Noise-induced synchronization in realistic models

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Noise-induced synchronization is studied numerically in two realistic models—the Pikovsky-Rabinovich circuit model and the Hindmarsh-Rose neuron model. A different feature, single-variable-Jacobian matrix, is found in these two models and conditions are found by which two noncoupled identical systems can been synchronized by forcing with a common Gaussian noise term.

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Noise-induced synchronization is a subject area that has created a good deal of interest of late [1–5]. Theoretical results have recently inspired experimental work and noise-induced synchronization was observed for the first time in a biological system among pairs of noncoupled sensory neurons [3]. From a theoretical perspective, one is usually concerned with three types of questions: Whether noise-induced complete synchronization (CS) can be realized in a certain chaotic system? Which variable or equation is the most effective to apply the common noise when attempting to achieve CS in identically forced chaotic systems? And *how* strong should the noise intensity be to achieve CS? Unfortunately it is usually hard, if not impossible, to answer any of these questions with rigour or with some type of general proof. At a minimum, the attractor of the system considered must have a large enough basin of attraction to suffer the perturbation of noise which often must be relatively strong for CS to be achieved. In this paper, these questions are examined for two realistic models: the Pikovsky-Rabinovich (PR) circuit and the Hindmarsh-Rose (HR) neuron model. As discussed here, common noise can induce CS in uncoupled identical PR (HR) models and the critical noise intensity proves to be roughly equal to the mean size of the original chaotic attractor. At this level, the models' deterministic dynamics are swamped out by the common stochastic noise, and the models synchronize to the same stochastic forcing.

It has previously been observed that noise-induced CS can be realized in two uncoupled identical Lorenz systems when common Gaussian noise is applied to the equation of motion of the y variable. For the standard parameters $\delta=10$, $\rho=28$, and $b=8/3$, the critical noise intensity is $D_c^y=33.3$. It is also important that the *attractor* with such strong noise on y equation is still similar to the original attractor [1,2]. Noise applied on the equation of x can also achieve CS, but much higher noise intensity ($D_c^x>100$) is needed and the distortion of the attractor is very severe. For the variable z, it has been explained in Ref. [1] that, due to the symmetries of the Lorenz system, noise acting only in the z direction often fails to push trajectories of identical systems into the same half space, and CS is, therefore, unachievable. By these criteria, we may say that the most effective variable is y and the critical intensity is $D_c^y=33.3$. It is interesting to note that D_c^y is also similar to the mean size of the attractor in the y direction, that is, $S_y\approx 35$. Here, the mean size is defined by the mean value of local maxima minus the mean value of local minima of y, i.e., the mean fluctuation amplitude of y. We have also found this interesting correspondence in the following two examples.

The Pikovsky-Rabinovich (1978) [6] circuit model has the following equations:

$$\dot{x}=y-\beta z,$$
$$\dot{y}=-x+2\gamma y+\alpha z, \tag{1}$$
$$\dot{z}=(x-z^3+z)/\mu+D\xi,$$

with $\beta=0.66$, $\alpha=0.165$, $\gamma=0.201$, and $\mu=0.047$. This model mimics a simple electronic circuit. The noise ξ is Gaussian with $\langle\xi(t)\xi(t-\tau)\rangle=\delta(\tau)$ and D denotes the noise intensity. The equations were integrated using the stochastic Euler method [2] with a time-step of $\Delta t=0.001$. The PR circuit model with parameters given above has a symmetric chaotic attractor [6]. The projection of the attractor in the z-y plane is given in Fig. 1(a) for the case of $D=0$. Figure 1(b) shows the same projection of the attractor when noise of intensity $D=3.0$. Let X_1 and X_2 denote the vectors (x_1,y_1,z_1), (x_2,y_2,z_2) of two identical systems. The largest Lyapunov exponent (LLE) and the average synchronization error $|X_1-X_2|$ versus noise intensity D are given in Fig. 1(c). It can be seen that the LLE becomes negative when $D_c^z\approx 2.9$ and the synchronization error vanishes when $D>D_c^z$. Interestingly, again the noise intensity is roughly equal to the mean size of the attractor in the z direction ($S_z\approx 2.8$). However, if the equations are modified with common noise applied either for the x or y equation, the models fail to synchronize. There is a critical intensity point $D\approx 0.25$, below which synchronization is absent, and above which the system becomes unstable and variables undergo explosive growth.

The ability for the z equation to better accommodate noise forcing is a consequence of the stabilizing cubic term $F(z)=-z^3+z$ (corresponding to the tunnel diode in the original circuit). Approximating in the first instance $x=\overline{x}=0$, and plugging this into the equation for $\dot{z}$ with $D=0$, we see that z has two locally stable equilibria $z^*=\pm 1$ separated by the unstable equilibrium $z^*=0$. For relatively small noise levels, the trajectory spirals relatively slowly in the $x-y$ plane

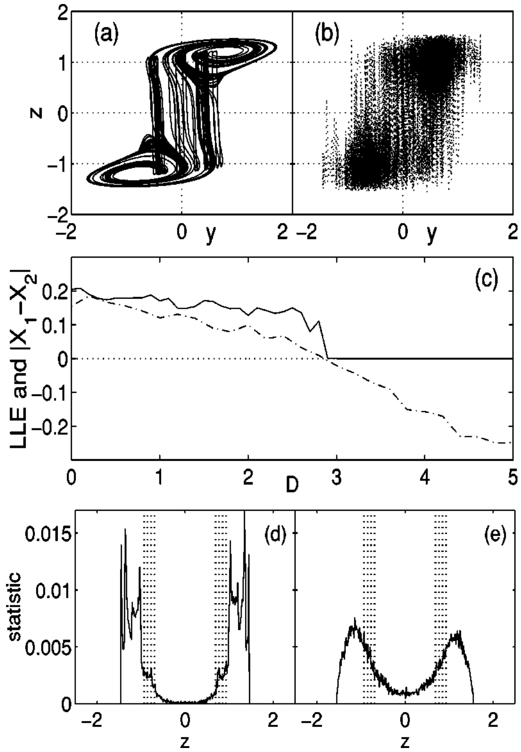

FIG. 1. Simulation result of the Pikovsky-Rabinovich circuit model. (a) Projection of the original attractor on the y-z plane ($D=0$); (b) with noise intensity $D=3.0$ on the y-z plane, noise added on the z direction; (c) the largest Lyapunov exponent (dash-dotted line) and average synchronization error $|X_1-X_2|$ (solid line) via noise intensity D; (d) sample data original distribution on z; (e) distribution with noise $D=3.0$. The contraction regions are shown by the dotted background.

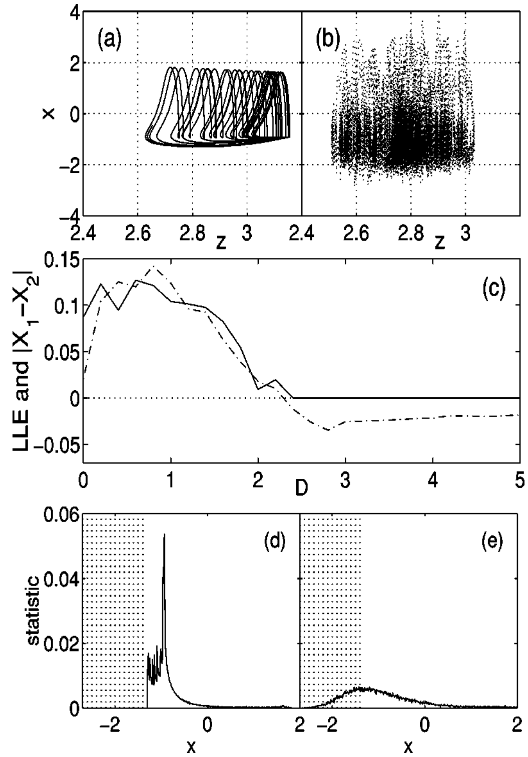

FIG. 2. Simulation result of the Hindmarsh-Rose neuron model. (a) Projection of the original attractor on the z-x plane ($D=0$); (b) with noise intensity $D=2.4$ on the z-x plane, noise added on the x direction; (c) the largest Lyapunov exponent (dash-dotted line) and average synchronization error $|X_1-X_2|$ (solid line) via noise intensity D; (d) distribution on x without noise; (e) distribution on x with noise $D=2.4$. The contraction regions are shown by the dotted background.

on one of the attractor wings, where either $z\approx 1$ or $z\approx -1$. However, the large noise levels associated with $D_c^z=2.9$ are able to push z from one stable equilibrium to the other with rapidity, so that it oscillates randomly between the two equilibria. Note that when close to synchronization, the system may be approximated as $\dot{\delta}=Df(x,y,z)\delta$, where $\delta=X_1-X_2$ and $Df(x,y,z)$ is the Jacobian matrix of the system evaluated on the synchronization manifold. By definition, the LLE is the long term time average of the logarithm of the norm of the product of the Jacobian along the trajectory. A simple calculation shows that $Df(x,y,z)$ is a function of the single variable z. As synchronization is ensured if the LLE is negative, synchronization is controlled exclusively by the dynamics of the z variable.

The reason that noise on z can induce CS may be understood by Figs. 1(d) and 1(e), where the distribution of z is plotted when the model is without noise and with noise, respectively. In Fig. 1(d), the distribution of z is thus for the case when the dynamics are strictly deterministic and the model's trajectory wanders on the chaotic attractor of Fig. 1(a). In Fig. 1(e), the distribution is given with noise intensity $D=3.0$ (i.e., slightly greater than the critical level $D_c^z=2.9$). At this noise level, the model's dynamics are largely swamped by the stochastic noise and the deterministic attractor appears largely washed out [Fig. 1(b)]. For noise levels lower than D_c^z, the deterministic attractor of Fig. 1(a) is more distinctive. Following Zhou and Kurths, define a *contraction region* [1] as that region on the attractor, where $Re(\Lambda_i)<0 (i=1,2,3)$ [Λ_i are the eigenvalues of $Df(x,y,z)$]. Figures 1(d) and 1(e) shows this contraction region projected in the z dimension as a dotted background. Noise has the effect of increasing the visiting frequency of the contraction region, thereby making the LLE negative.

We now turn to consider the Hindmarsh-Rose (1982) [7] neuron model,

$$\dot{x}=y-ax^3+bx^2-z+I+D\xi,$$

$$\dot{y}=c-dx^2-y, \tag{2}$$

$$\dot{z}=r[S(x-\chi)-z],$$

where $a=1.0$, $b=3.0$, $c=1.0$, $d=5.0$, $S=4.0$, $r=0.006$, $\chi=-1.56$, and $I=3.0$. The dynamics of this model is very different to that of the Lorenz attractor and the PR circuit. It exhibits a multi-time-scaled burst-rest behavior, a phenomenon which has great importance in neuroscience. The projection of the attractor in z-x plane is given in Fig. 2(a) without noise and in Fig. 2(b) with noise intensity $D=D_c^x=2.4$. Though the subtle structure of the attractor disappears in this strong noise regime, it still exhibits burst and rest behaviors in the x and y variables. The LLE and average synchronization error $|X_1 - X_2|$ versus noise intensity D is given in Fig. 2(c). It can be seen that the LLE becomes negative when $D_c^x = 2.4$ and the synchronization error vanishes for $D > D_c^x$. The mean size of the original attractor in the x direction is $S_x \approx 2.5$. Furthermore, CS can also be obtained by adding noise on the y equation, with $D_c^y \approx 6$, corresponding to the mean size of the attractor in the y direction $S_y \approx 6.2$. But small noise levels (e.g., $D = 0.3$) applied to the z equation severely distorts the attractor preventing meaningful synchronization.

Again the Jacobian matrix $Df(x,y,z)$ of the system is a function of the single variable, this time x. The reason that noise on x can induce CS may be understood by Figs. 2(d) and 2(e). Figure 2(d) gives the distribution of x for the deterministic chaotic attractor ($D=0$, $LLE>0$), with the dotted background indicating the contraction region. In Fig. 2(e), it can be seen that with noise $D=2.4$, x tends to visit the contraction region with much greater frequency, explaining why the LLE of the system is negative. Note that in the equation defining the motion of the variable x, there is also a cubic term $F(x) = -ax^3 + bx^2$. We see that when noise pushes x into the contraction region ($x < -1.4$), the cubic term acts as a large restoring force (stabilizing agent). This stabilizing action of the cubic term in the contraction regime is also responsible for the negative LLE when under noise forcing.

Finally, we study the kinetic rate neural spike equations with the parameters given in Ref. [5]. We found that CS can be induced through noise forcing of any single variable. The equation for y is the most effective with smallest critical noise intensity ($D_c^y = 0.01$) for CS. The variable z was the least effective with the highest critical noise intensity ($D_c^z = 0.025$). It is interesting that D_c^y and D_c^z corresponds to the mean size of the attractor in corresponding directions S_y and S_z, respectively.

In summary, we numerically analyzed noise-induced CS in several time-continuous chaotic models: the PR circuit, the HR neuron, and kinetic rate neural spike model. From these models, several interesting conclusions can be drawn: noise-induced CS, in general, can be obtained by adding noise on the "more stable" variable; for the PR and HR models, the one associated with the equation having a strong contractive cubic term. In addition, because the latter models are both linear apart from the aforementioned cubic term (polynomial), their Jacobian matrices are a function of a single variable only (the effective variable). The PR and HR models have single-variable-Jacobian matrix (but the kinetic rate model not). It is thus not surprising that the variable in the Jacobian matrix had so much influence. The critical noise intensity for CS is roughly equal to the mean size of the attractor measured with reference to the direction of the variable to which the noise is applied. At this intensity, the noise swamps the deterministic dynamics allowing identical models to synchronize to their common stochastic forcing. It has been found that noise-induced CS can be realized in Chua's circuit, but only with the application of *nonzero* mean noise forcing [4]. However, we found zero-mean Gaussian noise can realize synchronization in the PR circuit model, probably because of the tunnel diode in this circuit (which is the source of the cubic term discussed above).

It is possible that a designed experimental system of the same character can be effectively controlled by noise. We expect that the ideas presented in this paper may be helpful for further investigations of noise-induced synchronization in circuit and neuron experiments, and other practical applications.

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$$\dot{x} = 2hx + y - gz, \quad \dot{y} = -x, \quad \dot{z} = (x - f(z))/\epsilon + D\xi,$$

where $h = 0.07$, $g = 0.85$, $\epsilon = 0.2$, and $f(x) = 8.592z - 22z^2 + 14.408z^3$. Noise on the equation of z can induce CS, but noise on the equation of y blows up.
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