Derivation of nonlinear Fokker-Planck equations by means of approximations to the master equation

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Nonlinear Fokker-Planck equations (FPEs) are derived as approximations to the master equation, in cases of transitions among both discrete and continuous sets of states. The nonlinear effects, introduced through the transition probabilities, are argued to be relevant for many real phenomena within the class of anomalous-diffusion problems. The nonlinear FPEs obtained appear to be more general than some previously proposed (on a purely phenomenological basis) ones. In spite of this, the same kind of solution applies, i.e., it is shown that the time-dependent Tsallis's probability distribution is a solution of both equations, obtained either from discrete or continuous sets of states, and that the corresponding stationary solution is, in the infinite-time limit, a stable solution.

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I. INTRODUCTION

The master equation is one of the most important equations in statistical physics, with a wide range of applicability; essentially, it governs the dynamics of Markov processes, which are stochastic processes with a very limited memory of previous events [1,2]. The time evolution of a system of stochastic variables is characterized by transitions between the various realizations of these variables, in such a way that the probability of finding the system in a given state changes in time until it reaches a steady state, in which transitions do not produce changes in the probablity distribution. The master equation specifies how this probability distribution evolves in time due to such transitions between states.

For a system described in terms of discrete stochastic variables, the master equation for the probability $P(n,t)$ of finding the system in a state characterized by the variable n at time t, is given by

$$\frac{\partial P(n,t)}{\partial t}=\sum_{m=-\infty}^{\infty}[P(m,t)w_{m,n}(t)-P(n,t)w_{n,m}(t)], \tag{1.1}$$

where $w_{k,l}(t)$ represents the transition probability rate from state k to state l [i.e., $w_{k,l}(t)\Delta t$ is the probability for a transition from state k to state l to occur during the time interval $t\rightarrow t+\Delta t$]. The master equation may be written also for the case of a continuous stochastic variable x,

$$\frac{\partial P(x,t)}{\partial t}=\int_{-\infty}^{\infty}dx'[P(x',t)w(x'|x)-P(x,t)w(x|x')], \tag{1.2}$$

where $w(y|z)$ represents the transition rate from state y to state z.

By choosing conveniently the transition rates, both forms of the master equation [Eqs. (1.1) and (1.2)] lead, under certain approximations [1,2], to the linear Fokker-Planck equation (FPE),

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial[F(x)P(x,t)]}{\partial x}+D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{1.3}$$

where D is a constant (usually known as the diffusion constant) and $F(x)$ represents an external force. For the case of discrete stochastic variables, the FPE above may be obtained from Eq. (1.1), if one considers a random walk in which the step size is given by Δ; defining

$$w_{k,l}(\Delta)=-\frac{1}{\Delta}\delta_{k,l+1}F(k\Delta)+\frac{D}{\Delta^2}(\delta_{k,l+1}+\delta_{k,l-1}), \tag{1.4}$$

and taking the limit $\Delta\rightarrow 0$ [2], one gets Eq. (1.3). In the continuous case [Eq. (1.2)], if one defines $y=x'-x$ and the transition rate

$$w(x|x+y)=\gamma_1(x,y)+\gamma_2(x,y)D, \tag{1.5}$$

where

$$\gamma_1(x,y)=\begin{cases}\dfrac{F(x)}{\Delta^2} & \text{if}\quad 0\leqslant y\leqslant\sqrt{2}\Delta,\\[2ex] 0 & \text{otherwise}\end{cases} \tag{1.6a}$$

and

$$\gamma_2(x,y)=\begin{cases}\dfrac{1}{2\sqrt{6}\Delta^3} & \text{if}\quad -\sqrt{6}\Delta\leqslant y\leqslant\sqrt{6}\Delta,\\[2ex] 0 & \text{otherwise,}\end{cases} \tag{1.6b}$$

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one may obtain Eq. (1.3), after the limit $\Delta \to 0$ is taken [2]. It is important to mention that other transition rates, different from the ones defined above, may be used in order to obtain the FPE in Eq. (1.3); e.g., for the case of continuous stochastic variables, any distribution with the first and second moments given, respectively, by $F(x)$ and a constant, and with vanishing higher moments in the limit $\Delta \to 0$, will lead to Eq. (1.3). However, in these transition rates, the first contribution [associated with the external force $F(x)$] must be an asymmetric term, in order to yield the first derivative in the right-hand side (rhs) of Eq. (1.3), whereas the second contribution (associated with the diffusion constant D) must be a symmetric term, in order to lead solely to the second derivative in the rhs of Eq. (1.3) (with a cancellation of first-derivative terms).

Although the linear FPE is very important for the description of many natural phenomena—mainly those within the class of normal diffusion—it is currently accepted that such an equation is not appropriate to describe more complicated diffusion processes, like those inserted in the class of anomalous-diffusion problems [3,4]. Among such processes, one may single out the transport of a fluid in porous media [5], the dynamics of surface growth [5], diffusion of polymerlike breakable micelles [6], correlations in heartbeat interval increments [7], and financial transactions [8]. Nowadays, there is an increasing trend towards the use of nonlinear FPEs as good candidates for a proper description of anomalous-diffusion processes [3–6,9–16]. Some of the nonlinear FPEs proposed appear as simple phenomenological generalizations of the linear FPE in Eq. (1.3) [9–12], and their solution comes to be the powerlike probability distribution that maximizes the entropy proposed by Tsallis [17–20].

It is important to remind that it is also possible to describe anomalous transport processes within a linear theory, by introducing the anomalous nature of the process through correlations expressed in nonlocal operators. Such an approach has been much considered recently, through the study of the fractional Fokker-Planck equation (see Ref. [21] for a review), which may also be derived [22] from a generalized master equation [23,24].

In the present work, we derive nonlinear FPEs directly from the master equation, by introducing nonlinear effects on the transition rates. Such nonlinear effects are expected to appear in some processes within the anomalous-diffusion class, as we argue below.

Particle diffusion in a porous media. The randomness in the media induces a distribution for the time that the particle spends in each position; one expects that such a distribution should be related to the probability distribution $P(\vec{x},t)$, defined above. The probability for a jump between two given states in phase space, in this case, should take in consideration this time distribution (transitions become more unlikely when the particle spends more time in a given position). Therefore, it is reasonable to propose a transition rate between two given states with a dependence on $P(\vec{x},t)$.

Surface growth in fractals. Since the growth of the surface in a fractal system occurs by adding new particles, at random, in the existing surface, one has, like in the preceding example, a distribution for the time that it takes for a new particle to be placed in a given position of the growing surface. If one defines $P(\vec{x},t)$ as the probability for a new particle to be placed at a position $\vec{x}$ nearby the growing surface, at time t, one should expect a dependence of the transition rate on $P(\vec{x},t)$ as well.

Financial transactions. Let us consider a simple case where the state of the system is characterized by the total amount of money, i.e., the budget, owned by an individual, whereas transitions between states occur by means of financial transactions. The probability for the individual to perform a certain financial transaction depends, of course, on his budget, in such a way that unprobable transactions may become possible as his budget increases. The transition probability between states should depend, in this case, on the probability for the individual to be found in a state characterized by a budget compatible with the transaction. Similar arguments hold for other financial transactions, such as those between companies and stock market exchanges.

This paper is organized as follows. In the following section we consider the master equation for a system of discrete stochastic variables; by introducing transition rates with an explicit dependence on the probability of finding the system in a given state, we derive a nonlinear FPE. In Sec. III, a similar procedure is given for the case of a system described in terms of continuous stochastic variables. In both cases, we show that the time-dependent Tsallis's power-law probability distribution is a solution of the nonlinear FPEs obtained, and that in the infinite-time limit, it approaches a stationary solution. Finally, in Sec. IV we present our conclusions.

II. DISCRETE STOCHASTIC VARIABLES

Let us consider the master equation of Eq. (1.1) for a random walk in which the step size is given by Δ,

$$\frac{\partial P(n\Delta,t)}{\partial t} = \sum_{m=-\infty}^{\infty} [P(m\Delta,t)w_{m,n}(\Delta) - P(n\Delta,t)w_{n,m}(\Delta)]. \tag{2.1}$$

As argued before, we shall introduce a transition rate with a dependence on P; we do that by modifying the transition rate of Eq. (1.4) to the form

$$w_{k,l}(\Delta) = -\frac{1}{\Delta}\delta_{k,l+1}F(k\Delta) + \frac{1}{\Delta^2}(\delta_{k,l+1}+\delta_{k,l-1})[aP^{\mu-1}(k\Delta,t)+bP^{\nu-1}(l\Delta,t)], \tag{2.2}$$

where a and b are constants, which may depend, in principle, on the system under consideration. The first nonlinear contribution, $P^{\mu-1}(k\Delta,t)$, corresponds to a dependence on the probability of the state where the system is found, before the transition takes place; the motivation for this kind of contribution has been described, for several real systems, at the end of the preceding section. For the sake of generality, we introduce, as well, through the second nonlinear contribution, $P^{\nu-1}(l\Delta,t)$, a dependence on the probabilities of the states near state k; the relevance of this term will be discussed throughout this section. It is important to remind that the transition rate above reduces to the form of Eq. (1.4) either for $(a=D,b=0,\mu=1)$ or $(a+b=D,\mu=\nu=1)$; obviously, the nonlinear FPE to be obtained will recover the form of Eq. (1.3) in these two particular limits.

Substituting Eq. (2.2) in Eq. (2.1), carrying out the sums, and defining $x=n\Delta$, one gets

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{1}{\Delta}[P(x+\Delta,t)F(x+\Delta)-P(x,t)F(x)]\\&+\frac{a}{\Delta^2}[P^{\mu}(x+\Delta,t)+P^{\mu}(x-\Delta,t)]-\frac{2a}{\Delta^2}P^{\mu}(x,t)\\&+\frac{b}{\Delta^2}P^{\nu-1}(x,t)[P(x+\Delta,t)+P(x-\Delta,t)]\\&-\frac{b}{\Delta^2}P(x,t)[P^{\nu-1}(x+\Delta,t)+P^{\nu-1}(x-\Delta,t)].\end{aligned}\tag{2.3}$$

The limit $\Delta\to 0$ leads to the following nonlinear FPE:

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t}=&-\frac{\partial[F(x)P(x,t)]}{\partial x}+a\frac{\partial^2 P^{\mu}(x,t)}{\partial x^2}\\&+bP^{\nu-1}(x,t)\frac{\partial^2 P(x,t)}{\partial x^2}-bP(x,t)\frac{\partial^2 P^{\nu-1}(x,t)}{\partial x^2}.\end{aligned}\tag{2.4}$$

It should be noticed that the equation above recovers, for $b=0$, the nonlinear FPE proposed, on a phenomenological basis, by many authors [9–14], whereas the term $bP^{\nu-1}(x,t)\partial^2P(x,t)/\partial x^2$ coincides with one of the terms in the nonlinear diffusion equation proposed by Bouchaud *et al.* (considering $\nu-1=-\theta$, in their notation), for the description of the diffusion of polymerlike breakable micelles [6]. This gives an example where the second nonlinear contribution to the transition rate of Eq. (2.2) [term $bP^{\nu-1}(l\Delta,t)$] could be important.

In the Appendix we show that Tsallis's probability distribution,

$$P(x,t)=B(t)[\xi(x,t)]^{1/(1-q)},\tag{2.5a}$$

$$\xi(x,t)=1+\beta(t)(q-1)[x-x_0(t)]^2 \qquad (1<q<3),\tag{2.5b}$$

is a solution of Eq. (2.4), with the particular choice $\mu=\nu=2-q$, for the case of an external force $F(x)=k_1-k_2x$ (k_1 and k_2 constants, $k_2\geqslant 0$). It is shown that, if one considers a normalized probability distribution at some reference initial time, $t=t_0$, $\int_{-\infty}^{\infty}dxP(x,t_0)=1$, the above solution preserves the normalization condition for any arbitrary time t ($t>t_0$). Furthermore, the parameters of Eqs. (2.5) satisfy differential equations that may be solved to yield

$$x_0(t)=\frac{k_1}{k_2}+\left[x_0(t_0)-\frac{k_1}{k_2}\right]\exp[-k_2(t-t_0)],\tag{2.6a}$$

$$B(t)=\{(B^*)^{q-3}+\{[B(t_0)]^{q-3}-(B^*)^{q-3}\}\exp[-k_2(3-q)(t-t_0)]\}^{1/(q-3)},\tag{2.6b}$$

$$\beta(t)=\beta(t_0)\left[\frac{B(t)}{B(t_0)}\right]^2.\tag{2.6c}$$

In the limit $t\to\infty$, the above time-dependent parameters approach the stationary values,

$$x_0^*=\frac{k_1}{k_2},\tag{2.7a}$$

$$B^*=\left\{\frac{k_2[B(t_0)]^2}{2[a(2-q)+bq]\beta(t_0)}\right\}^{1/(3-q)},\tag{2.7b}$$

$$\beta^*=\beta(t_0)\left[\frac{B^*}{B(t_0)}\right]^2,\tag{2.7c}$$

which require $a(2-q)+bq>0$. In the Appendix, we also show that the above stationary solution is stable under small arbitrary perturbations.

With the choice $\mu=\nu=2-q$, mentioned above, the nonlinear FPE of Eq. (2.4) becomes

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + a\frac{\partial^2 P^{2-q}(x,t)}{\partial x^2} + bP^{1-q}(x,t)\frac{\partial^2 P(x,t)}{\partial x^2} - bP(x,t)\frac{\partial^2 P^{1-q}(x,t)}{\partial x^2}, \tag{2.8}$$

where one can identify the term relevant for the description of breakable-micelles difusion, $bP^{1-q}(x,t)\partial^2 P(x,t)/\partial x^2$, by considering $\theta = q-1$, which yields $q=5/2$, for the typical value $\theta = 3/2$ [6].

It is important to remind that the above solution is valid for $1<q<3$ and, for stability, one must have $a(2-q)+bq >0$. Besides that, each of the coefficients of the derivatives in Eq. (2.8) must change its sign, whenever the total power of $P(x,t)$ changes sign, leading to $a,b>0$ for $1<q<2$, and $a,b<0$ for $2<q<3$. Without loss of generality, the derivatives that appear in the above equation may be rewritten in such a way so as to yield an analytic continuation through $q=2$, e.g., the second term in rhs of Eq. (2.8) is equivalent to, apart from a multiplicative constant, $[a/(2-q)][\partial^2(P^{2-q}(x,t)-1)/\partial x^2]$, leading to $a\{\partial^2[\ln P(x,t)]/\partial x^2\}$, when $q\to 2$. The restrictions mentioned above result in

$$b > -\frac{2-q}{q}a \quad (1<q<2;\; a,b>0), \tag{2.9a}$$

$$|b| < \frac{q-2}{q}|a| \qquad (2<q<3; \quad a,b<0). \tag{2.9b}$$

The restriction in Eq. (2.9a), for $1<q<2$, is satisfied for any pair of coefficients a and b $(a,b>0)$; in particular, one can have systems for which the above solution may be valid, with $b \gg a$, implying a behavior dominated by transitions with a dependence on the probabilities of the states near state k [term $bP^{\nu-1}(l\Delta,t)$ in Eq. (2.2)]. On the other hand, the restriction in Eq. (2.9b), for $2<q<3$, shows that, for the above solution to be valid, transitions with a dependence on the probability of the state where the system is found, before the transition takes place [term $P^{\mu-1}(k\Delta,t)$ in Eq. (2.2)], are always dominant, particularly as $q\to 2$ from above. In this case, transitions with a dependence on $P^{\nu-1}(l\Delta,t)$ become more relevant as q moves away from $q=2$.

In the following section, we shall derive a nonlinear FPE from a master equation defined for a system of continuous stochastic variables.

III. CONTINUOUS STOCHASTIC VARIABLES

Introducing the variable $y=x-x'$, the master equation of Eq. (1.2) becomes

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} dy[P(x-y,t)w(x-y|x) - P(x,t)w(x|x+y)], \tag{3.1}$$

where we have changed $y\to -y$ in the first integral. Defining $\tau(x,y)=w(x|x+y)$ as the transition rate between states x and $x+y$, Eq. (3.1) may be written as

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} dy[P(x-y,t)\tau(x-y,y) - P(x,t)\tau(x,y)]. \tag{3.2}$$

Considering $\tau(x,y)$ sharply peaked around $y=0$, one may expand Eq. (3.2) [26,27,2],

$$\frac{\partial P(x,t)}{\partial t} = \int_{-\infty}^{\infty} dy\left\{\sum_{n=1}^{\infty}\frac{(-y)^n}{n!}\frac{\partial^n[P(x,t)\tau(x,y)]}{\partial^n x}\right\}. \tag{3.3}$$

Let us now define the transition rate for a transition between states x and $x+y$; we shall do this by modifying the one used for the derivation of the linear FPE [see Eqs. (1.5) and (1.6)] in such a way so as to introduce a dependence on P similar to the one used in the case of discrete stochastic variables [cf. Eq. (2.2)]. Let us consider

$$\tau(x,y) = \gamma_1(x,y) + \gamma_2(x,y)[aP^{\mu-1}(x,t) + bP^{\nu-1}(x+y,t)], \tag{3.4}$$

where $\gamma_1(x,y)$ and $\gamma_2(x,y)$ are the same as defined in Eqs. (1.6). Substituting the transition rate above in Eq. (3.3), expanding $P^{\nu-1}(x+y,t)$ for y small, and taking the limit $\Delta \to 0$, one gets the following nonlinear FPE:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial[F(x)P(x,t)]}{\partial x} + a\frac{\partial^2 P^{\mu}(x,t)}{\partial x^2} + b\frac{\partial^2 P^{\nu}(x,t)}{\partial x^2} - 2b\frac{\partial P(x,t)}{\partial x}\frac{\partial P^{\nu-1}(x,t)}{\partial x} - 2bP(x,t)\frac{\partial^2 P^{\nu-1}(x,t)}{\partial x^2}. \tag{3.5}$$

Although the equation above presents a few different terms with respect to the nonlinear FPE obtained in the preceding section [cf. Eq. (2.4)], the probability distribution of Eqs. (2.5) is also a solution in the present case, if one considers the same external force, i.e., $F(x)=k_1-k_2x$ (k_1 and k_2 constants, $k_2\geq 0$). Indeed, if one substitutes the derivatives of Eqs. (A2) (see Appendix) into Eq. (3.5), considering the particular choice $\mu=\nu=2-q$, and comparing equal powers, $[\xi(x,t)]^{\alpha/(1-q)}[x-x_0(t)]^m$, one finds, curiously, *exactly* the same set of differential equations of the previous case, for the parameters $(x_0(t),B(t),\beta(t))$ [cf. Eqs. (A3) in the Appendix]. It should be emphasized that the two nonlinear FPEs derived herein present a solution with the *same functional form* if an external force of the kind $F(x)=k_1-k_2x$ is considered in both cases. However, the parameters a and b that appear in such solutions have different meanings in each case, since they may be associated with distinct coefficients of the differential equations. Hence, the same analysis of the preceding section applies for the present case.

With the choice $\mu=\nu=2-q$, Eq. (3.5) becomes

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial[F(x)P(x,t)]}{\partial x}+(a+b)\frac{\partial^2 P^{2-q}(x,t)}{\partial x^2}-2b\frac{\partial P(x,t)}{\partial x}\frac{\partial P^{1-q}(x,t)}{\partial x}-2bP(x,t)\frac{\partial^2 P^{1-q}(x,t)}{\partial x^2} \tag{3.6}$$

in such a way that Tsallis's probability distribution is a solution of the nonlinear FPE above, with its validity restricted by the constraints of Eqs. (2.9).

IV. CONCLUSION

We have derived nonlinear FPEs directly from the master equation. The master equation was considered for cases of transitions among both discrete and continuous sets of states, leading to two different nonlinear FPEs. The nonlinear effects were introduced through a dependence of the transition rates on both probabilities of finding the system on the state before and after the transition. We have shown that the probability distribution that maximizes Tsallis's entropy is a solution of both equations; although the two nonlinear FPEs appear to be different from each other, the time-dependent parameters that appear in Tsallis's probability distribution satisfy the same differential equations. We have found the stationary solution, and have shown that, in the infinite-time limit, it represents a stable solution. Our FPEs appear to be more general than some nonlinear FPEs, introduced previously by many authors, in a purely phenomenological basis. The pertinence of the new terms was discussed, and it was argued that, depending on the system, such terms may become important, and even dominate the dynamics. The transition rates employed, as well as the FPEs obtained, are expected to be relevant for the description of many real phenomena, specially those included in the anomalous-diffusion category.

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APPENDIX

In this appendix we will solve Eq. (2.4) for the case of an external force $F(x)=k_1-k_2x$ (k_1 and k_2 constants, $k_2\geqslant 0$); in fact, we will show that the Tsallis's probability distribution,

$$P(x,t)=B(t)[\xi(x,t)]^{1/(1-q)}, \tag{A1a}$$

$$\xi(x,t)=1+\beta(t)(q-1)[x-x_0(t)]^2 \qquad (1<q<3), \tag{A1b}$$

is a solution of Eq. (2.4). Substituting the derivatives

$$\frac{\partial P(x,t)}{\partial t}=\frac{dB(t)}{dt}[\xi(x,t)]^{1/(1-q)}+B(t)[\xi(x,t)]^{q/(1-q)}\left[2\beta(t)[x-x_0(t)]\frac{dx_0(t)}{dt}-[x-x_0(t)]^2\frac{d\beta(t)}{dt}\right], \tag{A2a}$$

$$\frac{\partial[F(x)P(x,t)]}{\partial x}=-2(k_1-k_2x)B(t)[\xi(x,t)]^{q/(1-q)}\beta(t)[x-x_0(t)]-k_2B(t)[\xi(x,t)]^{1/(1-q)}, \tag{A2b}$$

$$\frac{\partial P^{\gamma}(x,t)}{\partial x}=-2\gamma B^{\gamma}(t)[\xi(x,t)]^{(\gamma-1+q)/(1-q)}\beta(t)[x-x_0(t)], \tag{A2c}$$

$$\frac{\partial^2 P^{\gamma}(x,t)}{\partial x^2}=-2\gamma B^{\gamma}(t)[\xi(x,t)]^{(\gamma-1+q)/(1-q)}\beta(t)+4\gamma B^{\gamma}(t)(\gamma-1+q)[\xi(x,t)]^{(\gamma-2+2q)/(1-q)}\beta^2(t)[x-x_0(t)]^2 \tag{A2d}$$

into Eq. (2.4) and equating equal powers, $[\xi(x,t)]^{\alpha/(1-q)}[x-x_0(t)]^m$, one has, for the particular choice $\mu=\nu=2-q$, the set of differential equations

$$\frac{dx_0(t)}{dt}=k_1-k_2x_0(t), \tag{A3a}$$

$$\frac{dB(t)}{dt}=k_2B(t)-2[a(2-q)+bq][B(t)]^{2-q}\beta(t), \tag{A3b}$$

$$\frac{d\beta(t)}{dt}=2k_2\beta(t)-4[a(2-q)+bq][B(t)]^{1-q}\beta^2(t). \tag{A3c}$$

Comparing Eqs. (A3b) and (A3c), one concludes that

$$\frac{d\beta(t)}{dt}=2\frac{\beta(t)}{B(t)}\frac{dB(t)}{dt}, \tag{A4}$$

which is the equation that ensures the preservation of the normalization condition for $P(x,t)$ [9]. In fact, if one integrates Eq. (A4), one gets a relation between $\beta(t)$ and $B(t)$,

$$\beta(t)=\beta(t_0)\left[\frac{B(t)}{B(t_0)}\right]^2, \tag{A5}$$

where t_0 is some reference initial time. If one chooses the normalization constant [25]

$$B(t_0)=\left[\frac{\beta(t_0)(q-1)}{\pi}\right]^{1/2}\frac{\Gamma[1/(q-1)]}{\Gamma[(3-q)/2(q-1)]} \quad (1<q<3), \tag{A6}$$

the normalization condition at $t=t_0$, $\int_{-\infty}^{\infty}dxP(x,t_0)=1$, may be expressed as

$$\begin{aligned}\int_{-\infty}^{\infty}dxP(x,t_0)&=\int_{-\infty}^{\infty}dxB(t_0)\{1+\beta(t_0)(q-1)[x-x_0(t_0)]^2\}^{1/(1-q)}\\&=B(t_0)[\beta(t_0)]^{-1/2}\int_{-\infty}^{\infty}dy[1+(q-1)y^2]^{1/(1-q)}=1,\end{aligned} \tag{A7}$$

where we have defined the variable, $y=[\beta(t_0)]^{1/2}[x-x_0(t_0)]$. Now, for an arbitrary time t ($t>t_0$),

$$\begin{aligned}\int_{-\infty}^{\infty}dxP(x,t)&=\int_{-\infty}^{\infty}dxB(t)\{1+\beta(t)(q-1)[x-x_0(t)]^2\}^{1/(1-q)}\\&=B(t)[\beta(t)]^{-1/2}\int_{-\infty}^{\infty}dz[1+(q-1)z^2]^{1/(1-q)}=1,\end{aligned} \tag{A8}$$

where, $z=[\beta(t)]^{1/2}[x-x_0(t)]$; the preservation of normalization obtained above follows by using Eq. (A5), when comparing the last integrals of Eqs. (A7) and (A8).

Let us now concentrate on the set of Eqs. (A3); due to the relation in Eq. (A5), it is necessary to solve either one of Eqs. (A3b) or (A3c). Substituting Eq. (A5) into Eq. (A3b), one gets

$$\frac{dB(t)}{dt}=k_2B(t)-2[a(2-q)+bq][B(t)]^{4-q}\frac{\beta(t_0)}{[B(t_0)]^2}. \tag{A9}$$

Let us first analyze the stationary solution, B^*, satisfying $(dB^*/dt)=0$,

$$B^*=\left\{\frac{k_2[B(t_0)]^2}{2[a(2-q)+bq]\beta(t_0)}\right\}^{1/(3-q)}, \tag{A10}$$

which requires $a(2-q)+bq>0$. In order to perform a stability analysis of the stationary solution B^*, we will consider a small arbitrary perturbation $\eta(t)$ around B^*, i.e., $B(t)=B^*+\eta(t)$; one gets from Eq. (A9),

$$\frac{d\eta(t)}{dt}=\Lambda\eta(t), \quad \Lambda=k_2-2[a(2-q)+bq](4-q)\times[B^*]^{3-q}\frac{\beta(t_0)}{[B(t_0)]^2}. \tag{A11}$$

Stability requires $\Lambda<0$; if one substitutes Eq. (A10) into Eq. (A11), one gets that $\Lambda=k_2(q-3)$, which is negative for $q<3$. Therefore, the stationary solution B^* of Eq. (A10) represents a stable solution of Eq. (A9), in the limit $t\to\infty$. As a consequence of this, the associated stationary solution, β^*, obtained from Eq. (A5), is also an stable solution in the infinite-time limit. A similar analysis follows trivially for the stationary solution of Eq. (A3a), $x_0^*=k_1/k_2$.

The differential equations (A3a) and (A9) may be solved easily [10]; one gets

$$x_0(t)=\frac{k_1}{k_2}+\left[x_0(t_0)-\frac{k_1}{k_2}\right]\exp[-k_2(t-t_0)], \tag{A12a}$$

$$B(t)=\{(B^*)^{q-3}+\{[B(t_0)]^{q-3}-(B^*)^{q-3}\}\exp[-k_2(3-q)\times(t-t_0)]\}^{1/(q-3)}, \tag{A12b}$$

which approach, in the limit $t\to\infty$, the stationary solutions (x_0^*,B^*) defined above.

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