Density waves in traffic flow of two kinds of vehicles

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Through the car-following model, the traffic flow of two types of vehicles (cars and trucks) on a single-lane flow is studied, in which drivers on different vehicles have different sensitivities and the safety distance is assumed to be the same for all vehicles. The linear analysis is carried out to determine the condition of critical stability. With the nonlinear analysis, it proves that the small fluctuation of the vehicle density near the critical stable state satisfies the Korteweg–deVries equation and different sensitivities affect only the soliton evolution. When the headway in the critical state is more than the safety distance, the density around the soliton peak exceeds the density of the critical stable state, which can be explained as the formation of traffic jam. Contrarily, when the headway state is less than the safety distance, drivers will increase the headway to avoid the jam. The direct approach of the soliton perturbation shows that drivers' sensitivity will increase the soliton's amplitude continuously. Moreover, the increase of the number of trucks in the traffic flow will slow down the evolution of the amplitude.

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I. INTRODUCTION

For many years, traffic problems have been physicists' enthusiasm. A lot of models have been proposed for them, for instance, the car-following model [1–3], the cell automaton model [4–6], the gas kinetic model [7,8], and the hydrodynamic model [9–12], among which the car-following model is particularly important owing to its conciseness.

In a traffic flow, the fluctuation of vehicle density may form a density wave due to the interaction between vehicles. Since the density wave on some sections of a highway can become high, people naturally consider its formation to be related to a traffic jam [11–15]. At present, traffic flows in different models are exhibited quite differently. In particular, the soliton waves that describe the density fluctuation in different traffic models may have different forms. Research in some models show the existence of a causality relation between the soliton wave and traffic congestion [12–15], but this relation becomes uncertain in others [11]. Therefore, it is necessary to determine whether there exists a causality relation between the soliton wave and traffic congestion for each model. In this paper, we use nonlinear analysis to study the relation for the car-following model of two types of vehicles. Because the typical car-following model is normally discontinuous [2], in order to carry out the analysis, a simple way is to transform the discontinuous form into a continuous one. Moreover, the headway in the car-following model is a function of the vehicle density and its high-order derivative [16], so the density fluctuation appears in the acceleration equation with an explicit expression [17]. The nonlinear theory figures out that solitons can come forth from a system with both nonlinear and dispersion effects when these effects are balanced [19,20]. With the nonlinear and dispersion term in the acceleration equation of car-following model, naturally, the soliton wave can be formed and affect the traffic flows.

The stimulation of the traffic flow is worked out, which indicates the asymptotic stability phenomenon in our model [21]. In addition, it also points out that increasing the proportion of trucks in the total number of vehicles will reduce the stablity of the traffic flow [21]. This interesting finding attracts us to study how the soliton wave is formed in the model and how the proportion of trucks affects the soliton wave.

Our model has stable solution with the constant density and vehicle velocity, in which the linear analysis determines the critical stable state. So long as the traffic flow deviates away from the critical state a little, we can work out a Korteweg–de Vries (KdV) equation of the fluctuation. When the headway of the critical stable flow is larger than the safety distance, namely, the critical density of the critical stable state is less than the density determined by the safety distance, the soliton solution obtained from the KdV equation indicates that the density around the soliton peak will exceed the critical density. This phenomenon shows that the soliton of the density wave promotes the formation of the traffic jam [2]. But on the contrary, when the headway is smaller than the safety distance, the solution then indicates that the density will be reduced by increasing the headway. This result means that drivers tend to increase the distance to avoid forming a traffic jam.

Unlike the usual physical models, drivers' subjective in response to the running of vehicle flows is considered specially in the traffic problem, which is determined by the sensitivity parameter a in Bando model [2]. It is obvious that drivers of different vehicles have different sensitivities in this model. With the nonlinear analysis, we find that different sensitivities only influence the evolution of the soliton wave through a little perturbation. The direct approach of the soliton perturbation demonstrates that the evolution of the soliton wave's amplitude is obtained and the perturbation increases the soliton's amplitude at all times. Moreover, the increase of the number of trucks in the traffic flow will slow down the amplitude's evolution.

II. MODEL AND STABILITY ANALYSIS

The car-following model numbers each vehicle by an integer n. The nth vehicle has the acceleration determined by its actual velocity v_n and the desired speed $V(b_n)$ as follows [2]:

$$\dot{v}_n = a[V(b_n) - v_n], \tag{1}$$

where b_n is the headway from it to the vehicle in front. In the above equation, the overdot represents the derivative with respect to time. The optimal velocity $V(b_n) = \tanh(b_n - h_c) + \tanh(h_c)$ in Bando model is employed as the desired speed in Eq. (1), in which h_c is assumed to be the safety distance. The parameter a stands for the driver's sensitivity and it is equal to the inverse of the driver's reaction time.

The traffic flow investigated here consists of two types of vehicles, in which m trucks are inserted in the column of n cars. In this flow, the cars are numbered from 1 to l and $l+m+1$ to $m+n$ while the trucks are numbered from $l+1$ to $l+m$. Because the drivers of different vehicles have different sensitivities, so a_C is the sensitivity parameter of car drivers and a_T is of truck drivers. Obviously, we can suppose $a_C > a_T$. In addition, the safety distance h_c is regarded as the same for all vehicles.

The basic equation (1) is discontinous. Its continuous form is more appropriate for the linear and nonlinear analysis. In accordance with the definition of the flow density $\rho(x,t)$, the continuity equation and headway function $b(x,t)$ in the car-following model can be written as follows [16]:

$$\rho_t + (\rho v)_x = 0 \tag{2}$$

and

$$b \sim \frac{1}{\rho} - \frac{\rho_x}{2\rho^3} - \frac{\rho_{xx}}{6\rho^4} + \cdots, \tag{3}$$

respectively. Moreover, the continuous dynamic equation satisfied by vehicle velocity can be rewritten as [16]

$$v_t + v v_x = a[\overline{V}(\rho) - v] + a\overline{V}'(\rho)\left[\frac{\rho_x}{2\rho} + \frac{\rho_{xx}}{6\rho^2}\right]. \tag{4}$$

All terms in the above equation have been defined clearly according to the traffic problem. It is obvious that Eqs. (3) and (4) have the trivial solution $\rho_h = A$ and $v_h = \overline{V}(A)$ for an arbitrary constant A.

According to the linear analysis method, the traffic flow is assumed to deviate from a trivial solution ρ_h infinitesimally [11]. We can decompose the density and the speed into a linear combination of Fourier mode, respectively, and each of them grows or decays with its own growth rate σ_k. Thus we have

$$\begin{pmatrix} \rho(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \rho_h \\ v_h \end{pmatrix} + \sum_k \begin{pmatrix} \hat{\rho}_k \\ \hat{v}_k \end{pmatrix} \exp(ikx + \sigma_k t). \tag{5}$$

Substituting these expressions into Eqs. (2) and (4), we can get the equations of the deviation $\hat{\rho}_k$ and $\hat{v}_k$ as follows:

$$\hat{\rho}_k \sigma_k + \rho_h \hat{v}_k (ik) + v_h \hat{\rho}_k (ik) = 0, \tag{6}$$

$$\hat{v}_k \sigma_k + v_h (ik) \hat{v}_k = a[\overline{V'}(\rho_h)\hat{\rho}_k - \hat{v}_k] + a\overline{V}'(\rho_h)\left[\frac{ik\hat{\rho}_k}{2\rho_h} - \frac{k^2\hat{\rho}}{6\rho_h^2}\right]. \tag{7}$$

The linear analysis shows that the critical disturbance travels with a speed [11]

$$c(\rho_h) = \overline{V}(\rho_h) + \rho_h \overline{V}'(\rho_h). \tag{8}$$

Assuming that the density fluctuation is very small, the stablility condition for a traffic flow of two types of vehicles can be written as [21]

$$n + m - 2V'(b)(n/a_C + m/a_T) > 0, \tag{9}$$

or

$$\mathrm{sech}^2(b - h_c) < \frac{n+m}{2(n/a_C + m/a_T)}. \tag{10}$$

For the critical stable state of a traffic flow, stability condition can be expressed as $a_C = 2\,\mathrm{sech}^2(b_C - h_c)$ and $1/\rho_{hC} = \mathrm{sech}^{-1}\sqrt{a_C/2} + h_c$ for a car flow, and $a_T = 2\,\mathrm{sech}^2(b_T - h_c)$ and $1/\rho_{hT} = \mathrm{sech}^{-1}\sqrt{a_T/2} + h_c$ for a truck flow. Otherwise, for the traffic flow of two types of vehicles, the expression (10) determines the vehicle density in the critical state to be $1/\rho_h = \mathrm{sech}^{-1}\sqrt{(n+m)/2(n/a_C + m/a_T)} + h_c$, which can be rewritten as [16]

$$1/\rho_h = \mathrm{sech}^{-1}\sqrt{\bar{a}} + h_c, \tag{11}$$

in which $\bar{a} = (n+m)/(n/a_C + m/a_T)$ is defined formally as the average sensitivity. Equation (11) shows that the traffic flow with more trucks will have the lower average sensitivity, or its critical stable state has the lower vehicle density. This phenomenon was observed in the previous simulation [21]. Supposing $\gamma = b_n - h_c$ is the difference between the headway of the critical stable state and the safety distance, we can obtain another equation from Eq. (11) as follows:

$$\gamma = b_n - h_c = \ln[\sqrt{\bar{a}} \pm \sqrt{\bar{a}}]. \tag{12}$$

The real value γ requires the average sensitivity to satisfy the inequality $\bar{a} \geqslant 2$ for a critical stable state. When the headway is more than the safety distance, $\gamma > 0$ can be derived. On the contrary, we can get $\gamma < 0$. Moreover, the relation $a_T < \bar{a} = 2\,\mathrm{sech}^2(b - h_c) < a_C$, $\rho_{hC} > \rho_h > \rho_{hT}$ and $b_C < b < b_T$ can be attained.

III. NONLINEAR ANALYSIS

The slowly varying behavior of a traffic flow near the critical stable state is one important concern for us. In order to describe this behavior, the slow scales of the space variable x and time variable t will be introduced as follows [11]. For the small parameter $\varepsilon > 0$, we can define the following slow variables X and T as

$$X=\varepsilon(x-ct),\quad T=\varepsilon^3 t. \tag{13}$$

c is the travel speed of the critical disturbance as determined by Eq. (8). In the reference frame that moves with the speed c, both the traffic flow's density $\rho(x,t)$ and the vehicle speed $v(x,t)$ near the critical stable state have small fluctuation, which can be written as [11]

$$\rho(x,t)=\rho_h+\epsilon^2\hat{\rho}(x,t) \tag{14}$$

and

$$v(x,t)=v_h+\epsilon^2\hat{v}(x,t). \tag{15}$$

The terms $\varepsilon^2\hat{\rho}(X,T)$ and $\varepsilon^2\hat{v}(X,T)$ represent the fluctuations of the density and the vehicle speed, respectively. Substituting the expressions (14) and (15) into Eqs. (2) and (4), we can get the dynamical equation about the density perturbation as

$$\hat{\rho}_T+[2\overline{V'}+\rho_h\overline{V''}]\hat{\rho}\hat{\rho}_X+\frac{\overline{V'}}{6\rho_h}\hat{\rho}_{XXX}=\varepsilon\frac{\rho_h}{a}\Bigg[\frac{\overline{V'}a}{2\rho_h}\hat{\rho}_{XX}-\left(\rho_h\overline{V'}\,\overline{V''}+\frac{\overline{V'}^2}{2}+\frac{a\overline{V''}}{4\rho_h}\right)\hat{\rho}_{XX}^2-\frac{\overline{V'}^2}{3\rho_h}\hat{\rho}_{4x}\Bigg], \tag{16}$$

where $a=-4\rho_h\overline{V'}$, $\overline{V'}=-\rho_h^{-2}\mathrm{sech}^2\gamma$, and $\overline{V''}=-2\overline{V'}\rho_h^{-1}(1-\tanh\gamma)$. This equation describes the density fluctuation in the flow. If this flow consists of only cars, we will have $a=a_C$, but if the flow consists of only trucks we will have $a=a_T$.

Equation (16) is the nonlinear KdV equation with the perturbation term (ε term); its solution is the perturbed soliton. In the following text, we will give the nonperturbed soliton solution of the KdV equation first, and then determine the soliton evolution by the perturbation calculation.

Neglecting the ε-order term, the equation remained is just the KdV equation with its one-soliton solution written as follows [18]:

$$\hat{\rho}=A\,\mathrm{sech}^2[k_1(X+vT)]. \tag{17}$$

In this equation, the parameter k_1 is determined by the initial perturbation. The soliton's amplitude is $A=k_1^2/\tanh\gamma$ and its velocity is $v=\frac{2}{3}\rho_h^{-3}k_1^2\mathrm{sech}^2\gamma$.

This soliton solution reflects how the vehicle density deviates from that of the critical stable state. When the headway is larger than the safety distance, namely, $\gamma>0$, the critical stable state has lower vehicle density, the positive soliton amplitude $A>0$ can be obtained. Such a solution means that the vehicle density around soliton peak exceeds the critical density and this phenomenon can be regarded as the formation of the traffic jam [2]. Contrarily, when the headway is smaller than the safety distance, namely, $\gamma<0$, the negative amplitude $A<0$ can be attained, which means that drivers tend to increase the distance to avoid the traffic jam.

In order to discuss the effect of the ε-order correction, we write $T=C_1t$, $X=C_2y$, and $\hat{\rho}=C_1u$. When

$$C_1=-6\rho_h^3C_2^3/\mathrm{sech}^2\gamma \tag{18}$$

and

$$C_2=-\left(\frac{\mathrm{sech}^2\gamma}{12\rho_h^3\tanh\gamma}\right)^{1/5}, \tag{19}$$

Eq. (16) can be transformed into

$$u_t+6uu_y+u_{yyy}=\varepsilon\left[6B\rho_h^{-1}\tanh\gamma u_{yy}-\left(\rho_h^{-1}\tanh\gamma-\frac{1}{2}\right)\frac{18\rho_h^3B^4}{\mathrm{sech}^2\gamma}u_{2y}^2+\frac{\rho_h^{-1}}{B}u_{4y}\right]. \tag{20}$$

When the ε-order term is neglected, the above equation is just the KdV equation. With the new parameter η, the one-soliton solution of the equation is written as $u=2\eta^2\mathrm{sech}^2Z$, here $Z=\eta(y-4\eta^2t)$. The ε-order term determines the time evolution of the parameter η. By applying the direct approach of the soliton perturbation, we have [22–25]

$$\frac{d\eta}{dt}=\frac{\varepsilon}{4\eta}\int_{-\infty}^{\infty}dZ\,H\,\mathrm{sech}^2Z, \tag{21}$$

where

$$H=6B\rho_h^{-1}\tanh\gamma u_{yy}-\left(\rho_h^{-1}\tanh\gamma-\frac{1}{2}\right)\frac{18\rho_h^3B^4}{\mathrm{sech}^2\gamma}u_{2y}^2+\frac{\rho_h^{-1}}{B}u_{4y}=\frac{\hat{H}}{a}.$$

For the traffic flow of two types of vehicles with the sensitivities a_C and a_T, the integral in Eq. (21) should be written as

$$\frac{d\eta}{dt}=\frac{\epsilon}{4\eta}\left[\int_{-\infty}^{J1}\frac{H}{a_C}dZ\,\mathrm{sech}^2Z+\int_{J1}^{J2}\frac{H}{a_T}dZ\,\mathrm{sech}^2Z+\int_{J2}^{\infty}\frac{H}{a_C}dZ\,\mathrm{sech}^2Z\right]. \tag{22}$$

The linear relation between spectrum parameters η and k_1 is $\eta=Bk_1$ with a proportion coefficient B. According to Eq. (22), the evolution equation of k_1 can be obtained as

$$\frac{dk_1}{dt}=Dk_1^3\varepsilon. \tag{23}$$

The factor $D=\rho_h^{-1}(\mathrm{sech}^2\gamma/12\rho_h^3\tanh\gamma)^{3/5}[M_1a/a_C+(M_2+M_3)(a/a_T-a/a_C)]$ is very complex. The symbols M_1, M_2, and M_3 are expressed as

$$M_1 = 16\tanh\gamma/5 + 4k_1^2(2\rho_h/7\tanh\gamma - 4/15), \quad (24)$$

$$M_2 = 6\tanh\gamma(U - 4U^3/3 + 3U^5/5), \quad (25)$$

$$M_3 = 4k^2[(3\rho_h/4\tanh\gamma - 1)(2U - 17U^3/3 + 6U^5 - 15U^7/7) + U^3/2 - 4U^5/5 + 5U^7/14], \quad (26)$$

with the definitions $U = \tanh J_2 - \tanh J_1$, $J_1 = k_1 m\varepsilon/2 + (k_1 \mathrm{sech}^2\gamma/\rho_h)\varepsilon(1 + 4k_1^2\varepsilon^2/6)t$, and $J_2 = -k_1 m\varepsilon/2 + (k_1\mathrm{sech}^2\gamma/\rho_h)\varepsilon(1 + 4k_1^2\varepsilon^2/6)t$.

According to Eq. (11), both the parameters γ and ρ_h are the functions of the trucks' number m. So the factor D is also the function of the number m. In the reasonable value range of ε and k_1, it can be found that the derivative of the factor D with respect to m is always less than zero, which means that $dk_1/dt = Dk_1^3\varepsilon$ tends to decrease with the increase of the number m. Due to this property of the factor D, the soliton's amplitude in the car flow changes the fastest. Otherwise, if the traffic flow is purely a column of trucks, the soliton's amplitude will change most slowly. Generally, increasing the proportion of trucks in the total number of vehicles will slow down the variation of the vehicle density.

IV. SUMMARY

In our real life, it is quite rare to see the traffic flow consisting of only one kind of vehicle. When various vehicles coexist in a system, the traffic flow will become very complicated. In this paper, analytical results of the car-following model for two types of vehicles are obtained, part of which accords with the simulation finished by Mason. With our results, we have proved that the difference between different types of vehicles only affects the evolution of the determined soliton amplitude.

A traffic jam may be formed where the headway of the critical stable state is larger than the safety distance, because the soliton describing the density fluctuation has a positive amplitude. On the other hand, when the headway is smaller than the safety distance, drivers tend to increase the distance to avoid the traffic jam.

The perturbation on the soliton solution coming from the difference between two types of vehicles is calculated, which concludes that increasing the proportion of trucks in the total number of vehicles will slow down the variation of the vehicle density.

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