Quantum manifestations of classical periodic orbits in a square billiard: Formation of vortex lattices

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We extend the presentation of the SU(2) coherent states to analytically construct the wave function concentrated on high-order classical periodic orbits in a square billiard. With the constructed wave function, the localization of the wave pattern is found to be very efficient. We also analyze the vortices arising from the singular points of the quantum phase for the constructed coherent states. It is found that the wave interference gives rise to the appearance of vortex lattices in the probability current density associated with the high-order periodic orbits. Moreover, the topological charge of the vortex is in general nonintegral except for the periodic orbits with the same winding number to the sides of the square.

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I. INTRODUCTION

In the classical-quantum interface, an area of much current interest is the study of quantum states in the case of nonintegrable classical systems [1–3]. One of the most interesting phenomena is that the wave patterns of the scarred eigenstates are concentrated along unstable periodic orbits instead of being randomly distributed in the system [4–6]. Furthermore, there are some striking phenomena in open quantum ballistic cavities associated with the wave functions in terms of classical periodic orbits [7–9]. It is therefore useful to make the connection between quantum wave functions and classical periodic orbits for understanding the classical-quantum correspondence. In particular, the classical-quantum connection of conceptually simple classical systems will be of great value for the analysis of the quantum transport in mesoscopic systems.

The two-dimensional (2D) square billiard is one of the simplest billiards that is completely integrable in classical mechanics [10,11]. In a square billiard each family of periodic orbits can be denoted by three parameters (p,q,ϕ), where p and q are two positive integers describing the number of collisions with the horizontal and vertical walls, and the parameter $\phi(-\pi\leq\phi\leq\pi)$ that is related to the wall positions of specular reflection points [12–14]. Some example orbit families are given in Fig. 1. It can be seen that the trajectory constitutes a single, nonrepeated orbit provided that p and q are relatively prime. On the other hand, if p and q have a common factor m, the orbit family can be recast as the primitive periodic orbit $(p/m,q/m,\phi/m)$ and m is the number of repetitions of the primitive periodic orbit. According to Bohr's correspondence principle, the classical limit of a quantum system should be achieved when the quantum numbers go to infinity. However, the conventional eigenstates of a square billiard in most quantum mechanics do not manifest the properties of classical periodic orbits even in the correspondence limit of large quantum numbers.

Recently, we have analytically constructed the wave functions related to the primitive periodic orbit $(1,1,\phi)$ in a two-dimensional (2D) square billiard by using the representation of SU(2) coherent states [15]. In this paper, we analytically construct the wave function concentrated on high-order periodic orbits (p,q,ϕ) by introducing the folding property into the SU(2) coherent states. With the SU(2) coherent state, we find that the localization of the wave pattern is very efficient; only a few nearly degenerate eigenfunctions are already sufficient to localize wave patterns on high-order periodic orbits. This finding explains the phenomenon that the wave patterns concentrated on periodic orbits frequently appear in the ballistic quantum dots [16,17] as well as in weakly perturbed integrable systems [18,19]. Furthermore, we analyze the property of phase singularities in the quantum probability current for the constructed wave function. The phase singularity is well known to give rise to the vortices in the wave. The prominent feature for the constructed wave function is the appearance of vortex lattices in the flow of probability current density associated with the high-order periodic orbits. The formation of vortex lattices is clearly found to be the result of quantum interference effects. The noticeable finding is that the topological charge of the vortex is nonintegral for the states related to the periodic orbit (p,q,ϕ) with $p\neq q$.

II. WAVE FUNCTIONS ASSOCIATED WITH CLASSICAL PERIODIC ORBITS

Recently, we used the presentation of the SU(2) coherent state to analytically construct a wave function that is well localized on the corresponding classical periodic orbits $(1,1,\phi)$ in a 2D quantum square billiard. Our construction is the analog of the one used in Refs. [20] and [21] to construct eigenstates in the 2D quantum harmonic oscillator, optimally localized on classical elliptic orbits. As in the Schwinger

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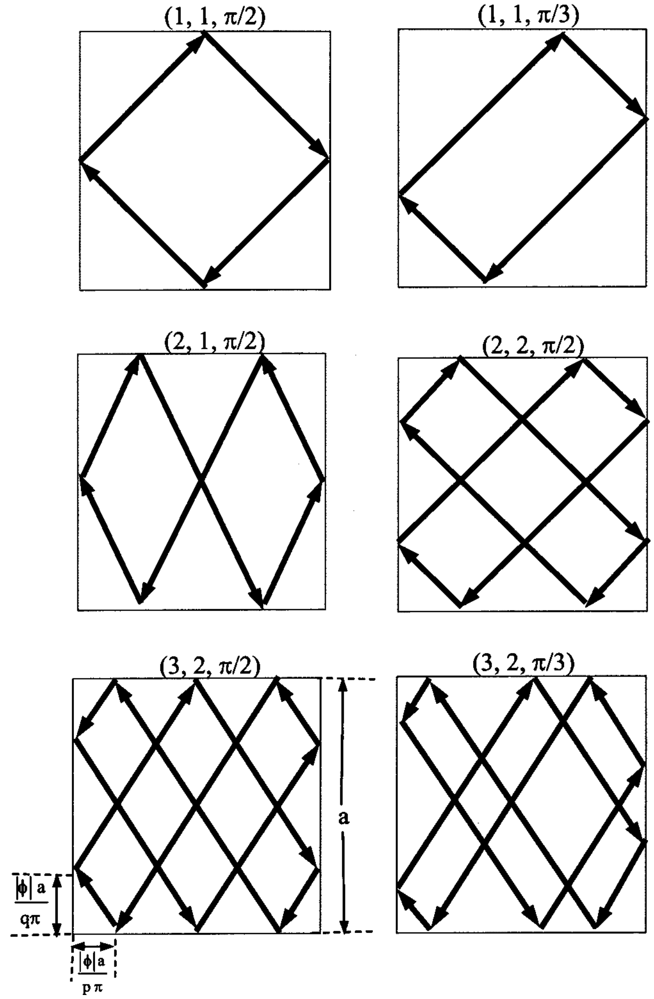

FIG. 1. Some classical periodic orbits (p,q,ϕ). p and q corresponding to the winding numbers of the orbit parallel to the sides of the square. The periodic orbits are in terms of the parameter ϕ that is related to the wall positions of specular reflection points.

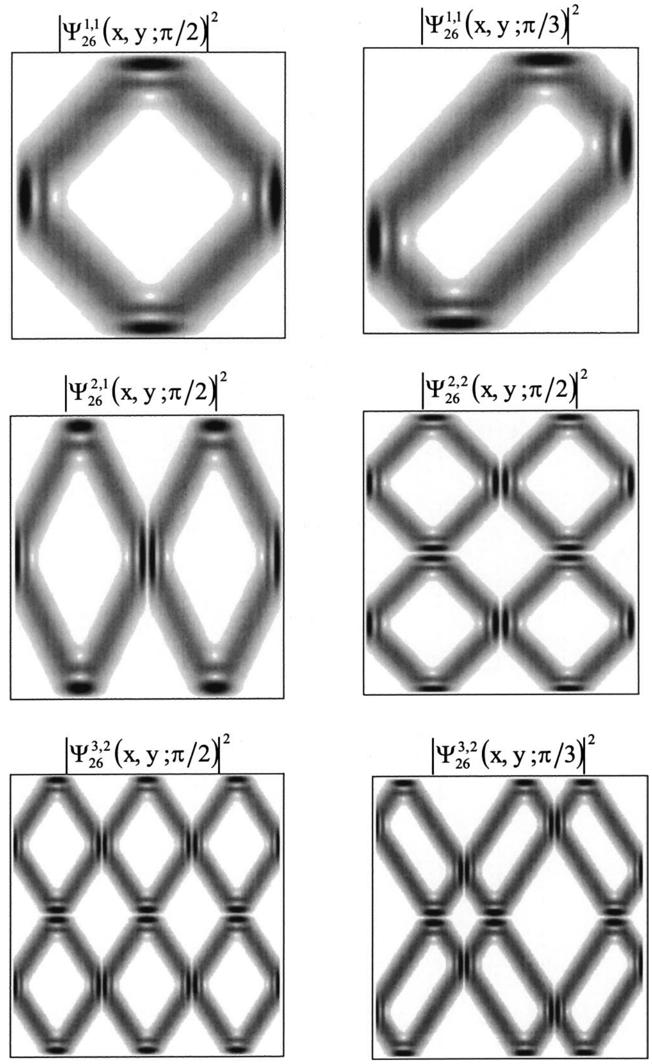

FIG. 2. The wave patterns of $|\Psi_N^{p,q}(x,y;\phi)|^2$ from Eq. (3) for $N=26$ corresponding to the classical trajectories displayed in Fig. 1.

representation of the SU(2) algebra, the wave function associated with the periodic orbit $(1,1,\phi)$ is given by

$$\Psi_N(x,y;\phi)=\frac{1}{2^{N/2}}\sum_{K=0}^{N}\binom{N}{K}^{1/2}e^{iK\phi}\psi_{K,N-K}(x,y), \quad (1)$$

where $\psi_{K,N-K}(x,y)$ is the eigenstates of the 2D square billiard,

$$\psi_{K,N-K}(x,y)=\frac{2}{a}\sin\left[(K+1)\frac{\pi x}{a}\right]\sin\left[(N-K+1)\frac{\pi y}{a}\right], \quad (2)$$

and a is the length of the square boundary. As seen in Fig. 1 the high-order periodic orbits (p,q,ϕ) can be folded to be a primitive cell $(1,1,\phi)$. Using this folding property and the periodicity of the sine function, the wave function associated with high-order periodic orbits (p,q,ϕ) can be analytically expressed as

$$\Psi_N^{p,q}(x,y;\phi)=\frac{1}{2^{N/2}}\sum_{K=0}^{N}\binom{N}{K}^{1/2}e^{iK\phi}\psi_{pK,q(N-K)}(x,y)$$
$$=\frac{(2/a)}{2^{N/2}}\sum_{K=0}^{N}\binom{N}{K}^{1/2}e^{iK\phi}\sin\left[p(K+1)\frac{\pi x}{a}\right]\times\sin\left[q(N-K+1)\frac{\pi y}{a}\right]. \quad (3)$$

Figure 2 depicts the wave patterns of $|\Psi_N^{p,q}(x,y;\phi)|^2$ with $N=26$ associated with the classical trajectories displayed in Fig. 1. It can be seen that the distributions of $|\Psi_N^{p,q}(x,y;\phi)|^2$ are in good agreement with the classical periodic orbits. Moreover, the behavior of $|\Psi_N^{p,q}(x,y;\phi)|^2$ illustrates geometrically Bohr's correspondence principle: the velocity of the classical particle is at a minimum at the specular reflection points of the motion, and therefore the distribution has a peak at these points. Note that the wave functions in Eqs. (1) and (3) are generally not stationary states because the eigenstate components are not degenerate for the Hamiltonian H. However, it can be easily shown that $\Delta H/\langle H\rangle$ for $N\geqslant 2$ is inversely proportional to N. Therefore, $\Delta H/\langle H\rangle \to 0$ as $N\to\infty$ for the wave function $\Psi_N^{p,q}(x,y;\phi)$. Namely, the coherent states in Eqs. (1) and (3) are stationary states in the classical limit. Note that the special states $\Psi_1^{p,p}(x,y;\phi)$ that contain two degenerate eigenstates of $\psi_{0,p}(x,y)$ and $\psi_{p,0}(x,y)$ are stationary states. As described in the following section, the $\Psi_1^{p,p}(x,y;\phi)$ states are pedagogically useful for understanding the formation of vortex patterns in the quantum probability current.

To understand how the parameter ϕ is determined for the different orbits, we use the identity of $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$ to rewrite the Eq. (3) as

$$\begin{aligned}\Psi_N^{p,q}(x,y;\phi)=\frac{(2/a)}{2^{N/2}}\{&e^{i\Theta_+(x,y)}F_+(x,y;\phi)\\&+e^{-i\Theta_+(x,y)}F_-(x,y;\phi)\\&-e^{i\Theta_-(x,y)}G_+(x,y;\phi)\\&-e^{-i\Theta_-(x,y)}G_-(x,y;\phi)\},\end{aligned}\tag{4}$$

where

$$F_\pm(x,y;\phi)=\sum_{K=0}^{N}\binom{N}{K}^{1/2}\exp\{iK[f_-(x,y)\pm\phi]\},\tag{5}$$

$$G_\pm(x,y;\phi)=\sum_{K=0}^{N}\binom{N}{K}^{1/2}\exp\{iK[f_+(x,y)\pm\phi]\},\tag{6}$$

and

$$\begin{aligned}f_\pm(x,y)&=\frac{p\pi}{a}x\pm\frac{q\pi}{a}y,\\ \Theta_\pm(x,y)&=\frac{p\pi}{a}x\pm(N+1)\frac{q\pi}{a}y.\end{aligned}\tag{7}$$

Since the property of the functions $F_\pm(x,y;\phi)$ and $G_\pm(x,y;\phi)$ is similar to the *Dirichlet kernel*, the wave function has the maximum value whenever $f_\pm(x,y)\pm\phi=2n\pi$ where n is an integer. It can be found that the lines of equation $f_\pm(x,y)\pm\phi=2n\pi$ coincide with the classical trajectories. Therefore, the relationship between the parameter ϕ and the periodic orbits is manifest.

Although the number of eigenstates used in the coherent state $\Psi_N^{p,q}(x,y;\phi)$ is $N+1$, the number of dominant eigenstates for wave localization is rather small for high-order states. To manifest the efficiency of wave localization, we modify $\Psi_N^{p,q}(x,y;\phi)$ to define a partially coherent state as

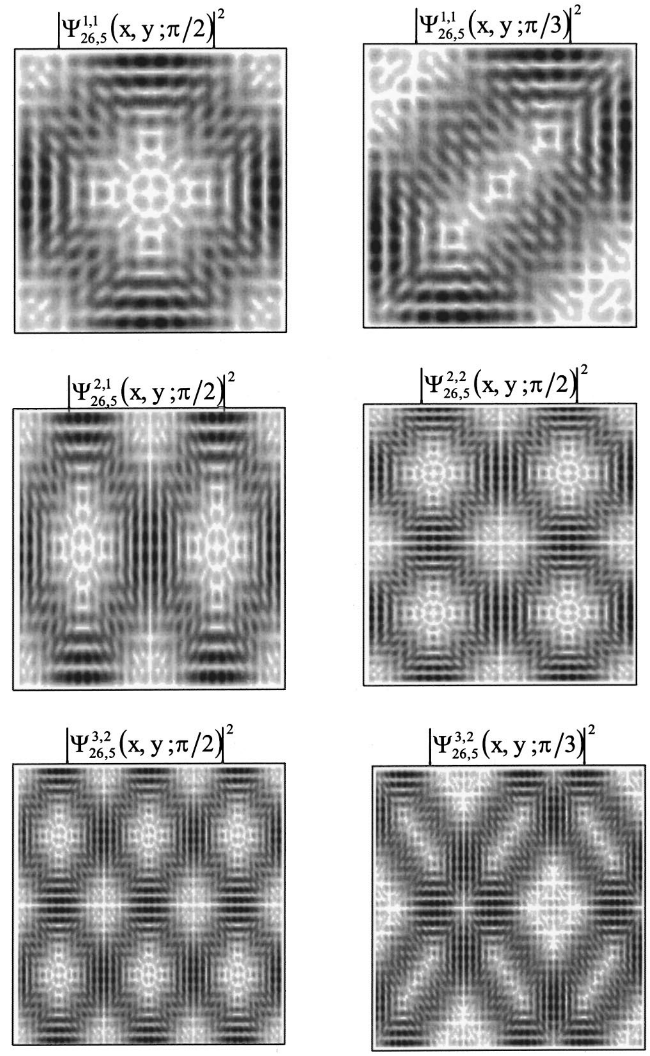

FIG. 3. The wave patterns of $|\Psi_{N,M}^{p,q}(x,y;\phi)|^2$ from Eq. (8) for $M=5$ and $N=26$ corresponding to the classical periodic orbits shown in Fig. 1.

$$\begin{aligned}\Psi_{N,M}^{p,q}(x,y;\phi)=&\frac{(2/a)}{\left[\sum_{K=J}^{N-J}\binom{N}{K}\right]^{1/2}}\sum_{K=J}^{N-J}\binom{N}{K}^{1/2}e^{iK\phi}\\&\times\sin\left[p(K+1)\frac{\pi x}{a}\right]\\&\times\sin\left[q(N-K+1)\frac{\pi y}{a}\right],\end{aligned}\tag{8}$$

where the index $M=N-2J+1$ represents the number of eigenstates used in the state $\Psi_{N,M}^{p,q}(x,y;\phi)$. Figure 3 displays the wave patterns of $|\Psi_{N,M}^{p,q}(x,y;\phi)|^2$ with $M=5$ and $N=26$ corresponding to the classical periodic orbits shown in Fig. 1. It is clear that only five eigenstates are already sufficient to localize the wave patterns on the classical trajectories, even for high-order periodic orbits. Since the partially coherent states $\Psi_{N,M}^{p,q}(x,y;\phi)$, in general, contain only a few nearly degenerate eigenstates, they often become the exact eigenstates in the weakly perturbed 2D square billiards [16,17] and usually appear in the ballistic quantum dot at resonances [18,19]. The present analysis indicates that the wave function obtained as a linear superposition of a few nearly degenerate eigenstates can provide a more physical description of a phenomenon than the true eigenstates in mesoscopic systems. Recently, Akis *et al.* [22] have shown how the scarred wave functions seen in open quantum dots may be interpreted as arising from single eigenstates of closed billiards. This finding is in good agreement with the present conclusion. Also, Hufnagel *et al.* [23] have shown that although the eigenstates of mixed-phase-space billiards can ignore the classical phase-space structures, semiclassically expected states and eigenstates will again coincide if the symmetry of the system is weakly perturbed.

III. FORMATION OF VORTEX LATTICES

Vortices are responsible for many observable phenomena known mainly to occur in macroscopic quantum systems, for example, superconductors or superfluids [24–26]. The order parameter equation in the study of these phenomena is the Ginzburg-Landau or Gross-Pitaevskii equation. The nonlinear character of the modeling equation greatly complicates the analysis of the solution. However, as pointed out already by Dirac [27], the vortices arising from the singular points of the quantum phase also manifest themselves in the linear Schrödinger equation. Recent works show that the vortex problems play an important role in quantum mechanics [28–30]. Therefore, it is of great interest to analyze the vortex behavior for the present wave function.

For analyzing the property of phase singularities associated with the classical periodic orbits, it will be convenient to separate $\Psi_N^{p,q}(x,y;\phi)$ into its real and imaginary parts,

$$\Psi_N^{p,q}(x,y;\phi)=\Phi_N^{p,q}(x,y;\phi)+i\,\Xi_N^{p,q}(x,y;\phi), \tag{9}$$

where

$$\Phi_N^{p,q}(x,y;\phi)=\frac{(2/a)}{2^{N/2}}\sum_{K=0}^{N}\binom{N}{K}^{1/2}\cos(K\phi)\sin\left[p(K+1)\frac{\pi x}{a}\right]\sin\left[q(N-K+1)\frac{\pi y}{a}\right], \tag{10}$$

$$\Xi_N^{p,q}(x,y;\phi)=\frac{(2/a)}{2^{N/2}}\sum_{K=0}^{N}\binom{N}{K}^{1/2}\sin(K\phi)\sin\left[p(K+1)\frac{\pi x}{a}\right]\sin\left[q(N-K+1)\frac{\pi y}{a}\right]. \tag{11}$$

In terms of the probability density $\rho(x,y)$ and the phase distribution $\chi(x,y)$, the probability current density is analytically given by [31]

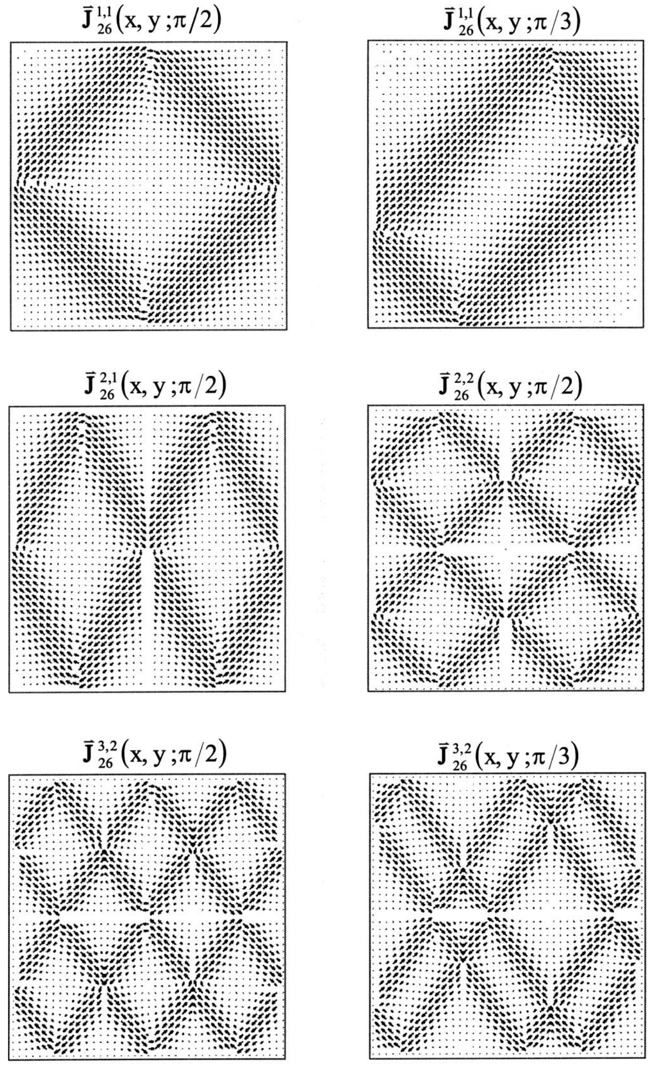

FIG. 4. The calculated results for the probability current densities corresponding to the wave functions displayed in Fig. 2.

$$\vec{J}(x,y)=\frac{\hbar}{m}\rho(x,y)\nabla\chi(x,y),$$
$$\rho(x,y)=|\Psi_N^{p,q}(x,y;\phi)|^2, \tag{12}$$
$$\chi(x,y)=\tan^{-1}[\Phi_N^{p,q}(x,y;\phi)/\Xi_N^{p,q}(x,y;\phi)].$$

Note that the coherent state $\Psi_N^{p,q}(x,y;\phi)$ is a standing wave and has no vortices when $\phi=\pm n\pi$ and n is an integer. In other words, the vortices can exist in the coherent state $\Psi_N^{p,q}(x,y;\phi)$ when $\phi\neq\pm n\pi$. Hereafter we focus on the case of $\phi\neq\pm n\pi$, unless otherwise specified.

Using Eqs. (9)–(12), the probability current densities have been calculated. Figure 4 shows the calculated results for the wave functions displayed in Fig. 2. The direction of probability flow corresponds to the classical rays shown in Fig. 1. Figure 5 shows the calculated results for the periodic orbit $(4,4,\pi/2)$ for the cases of $N=1$, 5, and 25. It can be seen that the order of the singularity is proportional to the index N. The order of the singularity, denoted also as the

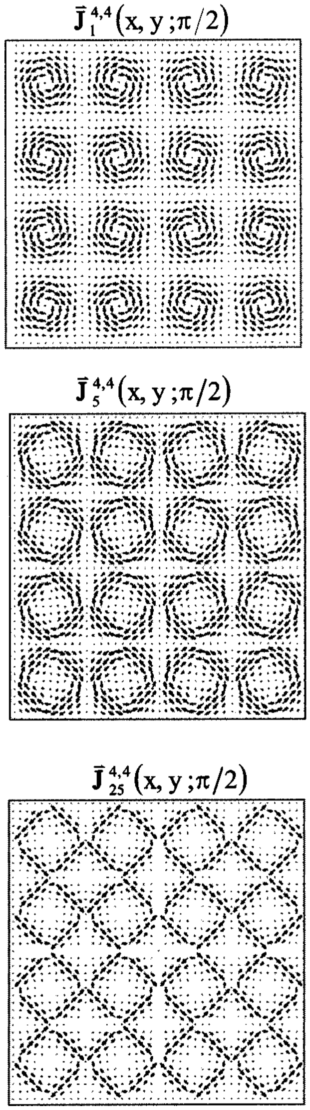

FIG. 5. The calculated results for the probability current densities corresponding to the periodic orbit $(4,4,\pi/2)$ of the coherent states with $N=1$, 5, and 25.

topological charge of the vortex, is calculated as the circulation of the phase gradient using a small closed curve encircling the singularity

$$Q=\frac{1}{2\pi}\oint \nabla\chi\cdot d\vec{\mathbf{r}}. \qquad (13)$$

Substituting Eqs. (10)–(12) into Eq. (13), the topological charge for the coherent state $\Psi_N^{p,q}(x,y;\phi)$ is found to be

$$Q=\pm\frac{p^2+q^2}{2pq}N. \qquad (14)$$

It can be seen that the topological charge is proportional to the index N. In the case of $p=q$, the topological charge is an integer and equal to $\pm N$. Although the coherent states $\Psi_1^{p,p}(x,y;\phi)$ hardly display the classical trajectories, they are stationary states in a 2D quantum square billiard, as mentioned in Sec. II. From Eq. (14), the topological charge of the stationary states $\Psi_1^{p,p}(x,y;\phi)$ is equal to ± 1. Even though the present result is based on the linear Schrödinger equation, it is interesting to notice that only vortex states with $Q=\pm 1$ are stable in the framework of the dissipative Ginzburg-Landau equation [32,33].

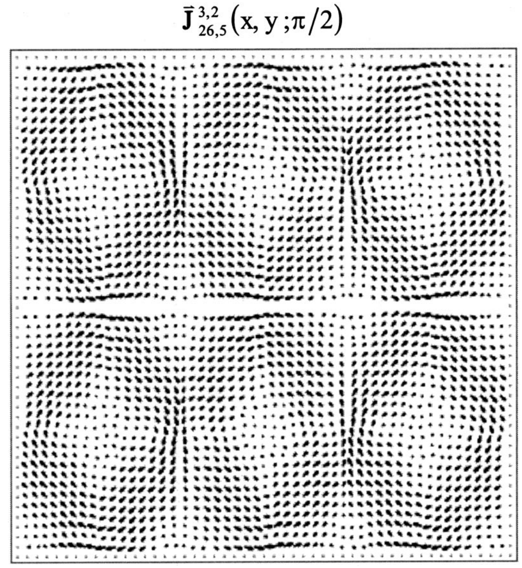

FIG. 6. The calculated results for the probability current densities corresponding to the partially coherent state $\Psi_{26,5}^{3,2}(x,y;\phi)$.

On the other hand, the topological charge is nonintegral when $p\neq q$. The nonintegral topological charge arises from the fact that the primitive periodic orbit $(p/m,q/m,\phi/m)$ for $p\neq q$ has at least one intersection at which the occurrence of the wave interference leads to the topological charge to be nonintegral [34]. However, for $p=q$ the primitive periodic orbit is $(1,1,\phi)$ and the intersection number for each trajectory is actually zero; therefore the quantization of the topological charge is the same as that of simple trajectories without intersections. The evidence for a fractional topological charge has been studied in the fields of high-energy physics [34,35] and nonlinear physics [36]. Finally, the nonzero circulation also comes out in the partially coherent states that are defined in Eq. (8). Figure 6 shows the calculated probability current density for the partially coherent state $\Psi_{26,5}^{3,2}(x,y;\phi)$. The structure of the vortices can be clearly seen, although only five eigenstates are used to localize the wave pattern on the classical trajectory.

IV. CONCLUSIONS

We have extended the SU(2) coherent states to analytically construct the wave function associated with the high-order periodic orbits (p,q,ϕ). We modify the constructed coherent state to investigate the efficiency of wave localization. It is found that only a few nearly degenerate eigenstates is already sufficient to localize wave patterns on high-order periodic orbits. The high efficiency of wave localization confirms that the wave patterns related to periodic orbits usually appear in the weakly perturbed integrable systems as well as in the ballistic quantum dots. Moreover, the property of phase singularities in the quantum probability current is analyzed. The singular points of the quantum phase is found to form the vortex lattices in the probability current density related to the high-order periodic orbits. An interesting result is that the topological charge of vortices is nonintegral for the quantum states related to the periodic orbit (p,q,ϕ) with $p\neq q$.

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