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Nonuniform non-neutral plasma in a trap

Sh. Amiranashvili
Department of Theoretical Physics, General Physics Institute, Moscow 119991, Russia

M. Y. Yu
Institut für Theoretische Physik I, Ruhr-Universität Bochum, D-44780 Bochum, Germany

L. Stenflo
Department of Plasma Physics, Umeå University, SE-90187 Umeå, Sweden
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An analytical model for breathing oscillations in a nonuniform non-neutral plasma slab is developed. The plasma is relatively small and warm with the smallest dimension only of several Debye lengths. Nonuniformity in the equilibrium results in a frequency shift associated with pressure and boundary effects. The plasma size and temperature, being related to the frequency shift, can therefore be evaluated from frequency measurements. In particular, for small nonuniform plasmas the frequency of the breathing mode is twice that predicted by the cold fluid theory. Nonlinear oscillations are also considered and the pressure is shown to have an important effect on the dynamics. Analytical solutions for linear and nonlinear oscillations are obtained and compared with that from one-dimensional particle-in-cell simulations. Good agreement is found.

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I. INTRODUCTION

A trapped cloud of identical charges can be considered as a non-neutral plasma [1,2] if its size is large compared to the interparticle spacing and Debye length. There has been much interest in linear electrostatic waves in trapped non-neutral plasmas [3]. These modes can be easily excited and measured, providing useful information on the plasma shape, size, and density. The plasma modes are usually studied in the background of an equilibrium state. The existence of such a state is an important feature of the trapped non-neutral plasma [4]. An important case, where the equilibrium state can be found analytically, is when the plasma is small compared to the dimensions of the trap and resides in a nearly quadratic trap potential (the harmonic trap). The cold-plasma-fluid approximation then leads to a constant-density equilibrium [5], which is invoked in most studies of electrostatic modes in non-neutral plasmas.

The pressure and temperature effects are completely ignored in the cold-plasma-fluid approximation. These effects can, however, be important if the equilibrium density is not constant. For example, at very low temperatures the fluid model is inadequate and the plasma equilibrium is markedly nonuniform. Corresponding corrections to the plasma frequencies were recently found [6]. On the other hand, even a constant-density plasma cloud behaves quite differently near the boundary. The size of the boundary region is usually comparable to the Debye length. If the latter is far less than the plasma size (as was the case for most previous studies of plasma oscillations), the boundary effects on the volume oscillations are negligible. However, some non-neutral plasmas, such as that of pure electron or positron, are often relatively small and warm, and the smallest dimension of the cloud is only several Debye lengths [7]. The equilibrium state is, therefore, significantly nonuniform. The goal of the present paper is to describe the oscillatory modes in such a plasma. As we will see, boundary effects result in a frequency shift proportional to the square of the ratio of the Debye length to the typical cloud dimension. If one keeps the plasma temperature on the same level and increases the number of trapped particles the shift disappears, indicating that it has a boundary origin. That is, the shift is negligible for cold or large plasmas, but it can significantly change spectrum for small and warm ones.

The simplest oscillatory mode of a trapped non-neutral plasma cloud corresponds to the center-of-mass motion. It is not affected by the plasma shape and density under the harmonic trap approximation. We shall, therefore, concentrate on the breathing (quadrupole) mode, which can be readily described in both the linear and nonlinear regimes. This mode is analogous to the well-known volume Langmuir oscillations in uniform infinite plasmas. For the sake of simplicity we consider a non-neutral plasma slab in a one-dimensional harmonic trap. Such a geometry appears naturally in experiments [8] and was investigated both analytically and numerically [9]. In contrast with the previous work we ignore the effect of nonharmonic terms in the trapping field and the fact that the aspect ratio of a thin oblate plasma, being small, is nonzero. Our slab is, therefore, perfectly one-dimensional and confined by a perfectly parabolic potential well. Making these simplifications we concentrate on temperature and boundary effects, both are treated analytically in a nonperturbative manner. The plasma equilibrium, and linear and nonlinear oscillations, are then investigated in a single framework. First, we neglect temperature effects and consider nonlinear oscillations of such a slab. Then we take pressure into account and consider the equilibrium state, as well as the linear and nonlinear breathing plasma oscillations, demonstrating the effects of nonuniformity. In connection to the nonlinear oscillations we also address an important issue concerning wave breaking and collapsing solutions. Such solutions are well-known artifacts of the cold-plasma approximation [10]. We will see how wave breaking is prevented by the pressure. Regular solutions exist formally for any amplitude, but it appears to be unstable if the amplitude is too large. The theoretical results are compared with that from particle-in-cell (PIC) simulations.

II. COLD-PLASMA SLAB

The equilibrium properties of a cold-plasma slab can be easily understood in the Lagrangian frame from a consideration of the forces acting on a single particle. We assume that a collection of particles with equal mass m and charge q is stored in a one-dimensional trap with a harmonic trapping force $m\omega_0^2 x$ on the particle at the point x, where ω_0 is referred to as the trap frequency. The trapping force may be thought of as resulting from an imaginary static background of opposite charges with constant density n_b such that $4\pi q^2 n_b/m=\omega_0^2$. The trapping force is opposed by Coulomb repulsion of the trapped particles. For a cold uniform plasma with the constant density n the condition of equilibrium simply indicates that $n=n_b$, so that the two forces equilibrate each other. The quantity n_b, being uniform, is also useful for normalizing the plasma density in more complicated situations.

When the particles are pushed, say at $t=0$, the equilibrium will be broken. Simple arguments first applied by Dawson [10] lead to a full description of the subsequent plasma oscillations. In the one-dimensional case the repulsion force acting on any particular particle remains fixed as long as the ordering of the particles is unchanged, whereas the trapping force increases with the particle coordinate. The motion of the particle follows the equation

$$\frac{d^2x(t)}{dt^2}=\omega_0^2 x(0)-\omega_0^2 x(t) \tag{1}$$

so that all particles undergo harmonic oscillations with the trap frequency ω_0. Breathing plasma motion occurs if the initial velocity of any particle is proportional to its position:

$$\left.\frac{dx(t)}{dt}\right|_{t=0}=a\omega_0 x(0), \tag{2}$$

where the dimensionless parameter $a=\text{const}$. The particle trajectories are then given by

$$x(t)=A(t)x(0) \quad \text{and} \quad A(t)=1+a\sin\omega_0 t, \tag{3}$$

where the dimensionless propagator $A(t)$ is identical for all particles. This propagator is in fact a Jacobian of the transformation from the Lagrangian variables to the Eulerian frame. A characteristic property of all quadrupole plasma modes is that the Jacobian depends on time, but not on space. The Jacobian must also remain positive for regular solutions. The well-known wave breaking phenomenon occurs then for $a>1$.

The plasma density $n(t)=n_b/A(t)$ remains uniform for the breathing mode. Recall that the trapped plasma always has a boundary and that the uniform density approximation requires the latter to be sharp. If initially the plasma is bounded in the region $|x|<R_0$, then at its boundary $R(t)=A(t)R_0$. The solution (3) is a particular one [11] as it is associated with a special velocity distribution (2). The distribution can be realized, for example, by a temporary shutdown of the trapping force of the stationary equilibrium state. The plasma expansion caused by Coulomb repulsion will then lead to the desired initial particle velocities. On the other hand, the simple solution (3) clearly exhibits two most important properties of cold-plasma oscillations, namely, the absence of any nonlinear frequency shift, and the appearance of wave breaking for perturbations with $a>1$. As we shall see, both these features are considerably modified by pressure and boundary effects.

III. NONUNIFORM EQUILIBRIUM

In this section we discuss the equilibrium properties of a plasma slab, assuming that the trapped particles have a uniform temperature T_0. We introduce the thermal velocity $v_T=(T_0/m)^{1/2}$ and the characteristic length $L=v_T/\omega_0$. The latter quantity being close to the Debye length is also uniform and useful for normalizing lengths. The equilibrium density profile is uniquely determined by the parameter

$$\Delta=\frac{\mathcal{N}}{2n_bL},$$

where $\mathcal{N}$ is the total number of trapped particles per slab area and $\mathcal{N}/(2n_b)$ is the cold-fluid value of the plasma radius R_0. The familiar case of sufficiently large or cold plasma corresponds to $\Delta\gg 1$. The case $\Delta\sim 1$ corresponds to sufficiently small and warm plasmas with the size of several Debye lengths. If $\Delta\ll 1$ the particles interactions are negligible compared with their thermal energy and the trap potential. In the latter limit we have a collection of independent particles rather than a real plasma with collective behavior.

To obtain the equilibrium space-charge electric field and particle distribution one can directly solve the Poisson equation, assuming that the particles obey the Boltzmann distribution in the space-charge and trapping electric fields. Such a solution has been intensively discussed in the literature for different geometries (see Ref. [12] for details) and here we only outline one-dimensional results in an easy-to-use form. The nonuniform equilibrium density $n=n_0(x)$ can be written as

$$n_0(x)=n_b\mathcal{F}(x/L), \tag{4}$$

where the dimensionless function $\mathcal{F}(\xi)$ is even and obeys the equation

$$\frac{d}{d\xi}\left[\frac{1}{\mathcal{F}(\xi)}\frac{d\mathcal{F}(\xi)}{d\xi}\right]=\mathcal{F}(\xi)-1, \tag{5}$$

with $d\mathcal{F}/d\xi|_{\xi=0}=0$ and $\mathcal{F}(0)$ being a free parameter determined by Δ. The function $\mathcal{F}(\xi)$ must be positive and tend to zero at infinity. Physically meaningful solutions correspond to $0<\mathcal{F}(0)<1$. Different solutions are possible for different

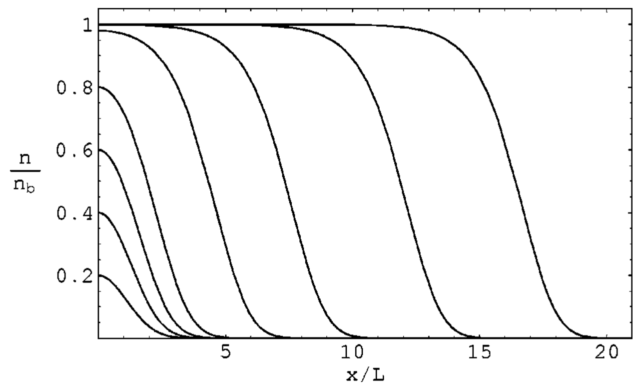

FIG. 1. Equilibrium plasma distribution $n=n_0(x)$ versus space coordinate for different initial values $n_0(0)/n_b$. As the total number of particles increases or the temperature decreases the Gaussian density profile gradually changes to a step profile.

$\mathcal{F}(0)$, as shown in Fig. 1. The particular solution should be chosen according to the integral condition

$$\int_0^\infty \mathcal{F}(\xi)d\xi=\Delta, \tag{6}$$

indicating that the trapped cloud contains $\mathcal{N}$ particles per slab area. The reader should guard against the apparent simple scaling form of Eq. (4), since $\mathcal{F}(\xi)$ depends implicitly on the plasma parameters because of Eq. (6).

Figure 1 shows how the Gaussian density profile changes to the step profile. Large or cold-plasmas correspond to $\mathcal{F}(0)\rightarrow 1$. Here the density is constant $n_0(x)=n_b$ as long as $|x|<\mathcal{N}/(2n_b)$ and it quickly tends to zero otherwise. The opposite case of independent particles with Gaussian distribution corresponds to $\mathcal{F}(0)<1$. In the following sections we trace in detail the evolution of the plasma oscillations as the plasma evolves to a small and warm state.

IV. BASIC EQUATIONS

We now generalize the problem of cold non-neutral plasma oscillations to include pressure and boundary effects. We assume the width of the plasma slab to be of the order of several Debye lengths and much larger than the interparticle spacing. The plasma equilibrium is significantly nonuniform. A nonuniform equilibrium density profile $n=n_0(x)$ invalidates the familiar periodic solutions of the form $\exp(ikx)$. Furthermore, the very definitions of the mean plasma density, plasma frequency, and actual cloud size requires separate consideration for small plasmas. The problem of linear and nonlinear breathing oscillations can nevertheless still be solved provided that the plasma is initially in an equilibrium and that the initial velocity of each particle is proportional to its position. In this case a direct generalization of the above solution (3) can be found.

Our approach, similar to that used by Chandrasekhar [13] for gravitating fluids, is to consider appropriate moments of the number density $n(t,x)$. This approach is known to be useful for a non-neutral plasma [6]. It also provides a clear definition of the mean slab width and density for arbitrary plasma distributions. Accordingly we *define* the density $N(t)$ and half-size $R(t)$ of the nonuniform plasma slab through the equations

$$\int_0^\infty n(t,x)dx=N(t)R(t)$$

and

$$\int_0^\infty n(t,x)x\,dx=\frac{1}{2}N(t)R(t)^2,$$

where $N_0=N(0)$ and $R_0=R(0)$ are calculated using the initial equilibrium profile. For cold or large plasmas $R_0=\mathcal{N}/(2n_b)$, whereas for the Gaussian limit $R_0=(8/\pi)^{1/2}L$. The corresponding plasma density is $N_0=\mathcal{N}/(2R_0)$. The quantity N_0 can now be used to *define* a mean plasma frequency

$$\overline{\omega_p}=\sqrt{4\pi q^2N_0/m},$$

which is equal to ω_0 for $\Delta\gg 1$ and tends to zero as $\omega_0\Delta^{1/2}$ for $\Delta\ll 1$.

Note that due to the conservation of particle number the product $N(t)R(t)$ is constant. We can then *define* a dimensionless parameter $A(t)$ such that

$$N(t)=\frac{N_0}{A(t)}\qquad\text{and}\qquad R(t)=R_0A(t), \tag{7}$$

where $A(0)=1$. One can get a simple equation for $A(t)$, starting from the standard one-dimensional plasma-fluid model

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0, \tag{8}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{q}{m}E-\omega_0^2x-\frac{1}{mn}\frac{\partial P}{\partial x}, \tag{9}$$

$$\frac{\partial E}{\partial x}=4\pi qn, \tag{10}$$

where the trapping force and pressure are included in Eq. (9). The equation of state for the pressure $P(t,x)$ will be specified later. We consider symmetric oscillations of the plasma slab, so that the velocity v and electric field E should vanish at $x=0$, and all the other plasma parameters should vanish sufficiently fast for $x\rightarrow\infty$. The equilibrium solution is $v=0$ and $n=n_0(x)$ with corresponding values for the pressure and electric field. The steady equilibrium state is then disturbed by introducing the initial velocity

$$v(0,x)=a\omega_0x, \tag{11}$$

where a is a constant with the same value and meaning as in Eq. (2). To proceed, we multiply Eq. (8) by $x\,dx$ and integrate from $x=0$ to $x=\infty$ to obtain

$$\frac{d}{dt}\int_0^\infty nx\,dx=\int_0^\infty nv\,dx, \tag{12}$$

which when evaluated at $t=0$ leads to the initial condition $dA/dt|_{t=0}=a\omega_0$. Thus we have generalized Eq. (2). We now multiply Eq. (9) by ndx and integrate. After some algebra and using Eqs. (10) and (12), we obtain a simple moment equation

$$\left(\frac{d^2}{dt^2}+\omega_0^2\right)\int_0^\infty nx\,dx=\frac{2\pi q^2}{m}\left(\int_0^\infty n\,dx\right)^2+\frac{P(t,0)}{m},$$

where the last term is the pressure at $x=0$.

The derivation up to this point is *exact* within the framework of the plasma-fluid equations. One can easily obtain a closed set of equations for a cold pressureless plasma and reproduce the solutions given by Eqs. (1)–(3). The problem is more complicated for a warm plasma. To include thermal effects we adopt a simple approach originally developed by Dubin [14]. The equilibrium pressure

$$P(0,x)=n_0(x)T_0 \tag{13}$$

is artificially replaced with

$$P(0,x)=\alpha N_0T_0\left(1-\frac{x^2}{R_0^2}\right) \tag{14}$$

for $|x|<R_0$ and $P(0,x)=0$ otherwise. The constant α can be chosen in different ways. Following Dubin we integrate Eqs. (13) and (14) over the plasma extent and require the results to be identical. That is, $\alpha=\frac{3}{2}$ for our infinite one-dimensional plasma slab. Assuming an adiabatic equation of state we finally obtain

$$P(t,x)=\alpha N_0T_0\left(\frac{N}{N_0}\right)^3\left(1-\frac{x^2}{R^2}\right), \tag{15}$$

where the parameter α is kept to make our formulas more flexible.

Of course, Eq. (14) is a crude approximation for large plasmas with constant density since the actual pressure gradient is not linear in x but concentrated at the edge of the plasma where the density falls to zero. On the other hand, Eq. (14) is quite reasonable for the small plasmas of interest here. We can check it by calculating the actual pressure at the plasma center directly from the numerical solutions of Eq. (5) and comparing it with that predicted by Eq. (14). Indeed these two values are close to each other for small plasmas, as shown in Fig. 2. Finally we evaluate the pressure at $x=0$, pass to the variable $A(t)$ in the moment equation, and obtain

$$\frac{d^2A}{dt^2}+\omega_0^2A=\overline{\omega}_p^2+\frac{2\alpha v_T^2}{R_0^2A^3}, \tag{16}$$

where we recall that $A(0)=1$, $dA/dt|_{t=0}=a\omega_0$, and $\alpha=\frac{3}{2}$, unless specified otherwise.

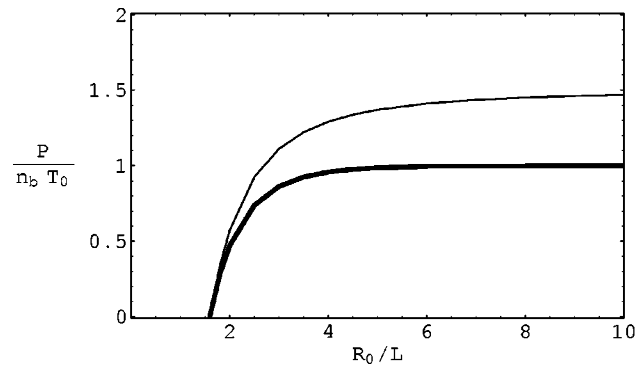

FIG. 2. The actual value of the equilibrium central pressure (thick line) from Eqs. (4) and (13) and that (thin line) predicted by Eq. (14) versus the size of the plasma cloud.

Equation (16) is seen to be a selfconsistent generalization of Eq. (1) for plasma oscillations. In what follows we solve it and compare the results with simulations.

V. SMALL OSCILLATIONS

In this section, we investigate thermal and finite-size effects in the theory of linear breathing oscillations. We also compare our results with that of other authors. Equation (16) should admit an equilibrium solution $A=1$, so that

$$\omega_0^2=\overline{\omega}_p^2+2\alpha\frac{v_T^2}{R_0^2} \tag{17}$$

is an additional constraint for equilibrium. For cold or large plasmas the last term in Eq. (17) is small and $\overline{\omega}_p=\omega_0$. In the opposite warm limit the plasma frequency can be neglected and we obtain the minimum possible $R_0=(2\alpha)^{1/2}L$. We see that the slab cannot be arbitrary thin because of the temperature effect. Note that for the Gaussian distribution $R_0=(8/\pi)^{1/2}L$ which is slightly different from our prediction for $\alpha=\frac{3}{2}$. The approximate character of Eq. (15) is responsible for this difference. One can then improve the choice of α (i.e., replace $\frac{3}{2}$ with $4/\pi$, the latter is 15% less) or even make it state dependent. Such an improvement, being tedious, has however no significant influence on the results obtained. We will, therefore, stay in the framework of the present model.

Note that generally both $\overline{\omega}_p$ and R_0 depend on $\mathcal{N}$, T_0, and the trap parameters. Such a dependence is determined by Eqs. (5) and (6) and cannot be extracted directly from Eq. (17). Nevertheless we can use Eq. (17) to check the validity of our approach. To this end we solved numerically Eqs. (5) and (6) and used the equilibrium profile $n_0(x)$ for calculating the density moments and for evaluating R_0, N_0, and $\overline{\omega}_p$. The results of each calculation can be presented as a point in the $(N_0/n_b, R_0/L)$ plane, as shown in Fig. 3. The theoretical expression (17) is also plotted in this plane. We see that the agreement between the equilibrium theory and our moment equation (17) is quite reasonable. Note that although the approximation (14) is not valid for large plasmas, the whole boundary effect also disappears in this limit. That is why a

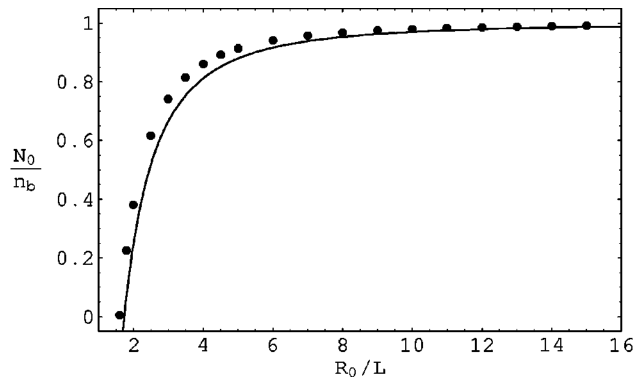

FIG. 3. Equilibrium plasma density versus slab dimension. The result from Eqs. (5) and (6) (points) is in good agreement with that of Eq. (17) (solid line).

good agreement is achieved in Fig. 3 despite of our crude approximation for the pressure.

Small oscillations around the equilibrium have a frequency

$$\omega^2 = \omega_0^2 + 6\alpha \frac{v_T^2}{R_0^2}, \tag{18}$$

where the last term is seen to have the usual Bohm-Gross form $3k^2 v_T^2$ with $k \sim 1/R_0$. For $v_T = 0$ we reproduce the cold-plasma result $\omega = \omega_0$. In the warm case we insert the minimum $R_0[=(2\alpha)^{1/2}L]$ to see that $\omega = 2\omega_0$. So the frequency increases with the temperature and is finally twice its cold value.

We are now in a good position to compare our predictions with that of other theories. The latter were developed for an ellipsoidal plasma cloud, the slab limit can be obtained when one of the semiaxes is much less than the others. Note, that the 3D version of Eq. (14) results in $\alpha = \frac{5}{2}$ when integrating over the ellipsoid. With this correction Eqs. (17) and (18) agree with the existing results (see, for example, Eqs. (26) and (28) from Ref. [15]).

The spectrum predicted by Eq. (18) is compared with that from the plasma simulations in Sec. VII. Before doing this let us first consider finite-amplitude oscillations.

VI. NONLINEAR OSCILLATIONS

In this section we investigate Eq. (16) to include thermal and finite-size effects in the theory of nonlinear breathing oscillations. We first consider two special cases of cold large sharply bounded plasmas and small warm Gaussian plasmas.

A. Sharply bounded plasmas

In this case we have $R_0 \gg L$ and $\overline{\omega}_p \to \omega_0$. Using Eq. (17) one can rewrite Eq. (16) in the form

$$\frac{d^2A}{dt^2} + \omega_0^2(A-1) = -2\alpha \frac{v_T^2}{R_0^2}\left(1 - \frac{1}{A^3}\right),$$

where the right-hand side is a small perturbation. The cold-plasma result

$$A(t) = 1 + a \sin \omega_0 t \tag{19}$$

is reproduced when the temperature term is ignored. Note that the solution (19) is nonlinear because the amplitude a need not be small. It is also clear that the expression (7) for the average density is nonlinear. The frequency $\omega = \omega_0$ of such a nonlinear oscillation happens to be the same for all amplitudes in the pressureless limit. The amplitude dependence of the frequency appears when the temperature term is not ignored, but included as a perturbation. Thus, by standard perturbation techniques [16], we obtain a modified frequency

$$\omega^2 = \omega_0^2 + 6\alpha \frac{v_T^2}{R_0^2}(1-a^2)^{-5/2}, \tag{20}$$

where for given $a < 1$ the temperature should be small enough for the pressure term to be a small correction. If the amplitude is small as well we obtain the frequency

$$\omega^2 = \omega_0^2 + 6\alpha \frac{v_T^2}{R_0^2} + 15\alpha \frac{v_T^2}{R_0^2} a^2,$$

where the familiar squared-amplitude nonlinear frequency shift appears in the right. For low temperature plasmas this shift is a second order correction. On the other hand, for $a \to 1$ the pressure cannot be considered as a perturbation, no matter how small the temperature is. In this case Eq. (19) is invalid and should be replaced by the full integral of Eq. (16). No blow-up solution then exists for $a \geqslant 1$. In particular, for $a \gg 1$ the pressure term behaves like an infinite repulsive potential wall located at $x = 0$. Thus, in the first approximation one has $A(t) = |a \sin \omega_0 t|$. We see that the linear frequency, which was close to ω_0, is replaced by $2\omega_0$, exhibiting a nonlinear frequency shift.

B. Gaussian case

Equation (16) can also be easily considered in the limit of small plasmas, i.e., when $R_0 \to (2\alpha)^{1/2}L$ and $\overline{\omega}_p \to 0$. Using Eq. (17) we rewrite Eq. (16) in the form

$$\frac{d^2A}{dt^2} + \omega_0^2 A - \frac{\omega_0^2}{A^3} = \overline{\omega}_p^2\left(1 - \frac{1}{A^3}\right),$$

where the right-hand side is a small perturbation. Ignoring the latter one obtains a nonlinear solution

$$A(t) = \sqrt{1 + \frac{a^2}{2} + a \sin 2\omega_0 t - \frac{a^2}{2}\cos 2\omega_0 t}, \tag{21}$$

which shows that the characteristic period $T = \pi/\omega_0$ regardless of the amplitude. Note, that the right side of Eq. (21) is always positive and no blow-up solution exists. A more general amplitude-dependent frequency can be obtained by tak-

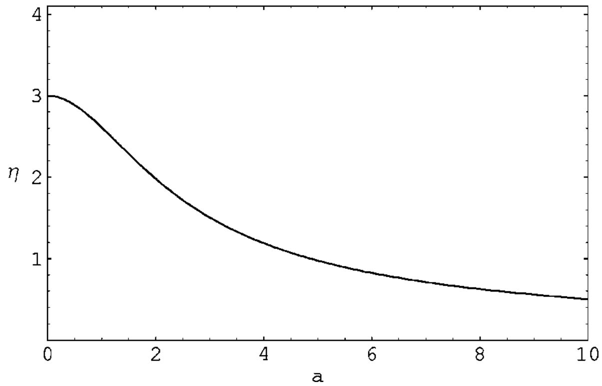

FIG. 4. The factor η from Eq. (22) versus amplitude a.

ing into account the $\overline{\omega}_p$ terms. A canonical version of perturbation theory is well suited for this calculation [17]. The modified frequency is

$$\omega^2=4\omega_0^2-\overline{\omega_p^2}\eta(a), \tag{22}$$

where

$$\eta(a)=\frac{32}{\pi}\left(\frac{df}{ds}\right)_{s=a^2},$$

$$f(s)=E(k)\sqrt{1+\frac{1}{2}s+\sqrt{s+\frac{1}{4}s^2}},$$

$$k^2=\frac{2\sqrt{s+\frac{1}{4}s^2}}{1+\frac{1}{2}s+\sqrt{s+\frac{1}{4}s^2}}$$

and $E(k)$ is the full elliptic integral of the second kind. The small amplitude limit of Eq. (22) is

$$\omega^2=4\omega_0^2-3\overline{\omega_p^2}+\frac{15}{32}\overline{\omega_p^2}a^2$$

and again the amplitude term is a second order correction. In contrast with Eq. (20), no modification of Eq. (22) is needed with an increase of the amplitude because the factor $\eta(a)$ is a decreasing function of the amplitude, as shown in Fig. 4.

C. General solution

It is also possible to find a general solution of Eq. (16) in terms of elliptical functions. To obtain such a solution we introduce the dimensionless time $\tau=\omega_0 t$ and the Hamiltonian function

$$H=\frac{1}{2}\left(\frac{dA}{d\tau}\right)^2+U(A),$$

with

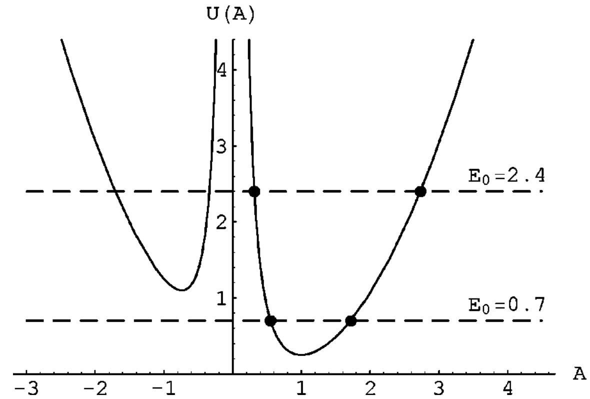

FIG. 5. The effective potential $U(A)$ for $\epsilon=0.5$. Black circles indicate minimum and maximum values of $A(t)$ for different choices of E_0.

$$U(A)=\frac{1}{2}A^2-(1-\epsilon)A+\frac{\epsilon}{2A^2}$$

and $\epsilon=2\alpha(L/R_0)^2\in[0,1]$. The two above considered limiting cases correspond to $\epsilon=0$ and $\epsilon=1$. To solve the general problem we note that the Hamiltonian should be equal to $E_0=\frac{1}{2}a^2+U(1)$ due to the initial conditions. A solution is then given by

$$\tau=\int_1^A\frac{dA}{\sqrt{2[E_0-U(A)]}}, \tag{23}$$

which is an elliptic integral. To calculate it, one has to investigate the roots of the corresponding polynomial. The oscillations are bounded in the region $A_1\leqslant A\leqslant A_2$, where $A_{1,2}(a,\epsilon)$ are two properly chosen real roots of the equation $U(A)=E_0$. A typical plot of the effective potential energy $U(A)$ and a graphical illustration of what is meant by proper roots for different values of E_0 is shown in Fig. 5. For given plasma parameters one can find these two roots and put Eq. (23) in the form

$$\tau=\int_1^A\frac{AdA}{\sqrt{(A_2-A)(A-A_1)Q(A)}},$$

where $Q(A)$ is a quadratic polynomial responsible for the two other roots of the equation $U(A)=E_0$. Figure 5 shows that these roots are initially complex, but become real and negative with increase of E_0. In any case their sum, say A_Σ, is a negative real number. It is also important to note that both $Q(A_1)$ and $Q(A_2)$ are positive. One can then pass to the new variable θ, where

$$\frac{p_1}{p_2}\tan^2\frac{\theta}{2}=\frac{A-A_1}{A_2-A},$$

and $p_{1,2}=[Q(A_{1,2})]^{1/2}$. The expression for the oscillation period T is then given by

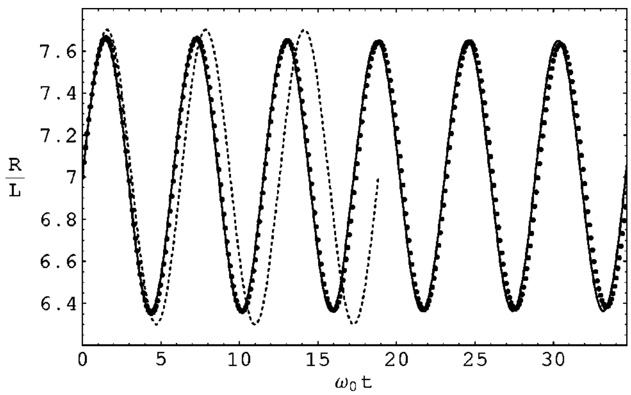

FIG. 6. Typical behavior of the plasma radius. The simulation (points) agrees with the theory (solid line). The pressureless solution (dashed line) is also shown for comparison.

$$2\omega_0 T=\frac{(A_1+A_2)K(k)}{\sqrt{p_1p_2}}+\frac{(p_1+p_2)^2[\Pi(\nu,k)-K(k)]}{\sqrt{p_1p_2}(A_1+A_2-A_\Sigma)},$$

where $K(k)$ and $\Pi(\nu,k)$ are full elliptic integrals of the first and the third kind, respectively. The corresponding parameters are

$$k^2=\frac{(A_2-A_1)^2-(p_2-p_1)^2}{4p_1p_2},\qquad \nu=\frac{(p_2-p_1)^2}{4p_1p_2},$$

and our definition for the elliptic integrals follows that of Ref. [18].

VII. SIMULATION

We have also performed one-dimensional PIC simulations for the problems considered above. For the computation, the time is normalized by ω_0, and the length by L. Being close to the plasma frequency and Debye length, these two normalization parameters are also well defined for nonuniform distributions. Thus, the dimensionless plasma density is n/n_b and the normalized electric field is $qE/m\omega_0^2L$ (i.e., the trapping force at $x=L$ is used to normalize forces). For different runs, we use from 10^2 to 10^6 particles per Debye length L.

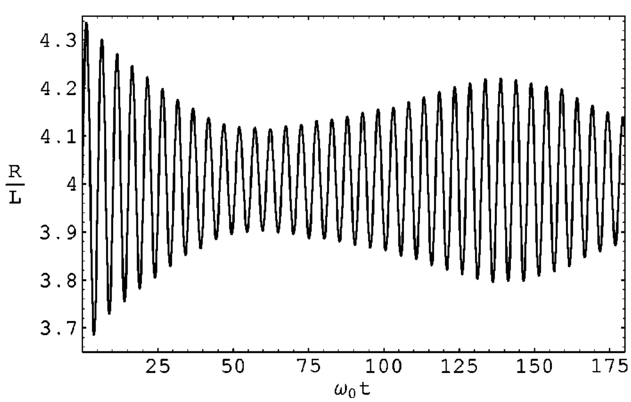

FIG. 7. The breathing oscillation for small plasmas is modulated. The present theory accurately predicts the frequency, but not the modulation.

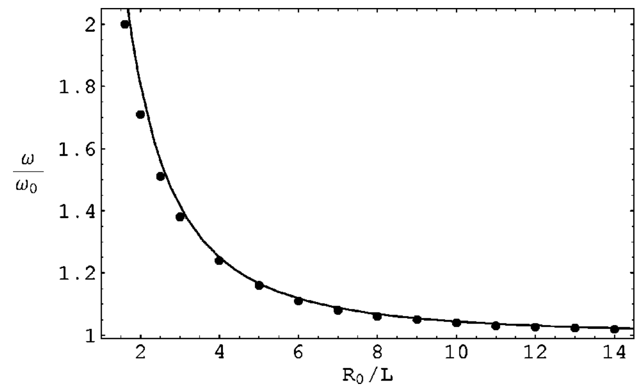

FIG. 8. Frequency of plasma oscillations versus slab dimension. The simulation result (points) is in good agreement with that of the theory (solid line).

As the theory describes only symmetric oscillations, only the region $x>0$ needs to be considered in the computation. The boundary at $x=0$ is assumed to be absorbing and reemitting. The total area covered by the computation is considerably larger than the plasma size, so that the particles never reach the right boundary (e.g., they never leave the trap).

To study breathing plasma oscillations we first prepared the above-described equilibrium distribution $n_0(x)$ with the desired size R_0 and Maxwellian velocities for the particles. The equilibrium retains on the background of small thermal fluctuations if the plasma is allowed to evolve freely. Breathing oscillations are initialized by forcing the particles to perform additional (nonthermal) motion with initial velocities proportional to their positions in accordance with Eq. (11). Properly chosen initial perturbations can generate oscillations that remain small (e.g., almost linear), but still considerably larger than the thermal fluctuations. The plasma is then allowed to move freely in both the self-consistent and trapping electric fields, and the moments of the density are evaluated to obtain $A(t)$.

An example of the oscillations is shown in Fig. 6, which shows the oscillation of the plasma half-size $R(t)$ for $R_0=7L$ and $a=0.1$. The behavior is in good agreement with the theoretical curve obtained from Eq. (16), whereas the constant frequency of the cold-plasma solution (3) is inaccurate.

After approximately one hundred periods a Fourier transform of the $A(t)$ is used to obtain the frequency. Then the whole procedure is repeated for the new R_0. As the plasma size decreases the oscillations become modulated as shown in Fig. 7 for $R_0=4L$ and $a=0.1$. Nevertheless, the corresponding Fourier transform still has a sharp peak and yields the correct frequency.

The R_0 dependence of the frequency is presented in Fig. 8 starting from the Gaussian $R_0[=(2\alpha)^{1/2}L]$. The frequency of the linear oscillations agrees well with the theory as given by Eq. (18).

To investigate the nonlinearity we also studied the behavior of the oscillations for the same R_0 but different a. As the amplitude of the excitation exceeds some critical value, bounce oscillations very similar to that shown in Fig. 7 begin

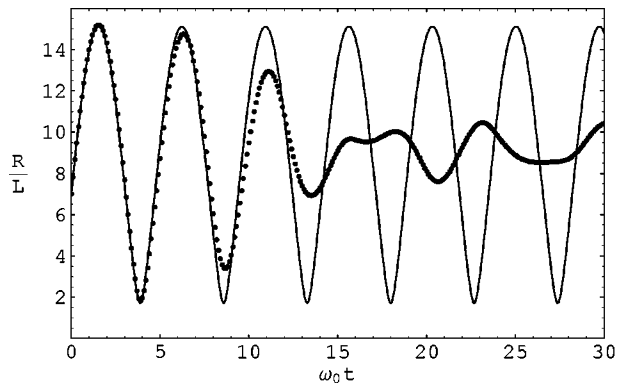

FIG. 9. The half size of the plasma cloud versus time for a solution with $a>1$ from theory (solid line) and simulation (points). Such a solution would inevitably blow up in the cold-plasma limit. Pressure accounts for the regular plasma behavior observed here, whereas the apparent instability is not described by the present theory.

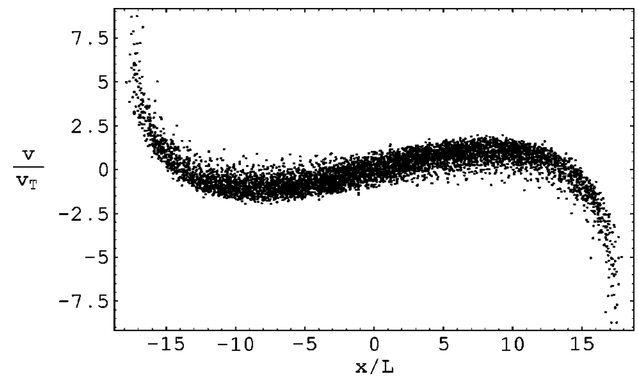

FIG. 10. Phase space of 3000 representative particles and their mirror images at $t=6.23\omega_0^{-1}$ for the solution displayed in Fig. 9. The spiral arms are developing and indicating instability of the solution.

to appear. Generally, the smaller the plasma the smaller the critical amplitude; for $R_0 \leqslant 5L$ the oscillations seem to be always modulated.

Figure 9 shows $R(t)$ for a strongly nonlinear regime with $a=1.2$ and $R_0=7L$. Note that no blow-up solution appears, in contrast to the cold-plasma case. The theory correctly predicts the maximum and minimum plasma densities as well as the nonlinear frequency. However, strong modulation is clearly observed, indicating that the solution is unstable. The instability destroys the oscillations within a few periods after its onset.

In general, several instabilities, such as trapping of the plasma fluid by the wave [19] or development of streaming instabilities [20], are possible in our system. To follow the evolution it is useful to consider the phase plane (x, mv_x). Each particle is then represented by a point, and the whole system forms a barlike cloud. Here, R_0 and mv_T are the half dimensions of the bar and the x axis is an "equilibrium position." The breathing mode corresponds to rocking oscillation $(a<1)$ or rotation $(a>1)$ of the barlike cloud around the origin. As the amplitude increases, such an oscillation is accompanied by development of spiral arms, as shown in Fig. 10. The phenomenon is similar to the formation of the galactic spiral arms, but it takes place in the phase plane rather then in the real space. A detailed description of the instability requires a kinetic approach and is beyond the scope of the present work.

VIII. CONCLUSION

In this paper we considered analytically the boundary effect on the linear and nonlinear breathing oscillations of a trapped non-neutral plasma slab. It is shown that the frequency of the oscillations differs from that predicted by the cold fluid approximation. The frequency depends on the plasma temperature and can be used to obtain the latter in an experiment. The effects considered here have a boundary related origin, i.e., they disappear with an increase of the plasma size. Our approach leads to simple analytical descriptions of both the linear and nonlinear oscillations despite the nonuniform plasma distribution. As expected, the familiar cold-plasma blow-up solutions are prevented by the pressure effects. Even for strongly nonlinear oscillation the theory accurately predicts the amplitude and frequency, but not the stability of the solution. The analytical results agree reasonably well with that from PIC simulations.

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