Resonant interaction of modulational instability with a periodic soliton in the Davey-Stewartson equation

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(Received 31 July 2002; published 19 December 2002)

The time evolution of the Benjamin-Feir unstable mode in two dimensions is studied analytically when it resonates with a periodic soliton. The condition for resonance is obtained from an exact solution to the hyperbolic Davey-Stewartson equation. It is shown that a growing-and-decaying mode exists only in the backward (or forward) region of propagation of the periodic soliton if the resonant condition is exactly satisfied. Under a quasiresonant condition, the mode grows at first on one side from the periodic soliton, but decays with time. The wave field shifts to an intermediate state, where only a periodic soliton in a resonant state appears. This intermediate state persists over a comparatively long time interval. Subsequently, the mode begins to grow on the other side from the periodic soliton.

DOI: 10.1103/PhysRevE.66.067601 PACS number(s): 05.45.Yv

I. INTRODUCTION

A uniform train of weakly nonlinear deep water waves is unstable to infinitesimal modulational perturbations, which is known as the Benjamin-Feir instability [1]. The time evolution of the unstable wave is governed by the nonlinear Schrödinger (NLS) equation [2,3,4]. Solving the NLS equation numerically with periodic boundary condition, Lake *et al.* [5] found that the unstable wave train recovers its initial state after modulation: the so-called Fermi-Pasta-Ulam recurrence. This problem was solved analytically by Tajiri and Watanabe [6].

Weakly nonlinear waves having two-dimensional property were studied by several authors [2,7,8]. The motion of a wave packet in a large scale of time was found to obey the Davey-Stewartson (DS) equation [8]:

$$iu_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0,$$

$$v_{xx} - pv_{yy} - r(|u|^2)_{xx} = 0, \tag{1}$$

where $p=\pm 1$ and r is constant. Equation (1) with $p=1$ or $p=-1$ are called the DS I and DS II equations, respectively.

Tajiri and Arai [9] obtained the analytical solution to the DS I equation for modulationally unstable mode as follows:

$$u = u_0 e^{i\zeta}(g/f), \quad v = -2(\ln f)_{xx}, \tag{2}$$

with

$$f = 1 - e^{-\Omega t + \sigma}\cos\eta + (M/4)e^{-2\Omega t + 2\sigma},$$

$$g = 1 - e^{-\Omega t + \sigma + i\phi}\cos\eta + (M/4)e^{-2\Omega t + 2\sigma + 2i\phi},$$

where

$$\zeta = kx + ly - \omega t, \quad \omega = k^2 + l^2 - ru_0^2,$$

$$\eta = \beta x + \delta y - \gamma t + \eta_0, \quad \Omega = (\beta^2 + \delta^2)\cot(\phi/2),$$

$$\gamma = 2k\beta + 2l\delta, \quad M = 2/(1+\cos\phi),$$

$$\sin^2(\phi/2) = (\delta^2 - \beta^2)/(2ru_0^2),$$

and σ and θ are arbitrary phase constants. We refer to this as a growing-and-decaying mode solution. The solution is nonsingular when M is greater than unity for real ϕ. This condition coincides with the one for the Benjamin-Feir instability:

$$0 < (\delta^2 - \beta^2) < 2ru_0^2. \tag{3}$$

The solution (2) shows that an unstable mode grows exponentially in its early stage. After reaching maximum modulation, it vanishes with time to reproduce the initial unmodulated state.

The resonance of a line soliton and a growing-and-decaying mode was studied by Pelinovsky [10]. Although the growing-and-decaying mode virtually exists only in a finite period of time, an infinite phase shift happens to the line soliton. Its mechanism has been clarified recently by showing that the mode develops only on a half plane divided by the line soliton if the condition for resonance is exactly satisfied [11]. The change of wave field with time is also investigated for this case under a quasiresonant condition. An unstable mode begins to grow only on one side from the soliton. Their interaction results in a line soliton in a transient state. Subsequently, the mode develops on the other side of the soliton, which decays finally.

In this paper, we investigate the interaction of the growing-and-decaying mode with a periodic soliton. The condition for resonance is obtained analytically and the interactive development of the waves is discussed.

II. QUASIRESONANCE BETWEEN PERIODIC SOLITON AND GROWING-AND-DECAYING MODE

The interaction of a periodic soliton with a growing-and-decaying mode is discussed on the basis of the DS I equation. Using the N-soliton solution of Satsuma and Ablowitz [12], we can get a solution for the present problem as follows:

$$u = u_0 e^{i\zeta}(g/f), \quad v = -2(\ln f)_{xx}, \tag{4}$$

with

$$f=1-\frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1-e^{\xi_2}\cos\eta_2+\frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}+\frac{M_2}{4}e^{2\xi_2}-\frac{1}{4}e^{\xi_1+\xi_2}\left\{\frac{M_1}{L_1L_2}e^{\xi_1}\cos(\eta_2+\Psi_1-\Psi_2)\right.$$
$$\left.+M_2e^{\xi_2}\cos(\eta_1+\Psi_1+\Psi_2)\right\}+\frac{1}{2L_1L_2}e^{\xi_1+\xi_2}\{L_1\cos(\eta_1+\eta_2+\Psi_1)+L_2\cos(\eta_1-\eta_2+\Psi_2)\}+\frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)}, \tag{5}$$

$$g=1-\frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i})-e^{\xi_2+i\phi_2}\cos\eta_2+\frac{M_1}{4L_1^2L_2^2}e^{2\xi_1+2i\phi_{1r}}+\frac{M_2}{4}e^{2\xi_2+2i\phi_2}-\frac{1}{4}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}$$
$$\times\left\{\frac{M_1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_2+\Psi_1-\Psi_2)+M_2e^{\xi_2+i\phi_2}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)\right\}+\frac{1}{2L_1L_2}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}$$
$$\times\{L_1\cos(\eta_1+\eta_2+i\phi_{1i}+\Psi_1)+L_2\cos(\eta_1-\eta_2+i\phi_{1i}+\Psi_2)\}+\frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}, \tag{6}$$

where

$$\xi_1=\alpha x+\kappa y-\Omega_1 t+\sigma_1,$$
$$\xi_2=-\Omega_2 t+\sigma_2,$$
$$\eta_1=\beta_1 x+\delta_1 y-\gamma_1 t+\eta_{10},$$
$$\eta_2=\beta_2 x+\delta_2 y-\gamma_2 t+\eta_{20},$$
$$\sin^2\frac{\phi_1}{2}=\frac{(\alpha+i\beta_1)^2-(\kappa+i\delta_1)^2}{2ru_0^2},$$
$$\sin^2\frac{\phi_2}{2}=\frac{\delta_2^2-\beta_2^2}{2ru_0^2},$$
$$\Omega_1=2k\alpha+2l\kappa-\mathrm{Re}\left\{\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\}\cot\frac{\phi_1}{2}\right\},$$
$$\gamma_1=2k\beta_1+2l\delta_1-\mathrm{Im}\left\{\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\}\cot\frac{\phi_1}{2}\right\},$$
$$\Omega_2=(\beta_2^2+\delta_2^2)\cot\frac{\phi_2}{2},$$
$$\gamma_2=2k\beta_2+2l\delta_2,$$
$$M_1=\frac{2ru_0^2\left|\sin\frac{\phi_1}{2}\right|^2\cosh\phi_{1i}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)}{2ru_0^2\left|\sin\frac{\phi_1}{2}\right|^2\cos\phi_{1r}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)},$$
$$M_2=\frac{2}{1+\cos\phi_2},$$

$$L_1e^{i\Psi_1}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}},$$
$$L_2e^{i\Psi_2}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}.$$

Here, θ_1, θ_2, σ_1, σ_2 are arbitrary real constants and ϕ_2 is assumed real. When we consider the case $0<\Omega_2$, $0<\alpha$, $0<\kappa$ and $0<\Omega_1$, the solution a long time before the growth of the growing-and-decaying mode is approximately given by

$$f=\frac{M_2}{4}e^{2\xi_2}\left\{1-e^{\xi_1}\cos(\eta_1+\Psi_1+\Psi_2)+\frac{M_1}{4}e^{2\xi_1}\right\}, \tag{7}$$

$$g=\frac{M_2}{4}e^{2(\xi_2+i\phi_2)}\left\{1-e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)\right.$$
$$\left.+\frac{M_1}{4}e^{2(\xi_1+i\phi_{1r})}\right\}. \tag{8}$$

On the other hand, the solution a long time after the interaction is given by

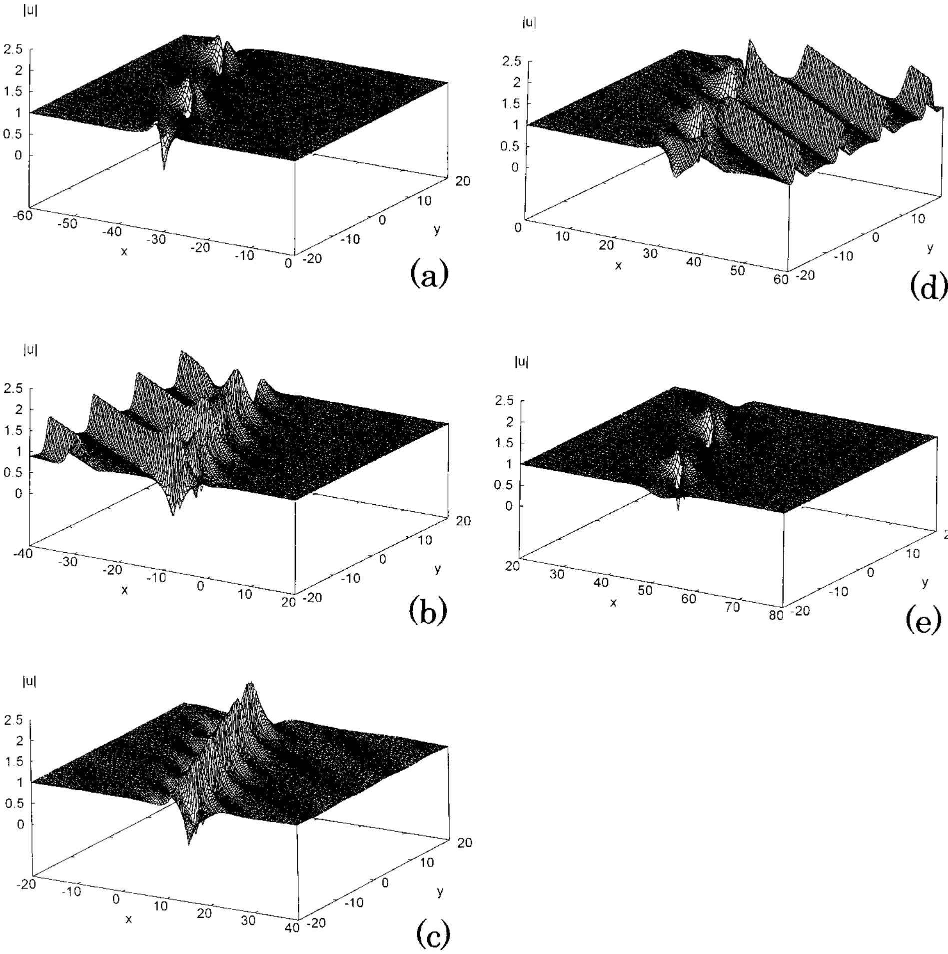

FIG. 1. The sequence of snapshots of the quasiresonant interaction between periodic soliton and growing-and-decaying mode. The parameters are $(k,l)=(1.0,1.0)$, $(\alpha,\kappa)=(0.42,0.20)$, $(\beta_1,\delta_1)=(0.22,0.45)$, $(\beta_2,\delta_2)=(0.27,0.55)$, and $(\phi_1,\phi_2)=(1/4\pi,2/9\pi)$. The time evolution is (a) $t=-7$, (b) $t=0.0$, (c) $t=3.8$, (d) $t=7.6$, and (e) $t=13.0$. In this figure, x, y, and u are all dimensionless.

$$f=1-\frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1+\frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}, \tag{9}$$

$$g=1-\frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i})+\frac{M_1}{4L_1^2L_2^2}e^{2(\xi_1+\phi_{1r})}. \tag{10}$$

Both formulas represent the structures of periodic soliton.

Comparing these solutions, we see that the phase of the periodic soliton shifts by the amount $\ln(L_1L_2)$ [or $-\ln(L_1L_2)$] due to the interaction with the growing-and-decaying mode. Therefore, $(L_1L_2)=\infty$ or 0 may be regarded as the condition for resonance between periodic soliton and growing-and-decaying mode. In this paper, we limit our discussion to the case in which L_1 is infinitely large, namely

$$D=2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}=0. \tag{11}$$

If we express α, κ, β_1, δ_1, β_2, and δ_2 in terms of ϕ_1, ϕ_2, θ_1, and θ_2 as follows:

$$\alpha+i\beta_1=i\sqrt{2ru_0^2}\sin(\phi_1/2)\sinh\theta_1,$$

$$\kappa+i\delta_1=i\sqrt{2ru_0^2}\sin(\phi_1/2)\cosh\theta_1,$$

$$\beta_2=\sqrt{2ru_0^2}\sin(\phi_2/2)\sinh\theta_2,$$

$$\delta_2=\sqrt{2ru_0^2}\sin(\phi_2/2)\cosh\theta_2,$$

Eq. (11) is rewritten as

$$D=2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\left\{\cos\frac{\phi_1+\phi_2}{2}-\cosh(\theta_1-\theta_2)\right\}. \tag{12}$$

Thus the condition for resonance is expressed in the form

$$\phi_2=\pm 2\theta_{1i}-\phi_{1r},\quad \theta_2=\theta_{1r}\pm(\phi_{1i}/2). \tag{13}$$

Evaluating Eqs. (5) and (6) approximately, we can characterize the process of interaction separately in the five specific periods in time.

(p_1) $t\to-\infty$ (before the mode grows). The solution is given by Eqs. (7) and (8). Only a periodic soliton exists in the wave field as shown in Fig. 1(a).

(p_2) $t\sim\sigma_2/\Omega_2;[e^{-\Omega_2t+\sigma_2}\sim O(1)]$. In this case, the functions f and g take approximate forms

$$f\simeq1-e^{\xi_2}\cos\eta_2+(M_2/4)e^{2\xi_2}, \tag{14}$$

$$g \simeq 1 - e^{\xi_2 + i\phi_2} \cos \eta_2 + (M_2/4) e^{2(\xi_2 + i\phi_2)}, \tag{15}$$

behind the periodic soliton and

$$f \simeq (M_1 M_2/16) e^{2(\xi_1+\xi_2)}, \tag{16}$$

$$g \simeq (M_1 M_2/16) e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}, \tag{17}$$

ahead of the soliton. The solutions corresponding to Eqs. (14), (15), (16), and (17) denote the growing-and-decaying mode and uniform state, respectively. This indicates that the mode grows only behind the periodic soliton, but does not grow ahead in this stage [Fig. 1(b)]. $(p_3)\; t \sim \sigma_2 + 1/2\, ln\, L_1 L_2/\Omega_2, [\sqrt{L_1 L_2} e^{-\Omega_2 t + \sigma_2} \sim O(1)]$. Approximate forms of Eqs. (5) and (6) are given by

$$f \simeq 1 + \frac{1}{2L_2} e^{\xi_1+\xi_2} \cos(\eta_1 + \eta_2 + \Psi_1) + \frac{M_1 M_2}{16} e^{2(\xi_1+\xi_2)}, \tag{18}$$

$$g \simeq 1 + \frac{1}{2L_2} e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)} \cos(\eta_1 + \eta_2 + i\phi_{1i} + \Psi_1) + \frac{M_1 M_2}{16} e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}. \tag{19}$$

It follows from these expressions that only a periodic soliton in a resonant state forms in the wave field as shown in Fig. 1(c). The mode which was produced in the backward region has already decayed. The x and y components of wave number, frequency and phase of the soliton are $\alpha + i(\beta_1+\beta_2)$, $\kappa + i(\delta_1+\delta_2)$, $\Omega_1+\Omega_2+i(\gamma_1+\gamma_2)$ and $\phi_1+\phi_2$, respectively. From the condition (11), we can obtain

$$\sin^2 \frac{\phi_1+\phi_2}{2} = \frac{[\alpha + i(\beta_1+\beta_2)]^2 - [\kappa + i(\delta_1+\delta_2)]^2}{2ru_0^2},$$

$$\begin{aligned}\Omega_1+\Omega_2+i(\gamma_1+\gamma_2) &= 2k[\alpha+i(\beta_1+\beta_2)] + 2l[\kappa+i(\delta_1+\delta_2)] \\ &\quad - \{[\alpha+i(\beta_1+\beta_2)]^2 + [\kappa+i(\delta_1+\delta_2)]^2\} \cot \frac{\phi_1+\phi_2}{2}.\end{aligned}$$

These equations give the dispersion relation for the periodic soliton in the resonant state.

$(p_4) t \sim \sigma_2 + ln\, L_1 L_2/\Omega_2; (L_1 L_2 e^{\xi_2} \sim O(1))$. The solutions in the backward and forward regions of the periodic soliton are given by

$$f \simeq 1, \quad g \simeq 1,$$

and

$$f \simeq \frac{M_1}{4L_1^2 L_2^2} e^{2\xi_1} \left\{ 1 - L_1 L_2 e^{\xi_2} \cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2 L_1^2 L_2^2}{4} e^{2\xi_2} \right\}, \tag{20}$$

$$g \simeq \frac{M_1}{4L_1^2 L_2^2} e^{2(\xi_1 + i\phi_{1r})} \left\{ 1 - L_1 L_2 e^{\xi_2 + i\phi_2} \cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2 L_1^2 L_2^2}{4} e^{2(\xi_2 + i\phi_2)} \right\}, \tag{21}$$

respectively. Equations (20) and (21) denote the growing-and-decaying mode. In this period, the mode is developed only in the forward region of the periodic soliton as shown in Fig. 1(d). $(p_5) t \to +\infty$. The solution is given by Eqs. (9) and (10), which shows appearance of the periodic soliton after the growth and decay of the mode as shown in Fig. 1(e). The periodic soliton has obtained the phase shift $-2\ln(L_1 L_2)$ as a result of interaction with the growing-and-decaying mode.

Similar asynchronous development of the growing-and-decaying mode may occur when $L_1 \to 0$. The condition for resonance in this case is given by

$$\phi_2 = \mp 2\theta_{1i} + \phi_{1r}, \quad \theta_2 = \theta_{1r} \pm \phi_{1i}/2. \tag{22}$$

III. CONCLUSIONS

The nonlinear evolution of a modulational instability is described by a growing-and-decaying mode solution to the DS I equation. We have investigated the time evolution of the resonant interaction between periodic soliton and growing-and-decaying mode. Under a quasiresonant condition, the mode develops first on one side from the periodic soliton. After the wave attains the maximum modulation, it returns to the unmodulated initial state. Then, the wave field shifts to an intermediate state affected by the growth and decay of the mode. Only a periodic soliton in the resonant state forms. This intermediate state persists over a comparatively long time interval. Next, the mode starts to grow on the other side from the periodic soliton.

The existence of periodic soliton changes the evolution of the growing-and-decaying mode drastically as if the periodic soliton dominates the instability in whole region of the wave field.

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