Breaking time for the quantum chaotic attractor

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A model of a quantum dissipative system is considered in the regime when the classical limit corresponds to a chaotic attractor, and the breaking time $\tau_\hbar$ of the classical-quantum correspondence is obtained. The model describes a periodically kicked harmonic oscillator (or a particle in a constant magnetic field) with a dissipation. Another analog of this problem is the dissipative kicked Harper model. It is shown that in the limit of the so-called dying attractor, the breaking time $\tau_\hbar$ can be arbitrarily large.

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The role of dissipation in quantum systems is an object of extensive research, especially due to the different practical needs of contemporary experimental and theoretical physics. Different aspects related to the quantum dissipative processes can be found in [1–6]. A special interest in dissipation is related to the decoherence [7] that can prevent quantum computing, since the common opinion is that dissipation should lead to losses of the quantum features of a system [8,2]. A detailed analysis of related problems can be found in [2], where a quantum counterpart of a dissipative kicked rotor map [9] is considered. It is practically shown in [2] that for small dissipation the time decay of different observables and the destruction of localization take place in the system.

The goal of this paper is to return to the question of the existence of a quantum analog of a classical chaotic attractor and to estimate the breaking time $\tau_\hbar$ of the semiclassical applicability in the presence of finite dissipation, i.e., of dissipation that does not go to zero. The breaking time $\tau_\hbar$ was derived for a nondissipative system in [10] and discussed in numerous papers [11–22]. Its general form

$$\tau_\hbar = \frac{1}{\Lambda}\ln(c/\hbar) \qquad (1)$$

with the Lyapunov exponent Λ and some constant c can be qualitatively estimated from the wave-packet dispersion. A more sophisticated expression of $\tau_\hbar$ appears in the case in which the classical limit consists of a possibility of a strong stickiness of trajectories to some hierarchical set of islands in phase space. The quantum breaking time appears to be

$$\tau_\hbar = \frac{c_1}{\hbar^\mu}\ln(c_2/\hbar) \qquad (2)$$

with μ standing as an anomalous transport exponent for classical diffusion [23,24]. The estimates of the constants c_1, c_2 in Eq. (2) and a simulation of Eq. (2) were performed in [24].

A typical situation for the appearance of a classical chaotic attractor includes a fairly strong dissipation and an external pumping [9] (see also the recent rigorous consideration in [25]). This type of chaotic attractor is related to systems that are close to the Hamiltonian ones. The quantization of such systems means that after the time $\tau_\hbar^{(d)}$, quantum corrections will be of the order of 1 and will destroy the (semi)classical behavior of the systems. In this paper, we will calculate the breaking time $\tau_\hbar^{(d)}$ for a dissipative quantum system. We consider the nonlinear dissipative dynamics of a particle in the presence of a constant magnetic field. The classical counterpart of the model that appeared in [26] is

$$\ddot{x} + \gamma\dot{x} + \omega_H^2 x = \epsilon k_0 T \sin(k_0 x)\sum_{n=-\infty}^{\infty}\delta(t-nT), \qquad (3)$$

which corresponds to the cyclotronic motion with the effective cyclotron frequency ω_H in the presence of dissipation of the rate γ, while the perturbation is a periodic set of kicks with the period T and the amplitude ϵ, and k_0 is a wave number. The model also corresponds to a kicked harmonic oscillator with dissipation and, with some modification, can be applied to nonlinear optics [27] or in the interaction of radiation with matter [27–29]. It was shown in [2] that dissipation destroys the localization length. Since the last one is a strong manifestation of classical chaos in the quantum situation, we select model (3) that does not have a localization for $\gamma=0$ and $\Omega T=\pi/2$ [30]. And, finally, one more reason to consider model (3) is that for the same conditions $\gamma=0, \Omega T=\pi/2$, it is as adequate as the kicked Harper model [31,32].

This quantum problem can be considered in the framework of the following non-Hermitian Hamiltonian:

$$H = (\Omega - i\gamma)a^\dagger a + \epsilon T\cos\tilde{k}_0(a^\dagger + a)\sum_{n=-\infty}^{\infty}\delta(t-nT). \qquad (4)$$

Here annihilation and creation operators have the commutation rule $[a,a^\dagger]=\hbar$. The complex frequency $\omega=\Omega-i\gamma/2$ determines the effective frequency

$$\omega_H = [\Omega^2 + \gamma^2/4]^{1/2} \qquad (5)$$

in the presence of a finite width γ of the levels. In the classical limit, the system has chaotic zones, and resonances between the perturbation and the linear oscillator leading to the unlimited pumping of energy [31]. The non-Hermitian nature of H does not change the operator's algebra, and it produces the following Heisenberg equations:

$$i\hbar\dot{a}=[a,H]=\hbar\omega a-\epsilon\tilde{k}_0 T\sin[\tilde{k}_0(a+a^\dagger)]\times\sum_n\delta(t-nT),$$

$$-i\hbar\dot{a}^\dagger=[H^\dagger,a^\dagger]=\hbar\omega^* a^\dagger-\epsilon\tilde{k}_0 T\sin[\tilde{k}_0(a+a^\dagger)]\times\sum_n\delta(t-nT). \tag{6}$$

It is convenient to transfer to the momentum-coordinate variables. Using the linear transformation

$$\hat{x}=(a+a^\dagger)/\sqrt{2m\Omega},\quad \hat{p}=i\sqrt{m\Omega/2}(a^\dagger-a), \tag{7}$$

we obtain from Eq. (7)

$$\dot{\hat{p}}=-\Omega^2 m\hat{x}-\frac{\gamma}{2}\hat{p}+k_0\epsilon T\sin(k_0\hat{x})\sum_n\delta(t-nT),$$

$$\dot{\hat{x}}=\hat{p}/m-\frac{\gamma}{2}\hat{x}. \tag{8}$$

Let us introduce a coherent state basis $|\alpha\rangle$ at the initial time $t=0$. Then we define the mean values of the operators at time t,

$$x(t)=\langle\hat{x}(t)\rangle=\langle\alpha|\hat{x}(t)|\alpha\rangle,$$

$$p(t)=\langle\hat{p}(t)\rangle=\langle\alpha|\hat{p}(t)|\alpha\rangle. \tag{9}$$

This definition also implies that the operators $\hat{x}(t)$ and $\hat{p}(t)$ are normally ordered, i.e., there exists a presentation

$$\langle\hat{x}(t)\rangle=\sum x_{k,l}(t)\langle(a^\dagger)^k a^l\rangle=\sum x_{k,l}(t)(\alpha^*)^k\alpha^l,$$

$$\langle\hat{p}(t)\rangle=\sum p_{k,l}(t)\langle(a^\dagger)^k a^l\rangle=\sum p_{k,l}(t)(\alpha^*)^k\alpha^l \tag{10}$$

with corresponding coefficients $x_{k,l}(t),p_{k,l}(t)$, while $\alpha\equiv\langle a(t=0)\rangle$ and $\alpha^*\equiv\langle a^\dagger(t=0)\rangle$. Using Eqs. (9) and (10), the Heisenberg equations (8) can be averaged over the coherent states $|\alpha\rangle$. It gives the Ehrenfest equations

$$\dot{p}=-m\Omega^2 x-\frac{\gamma}{2}p+k_0\epsilon T\langle\sin k_0 x\rangle\sum_n\delta(t-nT),$$

$$\dot{x}=p/m-\frac{\gamma}{2}x, \tag{11}$$

where $\langle\sin k_0 x\rangle\equiv\langle\alpha|\sin k_0\hat{x}|\alpha\rangle$. This quantity can be calculated only approximately [33,10] for an arbitrary t. In the first order of $\hbar$, it is

$$\langle\sin k_0 x\rangle\equiv\langle\sin(k_0\hat{x})\rangle=\sin(k_0 x)-\hbar k_0^2\sin(k_0 x)D(x,x),$$

$$D(x,x)=\frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\alpha^*}, \tag{12}$$

where $x\equiv x(t)$. The differentiation over α can be replaced by the differentiation over the initial momentum and coordinate. From Eqs. (7) and (9), we obtain

$$D(x,x)=\frac{1}{2m\Omega}\left[\left(\frac{\partial x}{\partial x_0}\right)^2+\frac{m\Omega}{2}\left(\frac{\partial x}{\partial p_0}\right)^2\right], \tag{13}$$

where $x_0=x(t=0)$ and $p_0=p(t=0)$. Following [26], we integrate the Ehrenfest equations (11) over the period T. Between two consequent kicks, the equation of motion is

$$\ddot{x}+\gamma\dot{x}+\omega_H x=0. \tag{14}$$

At a kick in the moment $t_n=nT$, the shift conditions are

$$x(t_n+0)=x(t_n-0),$$

$$p(t_n+0)=p(t_n-0)+k_0\epsilon T\langle\sin k_0 x_n\rangle,$$

where

$$\langle\sin k_0 x_n\rangle=\sin k_0 x(t_n-0)\times[1-\hbar k_0^2 D(x(t_n-0),x(t_n-0))].$$

The result of this integration is the following map:

$$p_{n+1}=e^{-(\gamma/2)T}\left\{(p_n+k_0\epsilon T\langle\sin k_0 x_n\rangle)\left(\cos\Omega T-\frac{\gamma}{2\Omega}\sin\Omega T\right)-\Omega x_n\sin\Omega T\left(1+\frac{\gamma^2}{4\Omega^2}\right)\right\},$$

$$x_{n+1}=\frac{1}{\Omega}e^{-(\gamma/2)T}\left\{(p_n+k_0\epsilon T\langle\sin k_0 x_n\rangle)\sin\Omega T+\Omega x_n\left(\cos\Omega T+\frac{\gamma}{2\Omega}\sin\Omega T\right)\right\}. \tag{15}$$

A variety of realizations for the classical counterpart has been studied in [26] for different values of the parameters $\Omega T,\gamma,\epsilon$. For simplicity, one considers $\Omega T=\pi/2$. Introducing the dimensionless variables

$$k_0 x=-v,\quad k_0 p/m\Omega=u, \tag{16}$$

we obtain [26]

$$u_{n+1}=e^{-(\gamma/2)T}\left\{-\frac{\gamma}{2\Omega}(u_n+K_H\langle\sin v_n\rangle)+v_n\left(1+\frac{\gamma^2}{4\Omega^2}\right)\right\}+\frac{\hbar k_0^2}{m\Omega}e^{-(\gamma/2)T}\frac{\gamma}{2}K_H\sin v_n D(v_n,v_n),$$

$$v_{n+1}=e^{-(\gamma/2)T}\left\{-(u_n+K_H\langle\sin v_n\rangle)+\frac{\gamma}{2\Omega}v_n\right\}+\frac{\hbar k_0^2}{m\Omega}e^{-(\gamma/2)T}\frac{\gamma}{2}K_H\sin v_n D(v_n,v_n). \tag{17}$$

Here

$$K_H=-\epsilon T k_0^2/m\Omega=|\epsilon|Tk_0^2/m\Omega \tag{18}$$

and

$$D(v_n,v_n)=\frac{1}{2}\left[\left(\frac{\partial v_n}{\partial v_0}\right)^2+\left(\frac{\partial v_n}{\partial u_0}\right)^2\right]. \tag{19}$$

Let us introduce the dimensionless quantum parameter

$$\tilde{h}=\frac{\hbar k_0^2}{m\Omega}. \tag{20}$$

When $\tilde{h}=0$, there is a chaotic attractor under the condition of strong chaos and strong dissipation [26]. A rough estimation of the classical chaotic attractor can be obtained from the stability condition of the initial point $(u,v)=(0,0)$. From Eq. (17) when $\tilde{h}=0$, the criteria are

$$K_H>2\cosh\left(\frac{\gamma}{2}T\right)>1. \tag{21}$$

The evolution of the quantum corrections (19) is determined by the local instability of classical trajectories $\partial v_{n+1}/\partial v_n \sim e^{-(\gamma/2)T}K_H$. An exponential growth of the quantum corrections $D(v_n,v_n)$ with time leads to the breakdown of semiclassical equations that describes quantum dynamics in the framework of the Ehrenfest equations (11). A radius of the convergence of the expansion (12) gives the following restriction on time:

$$t<\tau_\hbar^{(d)}\sim\frac{\ln(1/\tilde{h})}{2\ln K_H-\gamma T}=\tau_\hbar/(1-\gamma T/\Lambda). \tag{22}$$

This value $\tau_\hbar^{(d)}$ is the dissipative classical–quantum breaking time. For $\gamma=0$, it coincides with Eq. (1), where $\Lambda=2\ln K_H$. It follows from (21) that the denominator in Eq. (22) is always positive, but it can be arbitrarily small. The situation, when $2\ln K_H-\gamma T$ is very small, was called in [9] "the dying attractor." In this case, $\tau_\hbar^{(d)}$ is arbitrarily large but finite. The result of Eq. (22) expresses the fundamental correspondence principle. It establishes relations between the main parameters, namely the dimensionless semiclassical parameter $\tilde{h}$, the global chaos parameter K_H, and the decay γ, that determine the quantum dynamics of the system with the non-Hermitian Hamiltonian.

The final result can be commented on as follows. Since the dying attractor situation is

$$1-\gamma T/\Lambda\to 0+, \tag{23}$$

it means an increase of the validity of the semiclassical consideration simultaneously with the increase of time of the decay of correlations. Therefore, one meets two competing factors with respect to quantum computing: less decoherence due to chaos but more decoherence due to the dissipation. Our consideration has at least two weak points that are worth mentioning: we use an oversimplified model of dissipation and we are unable to characterize the situation for $t>\tau_\hbar^{(d)}$. The first point does not seem too serious and represents some technical features that can be overcome. The second point is more serious: what is the quantum manifestation of the classical chaotic attractor and does it mean that for $t>\tau_\hbar^{(d)}$ we can preserve the quantum features of the system necessary for quantum computing, and at the same time suppress the external dissipation? These problems will be studied in the future.

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