Dynamic critical behavior of the *XY* model in small-world networks

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The critical behavior of the *XY* model on small-world network is investigated by means of dynamic Monte Carlo simulations. We use the short-time relaxation scheme, i.e., the critical behavior is studied from the nonequilibrium relaxation to equilibrium. Static and dynamic critical exponents are extracted through the use of the dynamic finite-size scaling analysis. It is concluded that the dynamic universality class at the transition is of the mean-field nature. We also confirm numerically that the value of dynamic critical exponent is independent of the rewiring probability P for $P \gtrsim 0.03$.

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I. INTRODUCTION

In recent years, there has been a surge of activity in the field of complex networks among statistical and interdisciplinary physicists [1]. Quite naturally, various spin models of statistical mechanics have been studied on an underlying complex network [2–4]. These studies serve a twofold purposes: First, they aid studies of the static network structure. In many real-world situations, the network structure is an underlying infrastructure for a dynamical system, and nontrivial effects can emerge from the interplay between the dynamical system and the network [5]. Second, such studies of spins systems on complex networks can illuminate the properties of the spin model itself in certain extreme situations. For example, both the Ising and *XY* models can display a critical behavior similar to high dimensional regular lattices with a very low density of couplings (or edges in the network) between spins [2,3].

One of the most central complex network models is the Watts-Strogatz (WS) model of small-world networks [6]. Briefly, this model is controlled by a parameter P (the ''rewiring probability''), and by tuning P from 0 to 1 one goes from regular to random networks. The interesting region is that of intermediate P where the network is clustered (has a high density of short circuits, or more specifically, triangles) and a logarithmically increasing average path length (the path length of a pair of vertices is the smallest number of intervening edges). In the *XY* model, each vertex is associated with a two-dimensional spin angle. The *XY* model has mostly been used to study phase transitions in superconductors and superfluids, while it was also applied to, e.g., the formations of bird flocks [7]. The static properties of the *XY* model in the WS network have been studied in [3], where critical exponents characteristic of a mean-field transition have been found at any nonzero value of P. In the present paper, we study the *dynamic critical behavior* of the *XY* model on the WS small-world network with focus on the dynamic critical exponent.

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II. *XY* MODEL ON WS MODEL NETWORK

In the WS model for the small-world network [6], a regular network is first constructed by arranging N vertices in a one-dimensional circular topology and connecting each vertex to $2k$ neighbors. Then one goes through each edge one at a time, and with the rewiring probability P detaches the far side of the edge and reconnect it to a randomly chosen other vertex (with the restriction that loops and multiple edges must not be formed). In this manner, a small-world network with the size N is constructed with the model parameters k and P. This procedure is illustrated in Fig. 1. The former parameter k is not believed to give any significant change of the network structure for $k>1$, and thus we fix $k=3$ throughout the paper.

The *XY* model consists of planar spins interacting through the Hamiltonian

$$H = -\frac{1}{2}\sum_{i\neq j} J_{ij}\cos(\theta_i - \theta_j), \tag{1}$$

where $\theta_i \in (-\pi, \pi)$ at vertex i is the spin angle, corresponding to the phase of the superconducting order parameter in the Ginzburg-Landau theory of superconductivity. The coupling matrix J_{ij} is given by

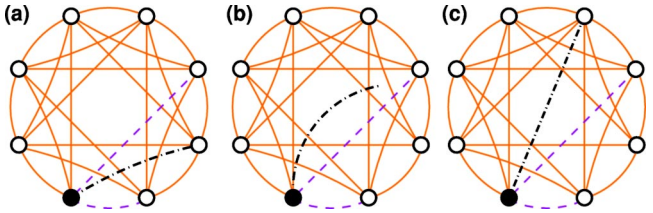

FIG. 1. The construction of a Watts-Strogatz model network. One starts from a regular one-dimensional lattice (a). For every vertex (we consider the black vertex specifically), one goes through the edges on one side (for the black vertex these are dashed). Then, with probability P one detaches the other end (b) and reattaches it (c) with the condition that no loops (edges starting and ending at the same vertex) or multiple edges must be formed.

$$J_{ij}=J_{ji}=\begin{cases}J & \text{if } (i,j) \text{ is an edge}\\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

For example, in the XY model on a two-dimensional square lattice where only nearest vertices interact, we have $J_{ij}=0$ except when i and j are nearest neighbors. For convenience, we measure the temperature T in units of J/k_B.

III. SHORT-TIME RELAXATION METHOD AND SCALING ANALYSIS

To investigate the dynamic critical behavior of the XY model on the WS network, we use the so-called ''short-time relaxation method,'' which utilizes the relaxation behavior of the system towards equilibrium from the a nonequilibrium initial state. By use of this method, several critical exponents have been successfully determined for the Ising model [8,9], for unfrustrated and fully frustrated Josephson junction arrays [10], and for the classical Heisenberg spin XY model [11]. The major advantage of the short-time relaxation method (compared to dynamical simulations in equilibrium) is the running time saved from the avoidance of equilibration.

The Monte Carlo (MC) scheme is based on the Hamiltonian (1) and the standard metropolis local update algorithm [12,13]. The key quantity we measure is [9,14]

$$Q(t)=\left[\left\langle \operatorname{sgn}\left(\sum_{i=1}^{N}\cos\theta_i(t)\right)\right\rangle\right], \tag{3}$$

where the time t is measured in units of one MC sweep, $\langle\cdots\rangle$ is the average over different time sequences from the same starting configuration, and average over different network configurations, denoted by $[\cdots]$, should also be taken. Here the sign function $\operatorname{sgn}(x)$ measures the sign(± 1) of x. The initial configuration is chosen as $\theta_i(0)=0$, giving $Q(0)=1$, and $Q(t\to\infty)=0$ since in equilibrium $(t\to\infty)$ there is no preferred angle direction. We chose the trial angle $\delta\theta=\pi/6$; the motivation is that it is sufficiently small in order to obtain good convergence rate of the quantity we measure while it is big enough to make simulations fast [13].

In order to obtain the dynamic critical exponent and detect the phase transition, we use the finite-size scaling of the quantity Q. Close to the critical temperature T_c, one expects that in a finite-sized system the characteristic time τ scales as $\tau\sim N^{\bar{z}}$, while the ratio of the correlation volume $\xi_V\sim|T-T_c|^{-\bar{\nu}}$ to the system size N gives the second argument of the scaling function [3,14–16]:

$$Q(t,T,N)=F(t/N^{\bar{z}},(T-T_c)N^{1/\bar{\nu}}), \tag{4}$$

where $F(x_1,x_2)$ is the scaling function with the property $F(0,x_2)=1$. At T_c, where the second scaling variable vanishes, the dynamic exponent $\bar{z}$ is easily determined from Eq. (4) by the requirement that the $Q(t)$ curves obtained for different sizes of the networks collapse onto a single curve when plotted against the scaling variable $tN^{-\bar{z}}$. It is also possible to determine T_c from Eq. (4) by applying an intersection method: Starting from the fully phase ordered nonequilibrium state, Q decays from 1 to 0 as time proceeds. For times t, where $0<Q(N,T,t)<1$, we can fix the parameter $a=tN^{-\bar{z}}$ to a constant for given N and $\bar{z}$. Then Q has only one scaling variable $(T-T_c)N^{1/\bar{\nu}}$ and can thus be written as

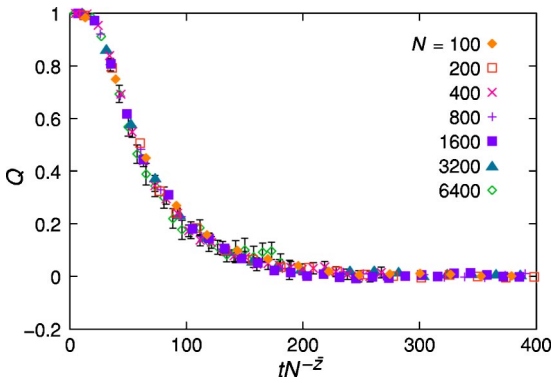

FIG. 2. Short-time relaxation of Q for $P=0.2$ at $T=T_c=2.23$. Q is shown as a function of the scaling variable $tN^{-\bar{z}}$. $\bar{z}=0.52(1)$ is found at the best data collapse (see the Appendix).

$$Q_a(T,N)=F(a,(T-T_c)N^{1/\bar{\nu}}). \tag{5}$$

If now we plot Q with fixed a as a function of T for various N, all curves should have a unique intersection point at $T=T_c$. Finally, we can check the consistency by using the full scaling form to collapse the data for different temperatures and networks sizes onto a single scaling curve in the variable $(T-T_c)N^{1/\bar{\nu}}$ at fixed $a=tN^{-\bar{z}}$. In addition, this is a consistency check of the value of the static exponent $\bar{\nu}$.

To discuss the finite-size scaling in more detail, the form (5) is based on the assumption that there is only two length scales in the system: the network size N (or the number of vertices in the network) and the correlation volume ξ_V diverging at T_c. However, it is known that in the small-world network there is an additional spatial length scale related with the distance between shortcut end points, given by $\zeta=(kP)^{-1}$ [17]. Accordingly, in the presence of the three competing scales (N, ξ_V, and ζ), the finite-size scaling function should take the form $\chi(t/N^{\bar{z}},\xi_V/N,\zeta/N)$ [17,18]. Here, we aim to use sufficiently large systems with N much larger than ζ (but, as we will see, this is difficult for small P), where $\chi(t/N^{\bar{z}},\xi_V/N,\zeta/N)$ may be approximated as $\chi(t/N^{\bar{z}},\xi_V/N,0)$. This leads to the above mentioned scaling forms (4) and (5) without ζ.

IV. SIMULATION RESULTS

We exemplify the critical behavior of Q for WS model networks with $P=0.2$. This value is quite representative for all P values of our simulations, but (as we will discuss later) small P requires larger system sizes, longer times series, and more averages. Figure 2 shows the finite-size scaling of the short-time relaxation given by Eq. (4) which at T_c turns into the simple form

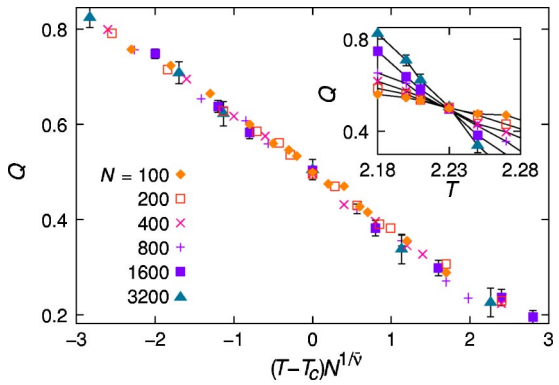

FIG. 3. Finite-size scaling of the short-time relaxation of Q with $P=0.2$ and at $T=2.18$, 2.20, 2.21, 2.23, 2.25, and 2.27. The inset an intersection plot with fixed $t/L^{z}=a$ and $(\bar{z},a)=(0.52,3.0)$; this is consistent with $T_c\approx 2.23$, while the main part of the graph displays the full scaling of $Q=F(t/N^{\bar{z}},(T-T_c)L^{1/\bar{\nu}})$ with the mean-field value of $\bar{\nu}=2$ Ref. [3].

$$Q(t)=F(t/N^{\bar{z}},0) \tag{6}$$

with the only one scaling variable $t/N^{\bar{z}}$. In Fig. 2 (as well as for all other P and T values), we have performed a sample average over 100 independent runs for 200 different network realizations. Instead of leaving both $\bar{z}$ and T_c as free parameters, we use T_c obtained from static MC simulations [3]. Figure 2 displays the best collapse onto a single curve in a broad range of the scaling variable $tN^{-\bar{z}}$ with $\bar{z}=0.52(1)$, where the number in the parenthesis is the error in the last digit (how $\bar{z}$ is obtained is described in detail in the Appendix) [19]. Just as for static quantities [3] the obtained $\bar{z}$ is consistent with higher dimensional regular lattices ($d\geqslant 4$ to be precise), where $\bar{z}=0.5$ is expected [16].

However, the above method presumes *a priori* knowledge of T_c. To check out the consistency of determination of T_c, one can use an intersection method described in Sec. III. In the inset of Fig. 3, we display Q as a function of T for different network sizes N with a fixed value of $a=tN^{-\bar{z}}$ in the first argument of the scaling form in Eq. (4). We find a unique crossing point at $T=T_c=2.23$ and $\bar{z}=0.52$. In some cases (typically for small P values), the T_c has to be slightly altered (from the values of Ref. [3]) to get both the collapse and intersection plots of Fig. 3 correct. We then use $\bar{z}$ and T_c estimated as above to make the full scaling plot for Q as displayed in the main part of Fig. 3. A very smooth collapse here is obtained with $\bar{\nu}=2.0$ which is again consistent with Ref. [3].

The procedure described above for $P=0.2$ is then repeated for various values of P to obtain Fig. 4. As one can see, except for $P\lesssim 0.03$, $\bar{z}=0.54(3)$ [20] throughout the broad range of P. We believe that the nature of the transition (and hence $\bar{z}$) is independent of P for all $P>0$. The larger values of $\bar{z}$ for small P is a result of a failure of the assumption that we can neglect the length scale ζ since $N\gg\zeta$ ($\sim 1/P$) cannot be valid for small P. The inset of Fig. 4 displays the dependence of critical temperature T_c (obtained as discussed above) upon P and is consistent with what has been obtained from static MC [3].

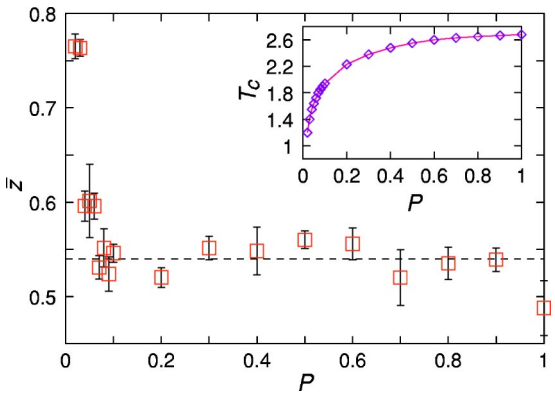

FIG. 4. The dynamic critical exponent $\bar{z}$ as a function of the rewiring probability P. The inset showing T_c as a function of P is consistent with Fig. 4 in Ref. [3]. The dashed line is $\bar{z}=0.54$.

V. SUMMARY

In conclusion, we have studied the dynamic critical behavior of the XY model on WS model networks by means of dynamic Monte Carlo simulations. We have used the short-time relaxation method, based on the relaxation from a non-equilibrium state, and determined the critical temperature T_c, the dynamic critical exponent $\bar{z}$, as well as the static correlation-volume exponent $\bar{\nu}$. The dynamic critical exponent was determined to be $\bar{z}=0.54(3)$ for the networks with rewiring probability $P\gtrsim 0.03$, while the static critical exponent was found to be $\bar{\nu}\approx 2.0$. We believe that this result will hold for any $P>0$ but that the system size needed to confirm this diverges as $P\to 0$. The exponent $\bar{\nu}$, as well as two others, critical exponents α and β of the specific heat and magnetization, respectively, have been obtained in Ref. [3]. The obtained values $\bar{\nu}=2$, $\beta=1/2$, and $\alpha=0$, which also have been shown to be independent from the value of P, establish the mean-field nature of the transition in XY model on WS networks. The result of the present paper support this picture and since the upper dimensionality for the mean-field theory is $d=4$, one can conclude that the phase transition in XY model on WS networks is in the same universality class as a regular lattice of dimensionality $d\geqslant 4$.

An interesting observation is that for a regular hypercubic lattice, this behavior requires a number of edges larger than $8N$, whereas in our simulations we have much fewer ($3N$) edges; and most probably $k=2$ (giving $2N$ edges) gives the same behavior. We also note that there is no additional critical behavior *induced* by the WS model other than the transition from linear (''large-world'') to logarithmic (small-world) behavior in average geodesic length as P becomes finite.

ACKNOWLEDGMENTS

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APPENDIX: DETERMINATION OF z

This appendix concerns the estimation of $\bar{z}$ from data collapses as illustrated in Fig. 2. The problem we are faced with is that we are looking for a collapse over a large range of $x=tN^{-\bar{z}}$, and that the functional form of $Q(x)$ is not easily expressed on a closed form or in low degree series expansions. To get around this problem, we partition the x range in N_{seg} segments X_i, $1\leqslant i\leqslant N_{\mathrm{seg}}$, and fit a line $(a_i+b_ix,\ x\in X_i)$ to the Q point set within each segment (cf. [21]). Then we sum the square of the deviations from the lines

$$\Lambda(\bar{z}')=\sum_{0\leqslant i\leqslant N_{\mathrm{seg}}}\ \sum_{x\in X_i(\bar{z}')}(Q(x)-a_i-b_ix)^2, \qquad \text{(A1)}$$

where $X_i(\bar{z}')$ is the set of all numerical values of $tN^{-\bar{z}'}$ and thus depends on the value of $\bar{z}'$ chosen. (Note that t and hence, x are discrete variables.) Now it is clear that if the segmentation can be done so that Q can be reasonably well approximated by the line segments a_i+b_ix, i.e., if $Q(x)$ is smooth enough, then $\bar{z}=\min_{\bar{z}'}\Lambda(\bar{z}')$ will converge to the correct value as the number of samples and N_{seg} are increased.

The remaining consideration is how to choose the segmentation. In general, one needs the segments large enough to get a small error in the linear regression, and small enough for the line-segment approximation to be feasible. In practice, the method seems to be rather insensitive for the partition method. We choose to partition the whole range of x in segments of equal length, with $N_{\mathrm{seg}}=30$. The minimization of Λ is conveniently done by a Newton-Raphson method [22]. The error in z is calculated by jackknife estimation [23].

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