correlation critical exponent ν takes on a nontrivial value on the critical line, numerically very close to $\nu_\theta=2/3$, the exponent at the Θ transition. It is not clear whether this is an accidental degeneracy or if it can apply also in the undirected case too. For example, in the three dimensional Sierpinski gasket, an exact renormalization leads to a nontrivial f dependence of ν [8]).

Our work is structured as follows. In Sec. II, we introduce the model and the basic quantities of interest. In Sec. III, we outline how the transfer matrix can be applied to our model, find explicitly the phase diagram (critical line), and give a rough estimate of the exponent ζ. A scaling argument is proposed to suggest that at criticality $\nu=\zeta$. In Sec. IV, we review the enumeration technique proposed in Ref. [16], which we use in Sec. V in order to estimate the value of ν on the critical line. In Sec. VI, we critically analyze our scaling ansatz and the hypothesis that $\nu=\zeta$. Finally, in Sec. VII, we draw our conclusions. In the Appendix, we derive the *exact* critical exponents in the continuum approximation through a technique developed in Ref. [17] and generalized for $f\neq 0$.

II. THE MODEL

The model is a directed SAW (DSAW) on a two-dimensional square lattice (see Fig. 1), with (nonconsecutive) nearest-neighbor interactions. A force **f**, directed along the same axis of the walk, is pulling on one end of the DSAW, the other one being fixed at the origin. Given a par-

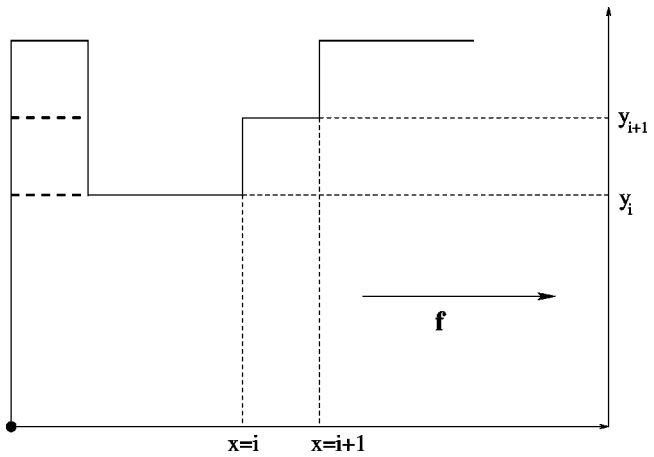

FIG. 1. An example of a DSAW configuration. The quantities y_i and the force direction are also displayed. Thick dashed lines indicate contacts.

ticular configuration $\mathcal{C}$, the energy is

$$E_{\mathcal{C}} = -\epsilon m - f R_x, \tag{1}$$

where m is the number of interacting pairs and ϵ the energy per pairs, f the modulus of the applied force, and R_x the longitudinal extension of the walk. Then, the canonical partition function can be written as

$$\mathcal{Q}_L = \mathcal{Q}_L(\beta\epsilon, \beta f) = \sum_{\mathcal{C}} e^{-\beta E_{\mathcal{C}}}, \tag{2}$$

where L is the number of the steps of the walk and $\beta^{-1} = T$ is the temperature in units of the Boltzmann constant. From now on, we will set $\epsilon = 1$ without loss of generality. From the canonical partition function, we construct the grancanonical partition function (generating function)

$$\mathcal{G}(T, f, z) = \sum_{L=1}^{\infty} \mathcal{Q}_L z^L, \tag{3}$$

z being the step fugacity. The (real) singularity closest to the origin, $z_c(T,f)$, of the generating function, Eq. (3), is related to the free energy per monomer as follows:

$$\ln z_c(T,f) = -\lim_{L\to\infty} \frac{\ln \mathcal{Q}_L}{L}. \tag{4}$$

From the singularities of the generating function where $f = 0$, a complete phase diagram can be extracted (see the following Section and also Refs. [14,15,17]). In particular, a singularity is found in the free energy at a value of $T = T_\theta$, called the Θ temperature.

III. TRANSFER MATRIX CALCULATIONS AND PHASE DIAGRAM

Starting from the definition of the generating function, Eq. (3), where $f = 0$, we observe that it can be conveniently rewritten [13,14] as

$$\mathcal{G}(T, f, z) = \sum_{L_x} \mathcal{G}_{L_x}(T, z) \exp(\beta f L_x), \tag{5}$$

where $\mathcal{G}_{L_x} = \Sigma_L \mathcal{Q}_{L,L_x} z^L$, $\mathcal{Q}_{L,L_x}$ being the partition function restricted to walks of total length L, of which L_x steps are along the x direction.

This is useful because now $\mathcal{G}_{L_x}$ can be written in terms of a transfer matrix T of dimensionality L_y^2, where L_y is the size of our system along the y direction [13,14]. Such a transfer matrix T is defined via its actions on the vectors $\{v_i\}_{i=1,\ldots,L_y^2}$, with $v_i = (y_i, y_{i+1})$, y_i being the height of the site in the ith row, which precedes the right-bound horizontal link in that column (see Fig. 1), as follows:

$$T(v_i, v_{i+1}) = \exp[\beta(\min\{|r_i|, |r_{i+1}|\})\,\theta(-r_i r_{i+1})] \times \exp[(|r_i| + 1)\ln z], \tag{6}$$

where $r_i = y_{i+1} - y_i$ and $\theta(x)$ is the Heaviside step function. It can be shown that $\mathcal{G}(T, f = 0, z)$ develops a singularity when λ, the largest eigenvalue of T, goes through 1 [20]. This means that for large L_x,

$$\mathcal{G}_{L_x} \propto [\lambda(T,z)]^{L_x}. \tag{7}$$

Consequently, the force-dependent singularity $z_c(T,f)$ occurs when

$$\lambda(T,z)\exp(\beta f) = 1. \tag{8}$$

Equation (8) has a rather deep consequence. In order for the critical fugacity and hence the free energy to display a singularity at a nonzero value of the force, i.e., in order for the force-induced unfolding transition to exist as a thermodynamic transition and not only as a crossover, it is necessary that $\lambda(T,z)$, the largest eigenvalue of the transfer matrix where there is no force, has itself a singularity as z approaches $z_c(T, f = 0) \equiv z_0$. Otherwise, from Eq. (8) it is clear that there can be no such singularity. If there is a transition, then we get the following equation for the critical force:

$$f_c(T) = -T \lim_{z\to z_0^-} \ln \lambda(T,z). \tag{9}$$

In Eq. (9), the value of λ to be put in the right hand side of the equation is the one pertaining to the infinite system. $\lambda(T,z)$ for z slightly less than z_0 is plotted in Fig. 3 with a lateral size L_y up to 40. It is rather clear that a singularity has to be expected at $z = z_0$ in the infinite size limit. It was indeed shown [14,15] that for $T < T_\theta$ ($T_\theta = 0.8205\ldots$ in this model), there is a singularity of the grand partition function for $z = z_0 = \exp(-\beta)$ and for this value of the fugacity, the largest eigenvalue is strictly smaller than 1, being [14]

$$\lambda(\beta, z = z_0 = \exp(-\beta)) \equiv \lambda(\beta) = \frac{z_0(1 + \sqrt{z_0})}{1 - \sqrt{z_0}}. \tag{10}$$

The Θ transition temperature is obtained when $\lambda(\beta, z = z_0 = \exp(-\beta)) = 1$. Consequently, the critical line $f_c(T)$ is obtained by putting $\lambda(\beta) = \exp[-\beta f_c(T)]$, i.e.,

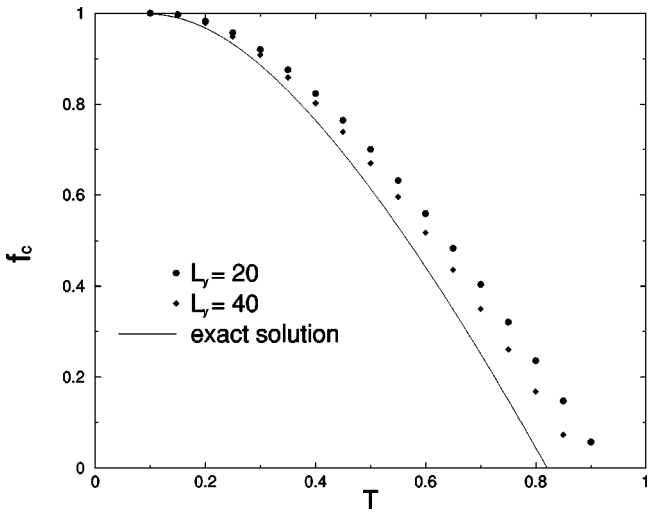

FIG. 2. Exact critical line as in Eq. (11) together with points corresponding to estimates with the transfer matrix calculation, with strip size $L_y=20$ and 40.

$$f_c(T)=T\ln\left[\frac{1-\exp(-\beta/2)}{\exp(-\beta)[1+\exp(-\beta/2)]}\right], \tag{11}$$

and is plotted in Fig. 2, where also the results obtained with the transfer matrix with system size up to $L_y=40$ are displayed. In view of Eqs. (8) and (9), we can define a new critical exponent ζ that characterizes the directed self-avoiding walk. From Eq. (9), if the largest eigenvalue approaches its limit value according to the law

$$\lambda(z_0^-)-\lambda(z)\sim(z_0-z)^{\zeta}, \tag{12}$$

then one straightforwardly obtains [via Eqs. (8) and (9)]

$$\lim_{L\to\infty}\frac{\langle R_x(L)\rangle}{L}\sim[f-f_c(T)]^{1/\zeta-1}, \tag{13}$$

where $\langle R_x(L)\rangle$ is the average projection of the end-to-end distance of the DSAW along the axis (1,0). From Fig. 3 we estimated $1/2<\zeta<1$, with $\zeta\simeq 0.7$ though a precise determination is difficult. If $\zeta<1$, in particular, the transition is of second order. It is widely accepted that for $d>2$, the transition is of first order and so $\zeta=1$. In Ref. [9], a renormalization group based argument in $d=2$, on the other hand, gave the (undirected) SAW $\zeta=1/2$. This argument would also apply to the present case. Given that the transition is of second order in our model, it is also sensible to look for the value of the critical exponent ν (defined as $R_g\sim L^{\nu}$ for large number of steps L, where R_g is the gyration radius of the L-site polymer. In Sec. VI, using a scaling argument, we shall demonstrate that $\nu=\zeta$.

In the following section, we shall study the complete canonical partition function, Eq. (2), using a powerful method of exact enumeration introduced in Ref. [16], which allows us to reach large values of L.

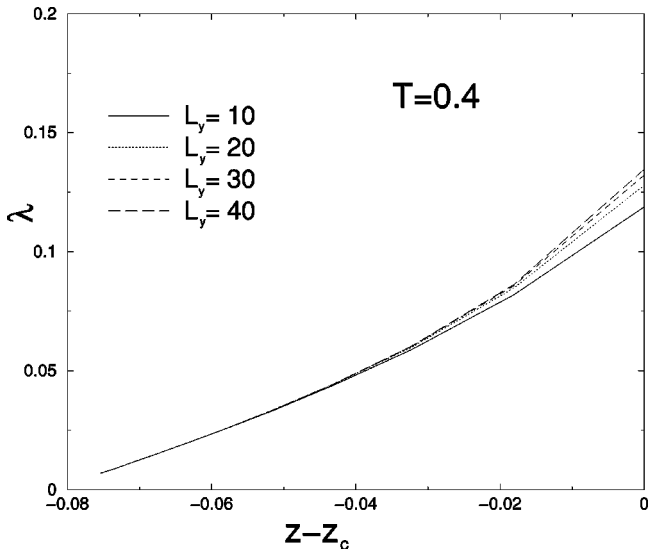

FIG. 3. Plot of the largest eigenvalue ($T=0.4<T_\theta$) vs z. It is apparent that the largest eigenvalue approaches a limit value as z approaches $z_0=\exp(-\beta)$ (<1) from below. Thus, a transition exists in the thermodynamic sense.

IV. THE METHOD OF ENUMERATION

As already mentioned, the configurations of the model are directed walks on a two-dimensional square lattice with nearest-neighbor interactions. For convenience, we demand that these walks end with a horizontal segment. Since the walks are directed in the x direction we can describe these configurations through a distance r_i between two horizontal steps, measured in the positive y direction. Thus, we associate to each configuration an N tuple $(r_1,r_2,\ldots,r_N)$, corresponding to a configuration of total length $L=\Sigma_{i=1}^{N}|r_i|+N$.

The energy due to the nearest-neighbor interactions for each of these configurations is [see Eq. (6)]

$$U(r_1,r_2,\ldots,r_N)=-\sum_{i=1}^{N-1}\min(|r_i|,|r_{i+1}|)\,\theta(-r_i r_{i+1}). \tag{14}$$

In the following, we assign weights x for steps in the horizontal direction and y for steps in the vertical direction. Then, the canonical partition function is

$$\mathcal{Q}_L(x,y,\omega)=\sum_{N=1}^{L}(xe^{\beta f})^N \times\sum_{|r_1|+|r_2|+\cdots+|r_N|=L-N} y^{L-N}\omega^{U(r_1,r_2,\ldots,r_N)}, \tag{15}$$

where $\omega=\exp(\beta)$.

Now, it is convenient to consider the partition functions $\mathcal{Z}_L^{(r)}=\mathcal{Z}_L^{(r)}(x,y,\omega)$ for walks of total length $L+1$ which start with a vertical segment of height r. Then, we have

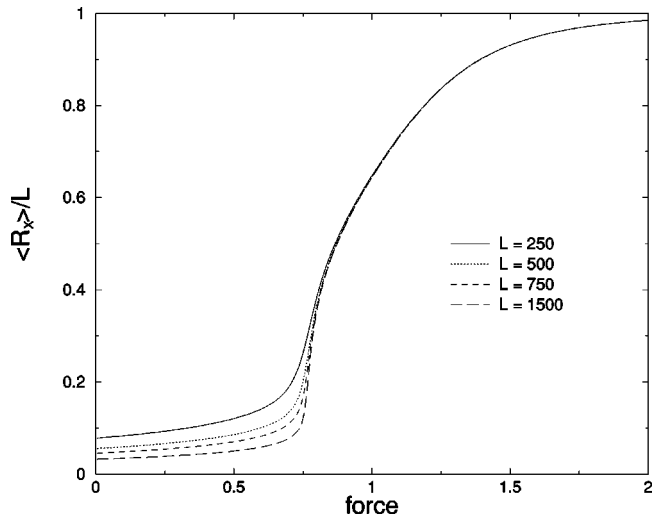

FIG. 4. Plot of $\langle R_x(L)\rangle/L$ vs f for a variable length L of the walk.

$$\mathcal{Q}_{L+1}(x,y,\omega)=\sum_{r=-L}^{L}\mathcal{Z}_L^{(r)}, \tag{16}$$

[note that $\mathcal{Z}_L^{(0)}=x\mathcal{Q}_L(x,y,\omega)$] which satisfies the following recursion relation:

$$\mathcal{Z}_L^{(r)}=xy^{|r|}\left\{\delta_{|r|,L}+e^{\beta f}\sum_{s=-L+|r|+1}^{L-|r|-1}\omega^{U(r,s)}\mathcal{Z}_{L-|r|-1}^{(s)}\right\}, \tag{17}$$

obtained concatenating these walks. In Eq. (17), $r=-L,\ldots,L$, with $L=0,1,2,\ldots$. Using the symmetry $\mathcal{Z}_L^{(r)}=\mathcal{Z}_L^{(-r)}$, Eq. (17) can be written only for non-negative r as

$$\mathcal{Z}_L^{(r)}=xy^{r}\left\{\delta_{r,L}+e^{\beta f}\sum_{s=0}^{L-r-1}\mathcal{Z}_{L-r-1}^{(s)}+e^{\beta f}\sum_{s=1}^{L-r-1}\omega^{\min(r,s)}\mathcal{Z}_{L-r-1}^{(s)}\right\}. \tag{18}$$

Setting $x=y=1$ in Eq. (18), we obtain on iteration scheme and the free energy $\mathcal{F}_L(\omega)=-(1/\beta L)\ln\mathcal{Z}_L^{(0)}$. The average longitudinal length of the walk $\langle R_x(L)\rangle$ is simply

$$\langle R_x(L)\rangle=\frac{\partial}{\partial(\beta f)}\ln\mathcal{Z}_L^{(0)}. \tag{19}$$

Then, we shall proceed as follows.

(1) We calculate the free energy using the iteration scheme proposed in Eq. (18).

(2) Using Eq. (19), we determine how the quantity $\langle R_x(L)\rangle/L$ varies against the applied force f.

V. RESULTS

The plot of $\langle R_x(L)\rangle/L$ vs f for various values of L is represented on Fig. 4 at $T=0.4$, which is below the Θ transition occurring at $T_\theta\simeq 0.8205\ldots$. From Eq. (11), we have $f_c(T=0.4)\simeq 0.764\ldots$.

TABLE I. Estimates for the critical exponents from a Padé approximant analysis. Note that in the $f>f_c$ case, the error is completely negligible.

f	ν	L
$<f_c$	0.501(7)	$\leqslant 1900$
$=f_c$	0.68(8)	$\leqslant 1900$
$>f_c$	1.00000	$\leqslant 1300$

From a careful examination of Fig. 4, we deduce that the quantity $\langle R_x(L)\rangle/L$ decreases as $L^{\nu-1}$, where the *critical exponent* ν might depend on the temperature T. In particular, the data are consistent with $\nu<1$ if $f\leqslant f_c(T)$ and $\nu=1$ if $f>f_c(T)$. In order to find more precise values for the critical exponent, we shall proceed along the same lines of Ref. [16].

An estimation of the critical exponent through the use of the Padé approximants [21] is given in Table I.

Our estimate of the critical exponent, at the critical force, is close to $\frac{2}{3}$, the ν value at the Θ point at $f=0$ [15,17]. As shown in the Appendix, this is the *exact* value.

To get a better insight, let us define an L-dependent critical exponent $\nu(L)$ through the formula

$$\nu(L)=\frac{\ln\langle R_x(L+1)\rangle-\ln\langle R_x(L)\rangle}{\ln(L+1)-\ln L}. \tag{20}$$

Plotting $\nu(L)$ vs an estimated correction-to-scaling term, a careful extrapolation to $L\to\infty$ can be performed, determining the critical exponent ν for all values of the force. Let us consider three different regimes:

(*a*) $f<f_c$. As an example, let us consider $f=0.4$. We have found that successive estimates for the exponent $\nu-1$ with increasing L follow a straight line when plotted against a correction-to-scaling term of $1/L^{0.5}$ (see also the case of Ref. [16] at $f=0$). The plot is shown in Fig. 5. The extrapolated value for $L\to\infty$ gives $\nu-1\simeq -0.4998$, then $\nu\simeq\frac{1}{2}$, the exponent typical of a compact phase.

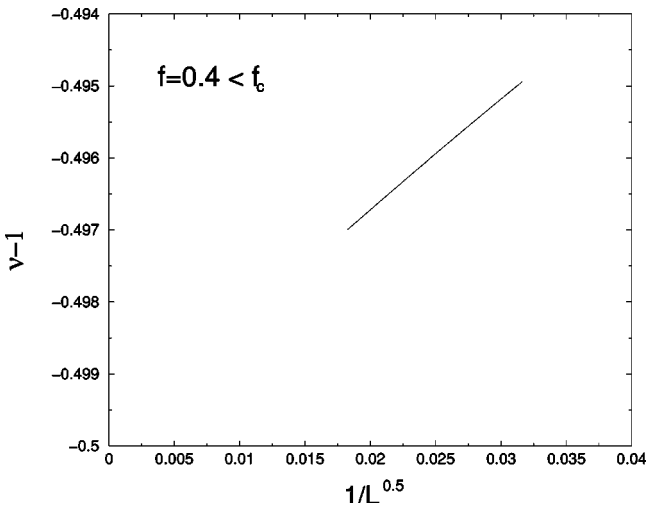

FIG. 5. Plot of $\nu-1$ vs $1/L^{0.5}$ for $f<f_c$ and L up to 3000.

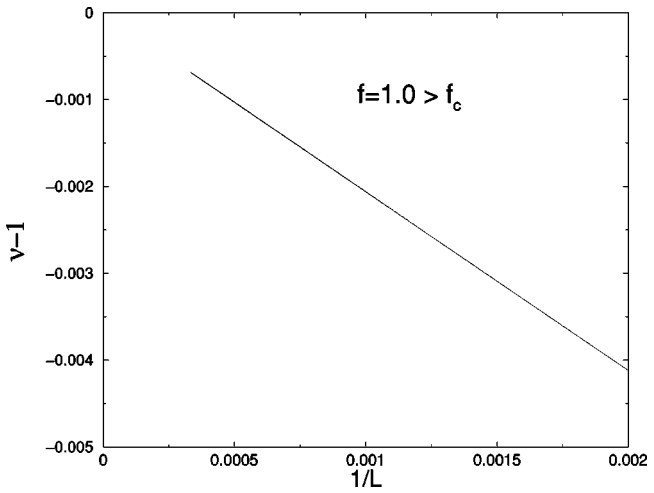

FIG. 6. Plot of $\nu-1$ vs $1/L$ for $f>f_c$ and L up to 3000.

(*b*) $f>f_c$. As before, we have plotted the exponent $\nu-1$ against a well-defined correction-to-scaling term. Now, this term is of the order of $1/L$. Figure 6 shows the $f=1.0$ case as a typical example. Now, the extrapolated value gives $\nu-1\simeq 3.0\times 10^{-6}$, then $\nu=1$ within the numerical precision.

(*c*) $f=f_c$. Now, the correction-to-scaling term is of the order of $1/L^{0.28}$ (see Fig. 7) and $\nu-1\simeq -0.3336$, which implies $\nu\simeq\frac{2}{3}$.

Thus, we find that the value of ν for $f=f_c(T)$ is equal (in the limit of numerical precision) to the value $\frac{2}{3}$, which corresponds to that for $f=0$ at $T=T_\theta$ [15,17]. This is a nontrivial result. In particular, we have to expect that along all the critical lines $f=f_c(T)$, ν takes the value $\frac{2}{3}$ (see also the Appendix). Moreover, as pointed out in Ref. [17], the correction-to-scaling term, where $f=0$ and $T=T_\theta$, is of the order of $1/L^{1/3}$. In our case, we found that this correction increases to $1/L^{0.28}$ for $T=0.4$. Within the numerical errors, this implies that the correction-to-scaling term depends on force as well as on temperature, and that the size of our system has to increase in order to find the right critical exponent.

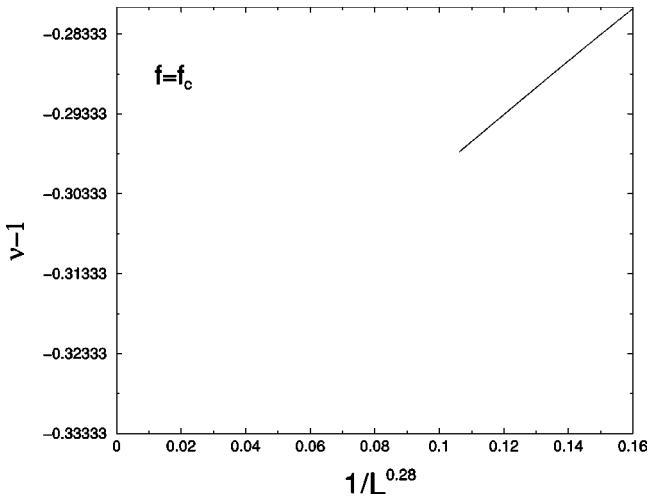

FIG. 7. Plot of $\nu-1$ vs $1/L^{0.28}$ for $f=f_c$ and L up to 3000.

TABLE II. Estimates for the critical exponents from an extrapolation to $L\to\infty$, obtained plotting the L-dependent critical exponent $\nu(L)$, Eq. (20), vs an estimated correction-to-scaling term.

f	ν
$<f_c$	1/2
$=f_c$	2/3
$>f_c$	1

Finally, in Table II we have summarized the above results.

In the following section, we shall introduce a scaling theory that rationalizes what we have found on the critical behavior of the average horizontal end-to-end displacement $\langle R_x(L)\rangle$.

VI. SCALING THEORY

Our previous results suggest the following scaling ansatz (see also Ref. [7]):

$$\langle R_x(L)\rangle = L^{\nu}\Phi(\Delta f L^{\psi}), \tag{21}$$

where $\Delta f\equiv f-f_c(T)$. The scaling function $\Phi(x)$ must have the following behavior:

$$\Phi(x)\sim\begin{cases} x^{(1-\nu)/\psi} & \text{if} \quad x\to+\infty \\ \Phi_0 & \text{if} \quad x\to 0 \\ |x|^{(1/2-\nu)/\psi} & \text{if} \quad x\to-\infty, \end{cases} \tag{22}$$

with Φ_0 a nonzero constant value. Then, the quantity $\langle R_x(L)\rangle$ obeys the equations

$$\langle R_x(L)\rangle\sim\begin{cases} L\Delta f^{(1-\nu)/\psi} & \text{if} \quad \Delta f>0, \quad \dfrac{1-\nu}{\psi}>0 \\ L^{\nu} & \text{if} \quad \Delta f=0 \\ L^{1/2}|\Delta f|^{(1/2-\nu)/\psi} & \text{if} \quad \Delta f<0, \quad \dfrac{1/2-\nu}{\psi}<0, \end{cases} \tag{23}$$

in agreement with the results found in the preceding section.

Now, let us observe that the free-energy contribution to the singular part is

$$\Delta F=\langle R_x(L)\rangle f-\langle R_x(L)\rangle f_c=\langle R_x(L)\rangle\Delta f=\Delta f L^{\nu}\Phi(\Delta f L^{\psi}), \tag{24}$$

where we have used Eq. (21), f is the applied force, and $\tilde{\Phi}(x)=x\Phi(x)$. Since ΔF is a contribution to the *total* free-energy (not a free energy density), we expect that it depends only on the ''dimensionless'' combination of the scaling fields Δf and L with appropriate exponents. This implies that $\nu=\psi$.

Comparing Eqs. (13) and (23), we deduce that $\zeta=\nu=2/3$. Then, Eq. (21) becomes $\langle R_x(L)\rangle=L^{2/3}\Phi(\Delta f L^{2/3})$.

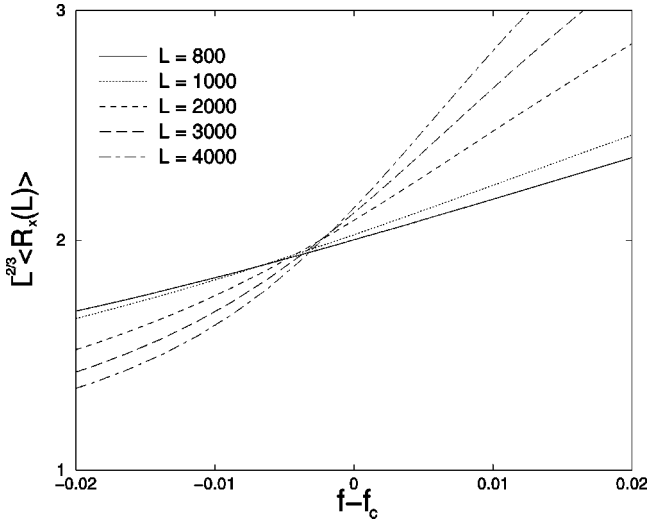

FIG. 8. Plot of $\langle R_x(L)\rangle$ scaled by $L^{2/3}$ vs $f-f_c$. Finite size scaling corrections to the critical force are evident.

To test this prediction, we have plotted in Fig. 8 the function $L^{-2/3}\langle R_x(L)\rangle$ vs $f-f_c$, where f_c is again determined from the exact formula, Eq. (11). It is evident that, apart from the obvious finite size scaling corrections, our ansatz is justified. Then, we have derived the scaling function $\Phi(x)$. The final result is shown in Fig. 9.

VII. CONCLUSIONS

In this work we have analyzed a model of self-avoiding, partially directed chains on a square lattice, with a force pulling along one of the two lattice directions. The model is simple enough to yield the exact form of the temperature dependent critical force $f_c(T)$ [see Eq. (11)]. However, the critical indices of the unfolding transition, which is of second order, are not trivial. The transition is characterized by two exponents: the usual correlation length critical exponent ν and one that we called ζ. In particular, the exponent ν at $f=f_c$ [see Eq. (23)] is different both from 1/2, the collapsed polymer value, and from 1, the extended polymer value [see again Eq. (23)]. The ζ exponent characterizes the singular behavior of the chain elongation per monomer along the force direction as the critical force is approached from above [see Eq. (13)]. Through a powerful enumeration technique taken from the literature [16], coupled with a finite size scaling to extrapolate our results to infinitely long chains, we find that ν is very close to 2/3. A scaling analysis also suggests that $\nu=\zeta$ at least within our numerical precision. Further investigations are required to extend our results to the undirected SAW case.

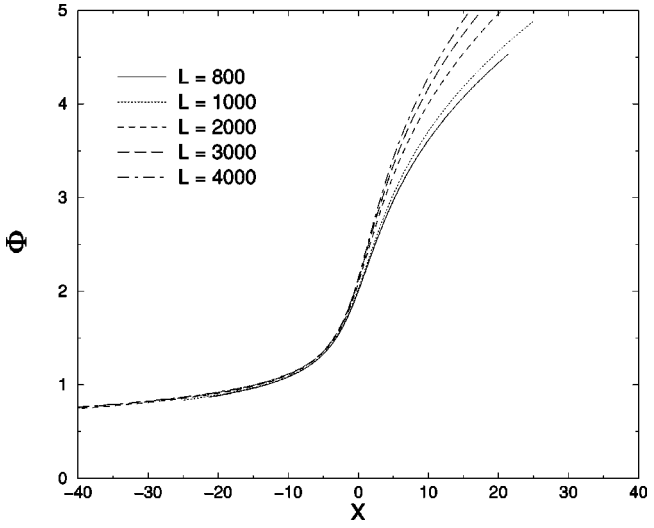

FIG. 9. Successive estimates for the scaling function $\Phi(x)$ are shown for increasing lengths of the walk.

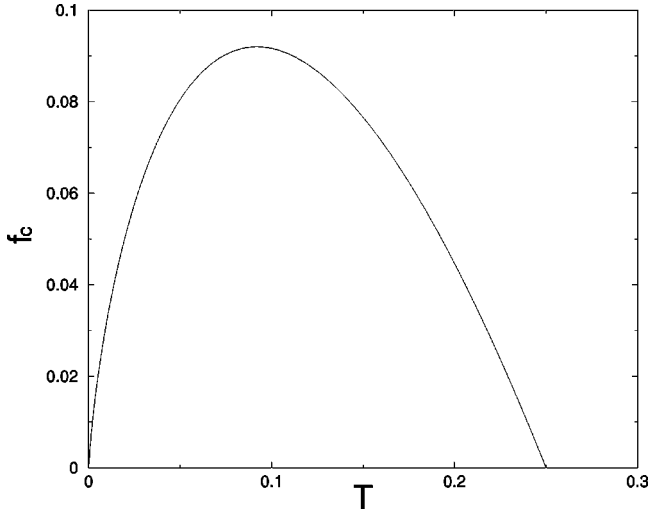

FIG. 10. Exact critical line in the continuum approximation.

APPENDIX: EXACT EVALUATION OF THE CRITICAL EXPONENTS

In this appendix we will show that, generalizing the continuum approximation of Ref. [17] for $f\neq 0$, the exact critical line $f_c(T)$ can be calculated. More importantly, we have obtained an exact derivation of the critical exponents at $f>f_c$ and $f=f_c$, and we will show that they do not depend on T for $T\leqslant T_c$.

First of all, we briefly outline the main results of Ref. [17]. The authors proposed a continuous model of DSAW, where the r_i's in the nearest-neighbor energy term, Eq. (14), are not restricted to integer values. As shown, this model is the continuous limit of the discrete DSAW and has the same critical exponents. After some algebra, they derived the following *exact* form for the *x*-generalized partition function

$$G(x,z;\omega)=-1+\sigma^{-1}\frac{J_\lambda(\sigma\lambda)}{J'_\lambda(\sigma\lambda)}, \tag{A1}$$

where x is the fugacity along the horizontal direction, $\tau=-\ln z$, $\sigma=(4x/\beta)^{1/2}$, and $\lambda=\beta/(\tau-\beta)$. J_λ and J'_λ are, respectively, the Bessel function of order λ and its derivative. The *desired* generalized partition function is given by

Eq. (A1), with $x=1$. The critical fugacity is given by the solution of the equation $J'_\lambda(\sigma\lambda)=0$ and the critical point is given by $\sigma=1$ or $\beta_c=4$.

From Eq. (A1), it is possible to define an average horizontal length as

$$\langle N\rangle = \left.\frac{\partial \ln G(x,z;\omega)}{\partial \ln x}\right|_{x=1}. \tag{A2}$$

When $f=0$, the *exact* critical exponents are $\nu=1$ for $T>T_\theta$ ($\sigma>1$) and $\nu=2/3$ for $T=T_\theta$ ($\sigma=1$).

To generalize for $f\neq 0$, we have replaced x with $xe^{\beta f}$ and, then, put $x=1$. Formally, Eqs. (A1) and (A2) still remain valid, with $\sigma=(4e^{\beta f}/\beta)^{1/2}$. The critical line, determined by the equation $\sigma=1$, gives $f_c=f_c(T)=(1/\beta)\ln(\beta/4)$ and is plotted in Fig. 10. It can be easily understood, once it is realized that Eq. (11) for $\epsilon\neq 1$ reads

$$f_c(T)=T\ln\left[\frac{1-\exp(-\beta\epsilon/2)}{\exp(-\beta\epsilon)[1+\exp(-\beta\epsilon/2)]}\right]. \tag{A3}$$

It is easy to verify that the continuum limit is formally achieved when $\epsilon\to 0$.

The exact critical exponents are $\nu=1$ for $f>f_c$ ($\sigma>1$) and $\nu=2/3$ for $f=f_c$, which agree with the results of the discrete model in Sec. V.

Notice that the shape of the transition line at low T is an unphysical feature of the continuum approximation, as discussed in Refs. [10,11].

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