Josephson junction coupled to a transmission line: A comparison of different approaches

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The case of a Josephson junction loaded by a transmission line is reexamined, according to the Green's function method, in order to compare the results with those that we previously obtained, analytically and numerically, following a different procedure.

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In a previous paper [1], we reported a different approach, based on a phenomenological analysis, for evaluating dissipation in macroscopic quantum tunneling. After reviewing several approaches given up till now, we focused our attention on the case of a Josephson junction coupled to a transmission line, a case that deserved particular interest also in relation with experimental results. By comparing the results reported in Ref. [2]—results obtained by a Green's function analysis—with those of Ref. [1], we concluded that the two approaches, which are both based on the inclusion of a distributed circuit model within the bounce formalism, led to similar results. This was especially true in the limit of high frequencies, or long lines, while there seemed to be a disagreement of roughly a factor of 2 within the opposite limit of small frequencies or short lines, i.e., within the capacitive limit. The purpose of this work is to further investigate this problem, in an attempt to find under what conditions a closer agreement between the two approaches can be raised. In order to facilitate the comparison, we shall adopt the same notations of Ref. [2], namely, ρ and σ are the capacitance and the inductance per unity of length, respectively, of the open line of length L, which loads the junction. The characteristic impedance of the line is $Z_0=(\sigma/\rho)^{1/2}$ and the wave velocity, is $c=(\sigma\rho)^{-1/2}$, the delay time is $\tau_0=L/c$.

According to Green's function method [3] we find that a modified form of the Fourier transform $g(\omega,z,z')$, given by Eq. (13) in Ref. [2], suitable to describe the case of a single loading line, z being the spatial coordinate $(0\leqslant z\leqslant L)$, is given by

$$g(\omega;z,z') = \frac{\sigma}{k\sinh(kL)}\begin{cases}\cosh[k(z-L)]\cosh(kz') & \text{if } z>z' \\ \cosh(kz)\cosh[(kz'-L)] & \text{if } z<z',\end{cases} \quad (1)$$

where $k=|\omega|/c$ [4].

By considering the derivative $g'\equiv\partial g/\partial z$, we have that for $z=z'$ the difference $g'(z+\epsilon,z')-g'(z-\epsilon,z')=-\sigma$, analogously to the case considered in Ref. [2].

By integrating the wave equation, Eq. (12) in Ref. [2], we have

$$\int_{-\epsilon}^{\epsilon}\mathrm{d}z\left[\rho\omega^2-\frac{1}{\sigma}\frac{\partial^2}{\partial z^2}\right]g(\omega;z,z')=\int_{-\epsilon}^{\epsilon}\delta(z-z')\mathrm{d}z, \quad (2)$$

that is,

$$\left[-\frac{1}{\sigma}\frac{\partial}{\partial z}g(\omega;z,z')\right]_{-\epsilon}^{\epsilon}=\frac{-\sigma}{-\sigma}=1, \quad (3)$$

as it is the second member of Eq. (2). We have to consider, however, that the integral of the $\delta(z-z')$ should be limited only to the positive domain of z (the line runs from $z=0$ to $z=L$), obtaining in this case the halving of the integral $\int_0^{\epsilon}\delta(z-z')\mathrm{d}z=1/2$. For the same reason, we have to reduce to a half also the function g of Eq. (1). Therefore, $g(\omega;0,0)$ becomes

$$g(\omega;0,0)=\frac{\sigma}{2k\tanh(kL)}=\frac{Z_0}{2|\omega|}\frac{1}{\tanh(kL)}, \quad (4)$$

where $\sigma/k=Z_0/|\omega|$. We note that, apart from the argument of tanh, Eq. (4) is formally identical to that of the symmetric case of Ref. [2]. This will allow us to obtain an agreement with the results of Ref. [1].

Proceeding, still according to the procedure of Ref. [2], the action of interaction $S_{\rm int}$ is expressed by Eq. (8) in Ref. [2], however, the integration over time will be taken only over half a domain of τ, according to the empirical evidence that in real tunneling processes only a half bounce, the one generating a positive voltage pulse, $V(\tau,z)\propto\dot{\varphi}(\tau)$, is actually traveled by the particle [5]. In other words, by assuming the bounce trajectory $\varphi(\tau)$ as being centered at $\tau=0$, we have that the positive voltage pulse , which for $z=0$ is situated in the negative domain of τ, is propagating along the line for $z,\tau\geqslant 0$. See Fig. 1, which is a modified version of that shown in Ref. [1], where Fig. 2(c) presented an inversion of sign. In the same lapse of time $(\tau\geqslant 0)$ the junction, at $z=0$, "feeds" the negative (virtual) part of the pulse, whose consideration becomes important only for evaluating the power supplied for $\tau\geqslant 0$. Adopting a symmetric transmission line, as in Ref. [2], we can follow the same reasoning (halving of the time) provided that a $g(\omega;0,0)$, one half with respect to that of Ref. [2] [that is, $g(\omega)$

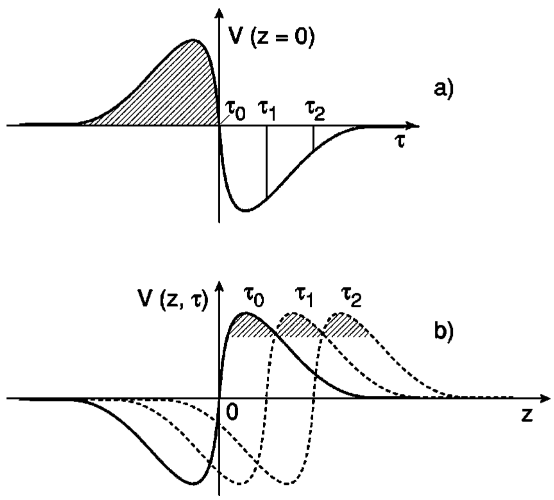

FIG. 1. Voltage pulse $V \propto \dot{\varphi}$ generated by a Josephson junction loaded by a transmission line with characteristic impedance $Z_0 = V/I$. In (a), the pulse $V(\tau)$ is represented as a function of time at $z=0$. In (b), the pulse $V(z-c\tau)$, which travels in the direction of increasing z, is represented as a function of the spatial coordinate at different instants $\tau_0 < \tau_1 < \tau_2$.

$=(Z_0/4|\omega|)\coth(kL/2)]$ is adopted, since Z_0 is now halved by the parallel of the two lines.

In this way we arrive at an expression for $S_{\rm int}$ of the type

$$\exp(-S_{\rm int}/\hbar) = \int \mathcal{D}\xi \exp\left[\frac{-\hbar}{4\pi}\left(\frac{1}{2}\right)\int_{-\infty}^{\infty} d\omega|\xi(\omega)|^2 g(\omega) - i\frac{\Phi_0}{2\pi}\left(\frac{1}{2}\right)\int_{-\infty}^{\infty} d\tau\varphi(\tau)\xi(\tau)\right], \tag{5}$$

where $g(\omega)$ is given by Eq. (4) for the case of only one line of length L, $\Phi_0(=2\pi\lambda$ in Ref. [2]) is the flux quantum.

Still according to Refs. [2,4], the integral in Eq. (5) can be rewritten as

$$\exp(-S_{\rm int}/\hbar) = \int \mathcal{D}\xi \exp\left\{-\int \frac{d\omega}{2\pi}\left[\left(\frac{1}{2}\right)\frac{\hbar}{2}|\xi(\omega)|^2 g(\omega) + i\frac{\Phi_0}{2\pi}\left(\frac{1}{2}\right)\varphi(\omega)\xi(\omega)\right]\right\}, \tag{6}$$

that is, in a form of the type $\int dx \exp(-ax^2 \pm ibx) = (\pi/a)^{1/2}\exp(-b^2/4a)$.

So, we arrive at the following expression for $S_{\rm int}$ [6]

$$S_{\rm int} = \left(\frac{\Phi_0}{2\pi}\right)^2\left(\frac{1}{2}\right)\int \frac{d\omega}{2\pi}\frac{|\varphi(\omega)|^2}{2g(\omega)} = \left(\frac{\Phi_0}{2\pi}\right)^2\int \frac{d\omega}{2\pi}|\varphi(\omega)|^2\frac{|\omega|}{2Z_0}\tanh(kL), \tag{7}$$

where $kL = |\omega|\tau_0$.

To facilitate the comparison with the results of Ref. [2], Eq. (7) can be rewritten in the following, more compact form:

$$S_{\rm int} = \left(\frac{\Phi_0}{2\pi}\right)^2\int \frac{d\omega}{2\pi}|\varphi(\omega)|^2 H(\omega) \tag{8}$$

with

$$H(\omega) = \frac{|\omega|}{2Z_0}\tanh(|\omega|\tau_0) \rightarrow \begin{cases} |\omega|/2Z_0 & \text{if} \quad |\omega|\tau_0 \gg 1 \\ \omega^2 C/2 & \text{if} \quad |\omega|\tau_0 \ll 1, \end{cases} \tag{9}$$

where $C=\rho L$ is the total capacitance.

By comparing Eq. (9) with the corresponding expressions, Eq. (21) of Ref. [2], we arrive at the worth noting result that indicates that the action $S_{\rm int}$ for a line with long delay (or length) is rightly one half of that of a couple of lines in parallel. In the opposite limit of short delay (capacitive limit), the results are the same, i.e., directly proportional to the total capacitance C of the line, independently of the position of the junction. These conclusions find confirmation also in the results of Ref. [1] since, in the capacitive limit [see Fig. (4) in Ref. [1] for small values of Ω] we obtain, by identifying Ω with $|\omega|$, nearly coincident results, and analogously, in the resistive limit [see Fig. (3) in Ref. [1] for large values of Ω] results again comparable.

A further confirmation of the correctness of the results can be found by considering the case of a line terminated by a short circuit. Along the lines of Refs. [2,4], $g(\omega;0,0)$ can be expressed as

$$g(\omega;0,0) = \frac{\sigma}{L}\sum_{n=1}^{\infty}\left(\frac{\omega^2}{c^2}+k^2\right)^{-1} = \frac{\sigma}{L}\sum_n\left[\frac{\omega^2}{c^2}+\frac{(2n-1)^2\pi^2}{4L^2}\right]^{-1}, \tag{10}$$

k being given, in this case, by $k=(n-1/2)\pi/L, n=1,2,$ etc.

By putting $x=2L|\omega|/\pi c = 2kL/\pi$, Eq. (10) becomes

$$g(\omega;0,0) = \frac{Z_0}{2|\omega|}\frac{4x}{\pi}\sum_{n=1}^{\infty}[x^2+(2n-1)^2]^{-1}, \tag{11}$$

which, according to Ref. [7], can be rewritten as

$$g(\omega;0,0) = \frac{Z_0}{2|\omega|}\tanh(kL) \rightarrow \begin{cases} Z_0/2|\omega|, & \text{if} \quad kL \gg 1 \\ \mathcal{L}/2, & \text{if} \quad kL \ll 1, \end{cases} \tag{12}$$

where $\mathcal{L}=\sigma L$ is the total inductance of the short-circuited line of length L. From Eq. (12) we have that for a sufficient length of the line, the result is independent of the termination of the line, a result anticipated by Eq. (4). Whereas, for short length, once $g(\omega;0,0)$ is substituted into Eq. (7), we obtain that $H(\omega)$ becomes $(1/2)\omega/\omega\mathcal{L}$, that is, one half of the ratio of the frequency to the inductive reactance $\omega\mathcal{L}$, analogously to the case of the capacitive limit for which this ratio is $\omega^2 C/2$, see Eq. (9).

These results are in agreement with Leggett's prescription mentioned in Ref. [2], although obtained in a completely different way (see also Ref. [1]). The main difference is that in the present analysis the results are obtained by considering a half bounce trajectory. The halving of the time, already adopted in the treatment of Ref. [1], can appear as a constraint, although compensated by the halving of the function $g(\omega)$. This, however, is the condition that allows the attainment of the agreement of the results obtained by different approaches.

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[5] The direct measurement of the delay in the switching events, in order to obtain the lifetime of the metastable state of the junction from the average, is indeed performed over positive pulses. See, for instance, P. Fabeni *et al.*, Solid State Commun. **97**, 439 (1996).
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