Relaxation and overlap-probability function in the spherical and mean-spherical models

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The problem of the equivalence of the spherical and mean-spherical models, which has been thoroughly studied and is understood in equilibrium, is considered anew from the dynamical point of view during the time evolution following a quench from above to below the critical temperature. It is found that there exists a crossover time $t^* \sim V^{2/d}$ such that for $t<t^*$ the two models are equivalent, while for $t>t^*$ macroscopic discrepancies arise. The relation between the off equilibrium response function and the structure of the equilibrium state that usually holds for phase ordering systems is found to hold for the spherical model but not for the mean-spherical one. The latter model offers an explicit example of a system which is not stochastically stable.

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I. INTRODUCTION

Let us consider a system with scalar continuous order parameter $\varphi(\vec{x})$ in a volume V and Hamiltonian

$$\mathcal{H}[\varphi(\vec{x})] = \frac{1}{2}\int_V d\vec{x}[(\nabla\varphi)^2 + r\varphi^2(\vec{x})], \tag{1}$$

where $r \geqslant 0$. The spherical model of Berlin and Kac [1] is obtained by considering the extensive random variable

$$\Psi = \frac{1}{V}\int_V d\vec{x}\varphi^2(\vec{x}) \tag{2}$$

and computing equilibrium properties with the Gaussian weight $\rho_g[\varphi] = (1/Z_g)e^{-(1/T)\mathcal{H}[\varphi]}$ under the microcanonical constraint $\Psi = \alpha$, where α is a given number. The mean spherical model of Lewis and Wannier [2], instead, is obtained by imposing the constraint in the mean, or canonically $\langle\Psi\rangle = \alpha$, which makes the model considerably easier to solve. The spherical model was originally introduced by Berlin and Kac as an exactly soluble model displaying critical phenomena. Subsequently, the enforcement of a spherical constraint (in either form, microcanonical or canonical) on the free part of nonlinear problems has become an extremely useful and practical way to generate mean-field approximations.

However, despite the great popularity of the method, it is usually overlooked that the equilibrium properties of the two models, as was recognized early on, coincide above but not below the critical point. The origin of the discrepancy was clarified first by Lax [3] and further investigated by Yan and Wannier [4]. Finally, Kac and Thompson [5] showed how to connect the averages in the two models. It is easy to understand that the microcanonical and canonical constraints are equivalent as long as the fluctuations $\langle\delta\Psi^2\rangle = \langle\Psi^2\rangle - \langle\Psi\rangle^2$ are negligible. Indeed, above the critical temperature T_C one has the usual behavior for thermodynamic quantities $\langle\delta\Psi^2\rangle \sim 1/V$. Not so below T_C, where fluctuations of Ψ turn out to be finite and independent of the volume $\langle\delta\Psi^2\rangle \sim \alpha^2$.

An important consequence of this, as we shall see in the following, is that the nature of the mixed state below T_C is quite different in the two models. It is then interesting to investigate whether the two models are equivalent or not when considering time dependent properties in the relaxation process following a quench from above to below the critical point. This is a relevant question since, in practice, dynamics can be solved only in the mean-spherical case and in the literature it is taken for granted that the mean-spherical form of dynamics applies to the spherical case as well. On the basis of the previous considerations, clarification of this point amounts to analyzing the time evolution of $\langle\delta\Psi^2\rangle$. As we shall see, this depends on the order of the limits $t\to\infty$ and $V\to\infty$; namely, it turns out that if V is kept finite during the relaxation there is a crossover from the preasymptotic behavior $\langle\delta\Psi^2\rangle \sim 1/V$ to the asymptotic one $\langle\delta\Psi^2\rangle \sim \alpha^2$ with the crossover time $t^* \sim V^{2/d}$. Instead, if $V\to\infty$ from the outset, then $\langle\delta\Psi^2\rangle$ stays negligible for any finite time since t^* diverges. In any case, in the scaling regime $t<t^*$, the time dependent evolution is identical in the two models, as usually assumed.

Although reassuring, this conclusion opens an unexpected and interesting problem when it comes to testing the connection between static and dynamic properties introduced by Cugliandolo and Kurchan [6] for mean-field spin glasses and then established in general by Franz *et al.* [7] for slowly relaxing systems. The dynamic quantities are the autocorrelation function $C(t,t_w)$ and the integrated linear response function $\chi(t,t_w)$, while equilibrium properties are encoded into the overlap-probability function $P(q)$ [8]. The statement is that, if $\chi(t,t_w)$ depends on time through the autocorrelation function $\chi(C(t,t_w))$, then one has

$$-T\frac{d^2\chi(C)}{dC^2}\bigg|_{C=q} = \tilde{P}(q), \tag{3}$$

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where $\widetilde{P}(q)$ is the overlap-probability function in the equilibrium state obtained when the perturbation giving rise to $\chi(t,t_w)$ is switched off. Therefore, the unperturbed overlap probability $P(q)$ can be recovered from the dynamics if $\widetilde{P}(q)$ coincides with or is simply related to $P(q)$. This happens for stochastically stable systems [7]. Formulated originally in the context of glassy systems, this connection between statics and dynamics applies also to phase ordering processes in nondisordered systems [9]. A detailed account of the latter case can be found in Ref. [10]. Now, the interesting point is that the spherical and mean-spherical models do share the same relaxation properties, and therefore the same $\chi(C)$ and the same $\widetilde{P}(q)$, while the corresponding unperturbed overlap probabilities are profoundly different. In particular, $P(q)$ can be recovered from $\widetilde{P}(q)$ in the spherical model, but not in the mean-spherical one.

The paper is organized as follows. In Secs. II and III the relevant properties of the spherical and mean-spherical models are presented. The relation between the two models is discussed in Sec. IV and comments on the connection between statics and dynamics are made in the concluding Sec. V.

II. SPHERICAL MODEL

The dynamical evolution is described by the Langevin equation

$$\frac{\partial\varphi(\vec{x},t)}{\partial t} = -\frac{\delta\mathcal{H}[\varphi]}{\delta\varphi(\vec{x},t)} + \eta(\vec{x},t), \tag{4}$$

where $\eta(\vec{x},t)$ is a Gaussian white noise. Taking for $\mathcal{H}[\varphi]$ the Gaussian model (1) and Fourier transforming with respect to space, we have

$$\frac{\partial\varphi(\vec{k},t)}{\partial t} = -\omega_k\varphi(\vec{k},t) + \eta(\vec{k},t) \tag{5}$$

with $\omega_k = k^2 + r$ and

$$\langle\eta(\vec{k},t)\rangle = 0,$$

$$\langle\eta(\vec{k},t)\eta(\vec{k}',t')\rangle = 2TV\delta_{\vec{k}+\vec{k}',0}\delta(t-t'), \tag{6}$$

where T is the temperature of the quench. Integrating Eq. (5), we obtain

$$\varphi(\vec{k},t) = R_g(\vec{k},t)\varphi(\vec{k},0) + \int_0^t dt' R_g(\vec{k},t-t')\eta(\vec{k},t'), \tag{7}$$

where $R_g(\vec{k},t) = e^{-\omega_k t}$. Considering an infinite temperature initial state with

$$\langle\varphi(\vec{k},0)\rangle = 0,$$

$$\langle\varphi(\vec{k},0)\varphi(\vec{k}',0)\rangle = \Delta V\delta_{\vec{k}+\vec{k}',0}, \tag{8}$$

and taking averages over the initial state and thermal noise, from Eq. (7) follows $\langle\varphi(\vec{k},t)\rangle = 0$, $\langle\varphi(\vec{k},t)\varphi(\vec{k}',t)\rangle = C(\vec{k},t)V\delta_{\vec{k}+\vec{k}',0}$, where the equal time structure factor is given by

$$C(\vec{k},t) = R_g^2(\vec{k},t)\Delta + \frac{T}{\omega_k}[1 - R_g^2(\vec{k},t)]. \tag{9}$$

With the initial condition (8) and the linear equation (5), the configuration $[\varphi(\vec{k},t)]$ executes a zero average Gaussian process whose probability distribution is given by

$$\rho_g([\varphi(\vec{k})],t) = \frac{1}{Z_g(t)} e^{-(1/2V)\sum_{\vec{k}} \varphi(\vec{k})C^{-1}(\vec{k},t)\varphi(-\vec{k})}, \tag{10}$$

where $Z_g(t) = \int d[\varphi(\vec{k})] e^{-(1/2V)\Sigma_{\vec{k}}\varphi(\vec{k})C^{-1}(\vec{k},t)\varphi(-\vec{k})}$. From Eq. (10) one can compute the one-time properties of the Gaussian model, including those at equilibrium obtained by letting $t \to \infty$.

Let us next define the joint probability of a configuration $[\varphi(\vec{k})]$ and a random variable Ψ by

$$\rho_g([\varphi(\vec{k})],\Psi,t) = \rho_g([\varphi(\vec{k})],t)\,\delta\!\left(\Psi - \frac{1}{V^2}\sum_{\vec{k}} |\varphi(\vec{k})|^2\right). \tag{11}$$

Clearly, $\rho_g([\varphi(\vec{k})],t)$ is recovered by integrating over Ψ, while the probability of Ψ is given by

$$\rho_g(\Psi,t) = \int d[\varphi(\vec{k})]\rho_g([\varphi(\vec{k})],\Psi,t). \tag{12}$$

Introducing the probability of $[\varphi(\vec{k})]$ conditioned to a given value of Ψ,

$$\rho_g(\Psi|[\varphi(\vec{k})],t) = \frac{\rho_g([\varphi(\vec{k})],\Psi,t)}{\rho_g(\Psi,t)} \tag{13}$$

we may also write

$$\rho_g([\varphi(\vec{k})],t) = \int d\Psi\,\rho_g(\Psi,t)\rho_g(\Psi|[\varphi(\vec{k})],t). \tag{14}$$

Notice that conditioning with respect to Ψ is tantamount to imposing the spherical constraint. Hence, the probability distribution for the Berlin-Kac spherical model [1] can be written as

$$\rho_s([\varphi(\vec{k})],t;\alpha) = \rho_g(\Psi=\alpha|[\varphi(\vec{k})],t). \tag{15}$$

III. MEAN-SPHERICAL MODEL

As stated in the Introduction, the mean-spherical model is obtained by imposing the constraint in the mean $(1/V^2)\Sigma_{\vec{k}}\langle|\varphi(\vec{k})|^2\rangle = \alpha$. This can be done by the Lagrange multiplier method or, which is the same, by modifying Eq. (5), letting the parameter r be a function of time,

$$\frac{\partial\varphi(\vec{k},t)}{\partial t}=-[k^2+r(t)]\varphi(\vec{k},t)+\eta(\vec{k},t), \tag{16}$$

where $r(t)$ is to be determined self-consistently through the constraint, which can be rewritten as

$$\frac{1}{V}\sum_{\vec{k}} C(\vec{k},t)=\alpha. \tag{17}$$

The structure of the solution of the equation of motion remains the same:

$$\varphi(\vec{k},t)=R_{\rm ms}(\vec{k},t,0)\varphi(\vec{k},0)+\int_0^t R_{\rm ms}(\vec{k},t,t')\,\eta(\vec{k},t'), \tag{18}$$

where now

$$R_{\rm ms}(\vec{k},t,t')=\frac{Y(t')}{Y(t)}e^{-k^2(t-t')} \tag{19}$$

with $Y(t)=e^{Q(t)}$ and $Q(t)=\int_0^t dt' r(t')$. The equal time structure factor is given by

$$C(\vec{k},t)=R^2_{\rm ms}(\vec{k},t,0)\Delta+2T\int_0^t dt' R^2_{\rm ms}(\vec{k},t,t'), \tag{20}$$

where, clearly, the initial value Δ must also be consistent with Eq. (17). This requires $V^{-1}\Sigma_{\vec{k}}\Delta=\int[d^dk/(2\pi)^d]\exp(-k^2/\Lambda^2)\Delta=\alpha$ where, as we shall always do in the following, in transforming the sum over $\vec{k}$ into an integral we make explicit the existence of a high-momentum cutoff Λ. Hence, eventually $\Delta=(4\pi)^{d/2}\Lambda^{-d}\alpha$. Finally, the probability distribution keeps the Gaussian form (10)

$$\rho_{\rm ms}([\varphi(\vec{k})],t)=\frac{1}{Z_{\rm ms}(t)}e^{-(1/2V)\sum_{\vec{k}}\varphi(\vec{k})C^{-1}(\vec{k},t)\varphi(-\vec{k})}, \tag{21}$$

where now $C(\vec{k},t)$ is given by Eq. (20).

In order to have an explicit solution the function $Y(t)$ must be determined. This is done in Appendix A where, for simplicity, the computation has been limited to the case $2<d<4$. For large time one finds

$$Y^2(t)=\begin{cases} Be^{t/\tau} & \text{for } T>T_C\\ B[e^{t/\tau}+(t/t^*)^{-\omega}] & \text{for } T\leqslant T_C,\end{cases} \tag{22}$$

where

$$T_C=\frac{2\Lambda^{d-2}}{(4\pi)^{d/2}(d-2)}\alpha \tag{23}$$

and the expressions for B, τ, ω, and t^* are listed in Appendix A. Here we point out that B and τ are independent of the volume for $T>T_C$, while B vanishes and τ diverges as $V\to\infty$ for $T\leqslant T_C$. Inserting Eq. (22) in Eq. (20), for $T>T_C$ we find

$$C(\vec{k},t)=\frac{\Delta}{B}e^{-2(k^2+1/2\tau)t}+\frac{T}{k^2+1/2\tau}[1-e^{-2(k^2+1/2\tau)t}] \tag{24}$$

and for $T\leqslant T_C$,

$$C(\vec{k},t)=\frac{\Delta}{B}\frac{(t/t^*)^{\omega}e^{-2k^2t}}{[1+(t/t^*)^{\omega}e^{t/\tau}]}+2T\int_{\hat{t}}^{t}dt' e^{-2k^2(t-t')}(t'/t)^{-\omega}\left[\frac{1+(t'/t^*)^{\omega}e^{t'/\tau}}{1+(t/t^*)^{\omega}e^{t/\tau}}\right], \tag{25}$$

where $\hat{t}$ is the microscopic time necessary to elapse for Eq. (22) to apply. Notice that from Eq. (A9) and from Eq. (A16) it follows that t^* and τ are both $O(V^{2/d})$ for $T=T_C$, while for $T<T_C$ and $d>2$ the two time scales are separated with $t^*\sim V^{2/d}\ll\tau\sim V$.

In any case, τ is the equilibration time. Taking $t\gg\tau$ from Eqs. (24) and (25) it follows that

$$C(\vec{k},t)=C_{eq}(\vec{k})=\frac{T}{k^2+\xi^{-2}}, \tag{26}$$

where the equilibrium correlation length ξ is related to τ by $\xi^2=2\tau$. Hence, using Eq. (A9) T_C may be identified with the static transition temperature separating the high-temperature phase where ξ is independent of the volume from the low-temperature phase where ξ diverges with the volume:

$$\xi\sim\begin{cases}\left(\frac{T-T_C}{T_C}\right)^{-1/(d-2)} & \text{for } 0<\frac{T-T_C}{T_C}\ll 1\\ V^{1/d} & \text{for } T=T_C\\ V^{1/2} & \text{for } T<T_C.\end{cases} \tag{27}$$

Finally, let us comment on the nature of the equilibrium state. As Eq. (21) shows, each $\vec{k}$ mode is Gaussianly distributed, with zero average, at any time including at equilibrium. Hence, in the low-temperature phase the system does not order. The static transition consists in the $\vec{k}=\vec{0}$ mode developing a macroscopic variance as the temperature is lowered from above to below T_C [11]:

$$\langle\varphi_0^2\rangle\sim\begin{cases} V & \text{for } T>T_C\\ V^{(d+2)/d} & \text{for } T=T_C\\ V^2 & \text{for } T<T_C.\end{cases} \tag{28}$$

For the sake of illustration let us consider $T=0$, where from $C_{eq}(\vec{k})=\alpha V\delta_{\vec{k},0}$ it follows that:

$$\rho_{ms}[\varphi(\vec{k})]=\frac{1}{\sqrt{2\pi\alpha V^2}}e^{-\varphi_0^2/2\alpha V^2}\prod_{\vec{k}\neq 0}\delta(\varphi(\vec{k})). \tag{29}$$

IV. THE CONNECTION BETWEEN THE TWO MODELS

In order to explore how the spherical and the mean-spherical models are connected, let us rewrite Eq. (21), following the same steps which led to Eq. (14) and obtaining

$$\rho_{\rm ms}([\varphi(\vec{k})],t)=\int d\Psi\,\rho_{\rm ms}(\Psi,t)\rho_{\rm ms}(\Psi|[\varphi(\vec{k})],t) \quad (30)$$

which yields the relation between the two models since the conditional probability $\rho_{\rm ms}(\Psi|[\varphi(\vec{k})],t)$ enforces the constraint of Berlin and Kac. Actually, to be precise, this quantity is not exactly the same as the spherical model distribution, since in Eq. (15) the constraint is imposed on the Gaussian model, while here it is imposed on the mean-spherical model. In the following we will ignore the difference.

Looking at Eq. (30) the state in the mean-spherical model can be regarded as a mixture of states in the spherical model with constraint values weighted by $\rho_{\rm ms}(\Psi,t)$. The properties of the two models are the same if this weight is narrowly peaked about the mean value $\langle\Psi\rangle=\alpha$, while discrepancies are to be expected if the weight spreads over significantly different values of Ψ. Therefore, the key quantity controlling the connection between the two models is $\rho_{\rm ms}(\Psi,t)$. The corresponding characteristic function is

$$\Theta(x)=\langle e^{ix\Psi}\rangle=e^{-(1/2)\sum_{\vec{k}}\ln[1-(2/V)ixC(\vec{k},t)]}, \quad (31)$$

where $\langle\cdot\rangle$ stands for the average computed with Eq. (21). The moments of Ψ are then given by $\langle\Psi^n\rangle=[d^n\Theta(x)/d(ix)^n])_{x=0}$ and, in particular,

$$\langle\Psi(t)\rangle=\frac{1}{V}\sum_{\vec{k}}C(\vec{k},t)=\alpha, \quad (32)$$

$$\langle\delta\Psi^2(t)\rangle=\langle\Psi^2(t)\rangle-\langle\Psi(t)\rangle^2=\frac{1}{V^2}\sum_{\vec{k}}C^2(\vec{k},t). \quad (33)$$

Let us first consider what happens at equilibrium by letting $t\to\infty$ and inverting Eq. (31). Following Kac and Thompson [5] and using Eqs. (26) and (27) for $T>T_C$ one finds

$$\rho_{\rm ms}(\Psi)=\frac{1}{\sqrt{2\pi\sigma}}e^{-(\Psi-\alpha)^2/2\sigma} \quad (34)$$

with $\sigma=(1/V^2)\Sigma_{\vec{k}}C_{eq}^2(\vec{k})$. For large volume this can be rewritten as $\sigma=(2/V)\int[d^dk/(2\pi)^d]e^{-k^2/\Lambda^2}/(k^2+\xi^{-2})^2$ and since the integral is finite we have $\langle\delta\Psi^2\rangle\sim O(1/V)$. Instead, for $T<T_C$ one finds

$$\rho_{\rm ms}(\Psi)=\begin{cases}0 & \text{for } \Psi<\alpha T/T_C\\ \dfrac{e^{-(\Psi-\alpha T/T_C)/2\alpha(1-T/T_C)}}{\sqrt{2\pi(\Psi-\alpha T/T_C)\alpha(1-T/T_C)}} & \text{for } \Psi>\alpha T/T_C,\end{cases} \quad (35)$$

which gives

$$\langle\delta\Psi^2\rangle=2[\alpha(1-T/T_C)]^2. \quad (36)$$

Hence, we have that in going from the high-temperature to the low-temperature phase the fluctuations of Ψ from microscopic become macroscopic and, as anticipated above, significant differences must be expected in the equilibrium states of the two models. In order to see this explicitly, let us consider $T=0$ where the equilibrium state in the mean-spherical model is given by Eq. (29). The joint probability of $[\varphi(\vec{k})]$ and Ψ is given by

$$\rho_{\rm ms}([\varphi(\vec{k})],\Psi)=\frac{1}{\sqrt{2\pi\alpha V^2}}e^{-\varphi_0^2/2\alpha V^2}\prod_{\vec{k}\neq 0}\delta(\varphi(\vec{k}))\times\delta\left(\Psi-\frac{1}{V}\int d\vec{x}\,\varphi^2(\vec{x})\right) \quad (37)$$

which, due to the presence of the δ functions for $\vec{k}\neq\vec{0}$, can be rewritten as

$$\rho_{\rm ms}([\varphi(\vec{k})],\Psi)=\frac{1}{\sqrt{2\pi\alpha\Psi}}e^{-\Psi/2\alpha}\prod_{\vec{k}\neq 0}\delta(\varphi(\vec{k}))\times\frac{V}{2\sqrt{\Psi}}[\delta(\varphi_0-V\sqrt{\Psi})+\delta(\varphi_0+V\sqrt{\Psi})]. \quad (38)$$

Identifying the first factor in the right hand side with the $T\to 0$ limit of Eq. (35), we find the conditional probability

$$\rho_{\rm ms}(\Psi|[\varphi(\vec{k})])=\frac{1}{2}[\delta(\varphi_0-V\sqrt{\Psi})+\delta(\varphi_0+V\sqrt{\Psi})]\prod_{\vec{k}\neq 0}\delta(\varphi(\vec{k})), \quad (39)$$

which gives the $T=0$ state in the spherical model with a value Ψ of the constraint. Comparing Eqs. (29) and (39), the difference in the low-temperature equilibrium states of the two models is quite evident. While in the mean-spherical model φ_0 is Gaussianly distributed, in the spherical model the probability of φ_0 is bimodal. The large fluctuations (36) around the average of Ψ do appear below T_C since it is necessary to spread out the weight of Ψ in order to reconstruct a Gaussian distribution by mixing bimodal distributions.

This difference in the structure of the low-temperature equilibrium states shows up quite clearly in the corresponding overlap-probability functions [8]

$$P(q)=\int d[\varphi]d[\varphi']\rho[\varphi]\rho[\varphi']\delta(Q[\varphi,\varphi']-q), \quad (40)$$

where

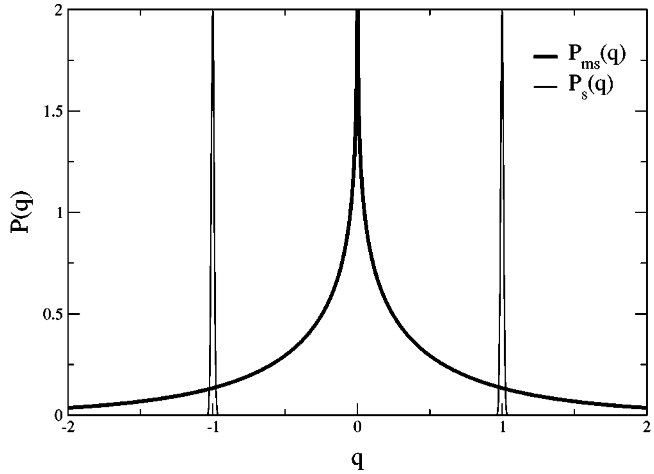

FIG. 1. The $T=0$ overlap probability function for the spherical and the mean-spherical models with $\alpha=1$.

$$Q[\varphi,\varphi']=\frac{1}{V}\int d\vec{x}\varphi(\vec{x})\varphi'(\vec{x})=\frac{1}{V^2}\sum_{\vec{k}}\varphi(\vec{k})\varphi'(-\vec{k}). \tag{41}$$

Keeping on considering, for simplicity, the $T=0$ states and looking at the characteristic function $\Theta(\lambda)=\int dqP(q)e^{i\lambda q}$ from Eqs. (29) and (39) with $\Psi=\alpha$ it follows that (Appendix B):

$$\Theta_{ms}(\lambda)=\frac{1}{\sqrt{1+(\lambda\alpha)^2}} \tag{42}$$

and

$$\Theta_s(\lambda)=\cos(\lambda\alpha). \tag{43}$$

Inverting, we have

$$P_{ms}(q)=\frac{1}{\alpha\pi}K_0(|q|/\alpha), \tag{44}$$

where K_0 is a Bessel function of imaginary argument and

$$P_s(q)=\frac{1}{2}[\delta(q-\alpha)+\delta(q+\alpha)]. \tag{45}$$

The plots of these two functions in Fig. 1 illustrate the great difference in the ground states.

Having analyzed the properties in the final equilibrium states, let us now go back to the dynamical problem. Notice that at $t=0$, with $C(\vec{k},0)=\Delta$, from Eq. (31) it follows that $\rho_{ms}(\Psi,t=0)$ is given by Eq. (34) with $\sigma=2\Delta^2\Lambda^d/(4\pi)^{d/2}V$. Hence, in a quench to $T>T_C$ fluctuations of Ψ remain microscopic throughout the time evolution, while in a quench to $T<T_C$ at some point $\langle\delta\Psi^2(t)\rangle$ must cross over from $O(1/V)$ to $O(1)$. Again, for simplicity, we show this in the case of the $T=0$ quench. From Eq. (25) in the limit $T\to 0$ we have

$$C(\vec{k},t)=\frac{\Delta(t/t_0)^{d/2}e^{-2k^2t}}{[1+(t/t^*)^{d/2}]}, \tag{46}$$

where $t_0=(2\Lambda^2)^{-1}$. Inserting this into Eq. (33) we find for $t/t_0\gg 1$ and $t^*/t_0\gg 1$,

$$\langle\delta\Psi^2(t)\rangle=2\alpha^2(t/t^*)^d\frac{[1+(t^*/2t)^{d/2}]}{[1+(t/t^*)^{d/2}]^2} \tag{47}$$

which gives

$$\langle\delta\Psi^2(t)\rangle=\begin{cases}2\alpha^2(t/2t^*)^{d/2} & \text{for } t\ll t^*\\ 2\alpha^2[1-(2-2^{-d/2})(t^*/t)^{d/2}] & \text{for } t\gg t^*.\end{cases} \tag{48}$$

Therefore, t^* is the crossover time separating the time regime $t\ll t^*$ with $\langle\delta\Psi^2(t)\rangle\sim O(1/V)$ from the time regime $t\gg t^*$ with $\langle\delta\Psi^2(t)\rangle\sim O((t^*)^{-d/2})\sim O(1)$.

We may now comment on the noncommutativity of the limits. If V is kept finite and the limit $t\to\infty$ is taken first, the system equilibrates and the two models at equilibrium are equivalent above T_C but not below. This conclusion remains valid if, after having reached equilibrium, the limit $V\to\infty$ is taken. If, instead, the limit $V\to\infty$ is taken first, then $\langle\delta\Psi^2\rangle\sim 1/V$ for quenches both above and below T_C, since in the latter case the crossover time t^* diverges. Hence, the two infinite volume models are always equivalent during relaxation. Therefore, as stated in the Introduction, all dynamical quantities are the same in the two models if the $V\to\infty$ limit is taken from the outset.

V. CONCLUDING REMARKS

We have analyzed the relationship between the spherical and the mean-spherical models, at equilibrium and during relaxation from an initial high-temperature state to a lower-temperature state. By monitoring the behavior of the fluctuations of Ψ we have found that the two models are equivalent in a quench above T_C, while discrepancies arise in a quench below T_C if the volume is kept finite and the time is larger than the crossover time $t^*\sim V^{2/d}$. In the infinite volume limit t^* diverges and the relaxation dynamics is the same in the two models for all times. In particular, the integrated autoresponse function

$$\chi(t,t_w)=\int_{t_w}^{t}dt'\int\frac{d^dk}{(2\pi)^d}R(\vec{k},t,t')e^{-k^2/\Lambda^2} \tag{49}$$

is expected to have the same form in both models. As anticipated in the Introduction, this poses an interesting problem when considering the connection between statics and dynamics.

In order to explain this it is necessary to expand somewhat on the behavior of the autocorrelation and response functions in a phase ordering process [9,10,13]. The basic feature is the split of both these quantities into the sum of a stationary and an aging contribution:

$$C(t,t_w)=C_{st}(t-t_w)+C_{ag}(t/t_w), \tag{50}$$

$$\chi(t,t_w)=\chi_{st}(t-t_w)+\chi_{ag}(t,t_w), \tag{51}$$

respectively, due to thermal fluctuations and defect dynamics. The time scales of these contributions are widely separated. In particular $C_{st}(t-t_w)$ decays rapidly from $M_0^2-M^2$ to zero (where M is the spontaneous magnetization at the temperature of the quench and M_0 is the zero temperature spontaneous magnetization) while $C_{ag}(t/t_w)$ decays from M^2 to zero on a much longer time scale. The stationary terms in Eqs. (50) and (51) are related by the equilibrium fluctuation dissipation theorem

$$T\chi_{st}(t-t_w)=C_{st}(t,t)-C_{st}(t-t_w). \tag{52}$$

Rewriting the right hand side in terms of the full autocorrelation function one finds [9,10]

$$T\chi_{st}(t-t_w)=\begin{cases} M_0^2-C(t,t_w) & \text{for } M^2\leqslant C\leqslant M_0^2 \\ M_0^2-M^2 & \text{for } C\leqslant M^2, \end{cases} \tag{53}$$

which yields

$$-T\frac{\partial^2\chi_{st}(C)}{\partial C^2}\bigg|_{C=q}=\delta(q-M^2). \tag{54}$$

The next statement is that the aging contribution of the response function obeys the scaling form

$$\chi_{ag}(t,t_w)=t_w^{-a}\hat{\chi}(t/t_w) \tag{55}$$

and therefore vanishes for $t_w\to\infty$ if $a>0$. As of yet knowledge of the exponent a remains limited. According to heuristic arguments [9,12] a ought to coincide with the exponent θ controlling the defect density $\rho(t)\sim t^{-\theta}$, namely, $\theta=1/2$ for a scalar order parameter and $\theta=1$ for a vector order parameter [13], independent of dimensionality. However, exact analytical results for the one-dimensional Ising model [14,15], careful numerical computations [10] for the Ising model in dimensions $d=2,3,4$, and exact analytical results for the large N model [16] (which is equivalent to the mean-spherical model) give the nontrivial behavior

$$a=\begin{cases} \theta\left(\dfrac{d-d_L}{d_U-d_L}\right) & \text{for } d<d_U \\ \theta & \text{for } d>d_U \end{cases} \tag{56}$$

with logarithmic corrections at $d=d_U$. Here $d_L=1$ and $d_U=3$ for the Ising model, while $d_L=2$ and $d_U=4$ in the large N case. What happens at $d=d_L$, where $a=0$, has been analyzed in detail in [15,10,16]. Let us here consider $d>d_L$, where $a>0$ and the aging contribution of the response function does asymptotically vanish. Then, putting together Eqs. (3) and (54) one finds [16,17]

$$\tilde{P}(q)=\delta(q-M^2). \tag{57}$$

Now, as stated in the Introduction, if stochastic stability holds $\tilde{P}(q)$ equals the overlap probability function $P(q)$ of the unperturbed system, up to the effects of global symmetries which might be removed by the perturbation. This is what usually happens in a phase ordering system, like the Ising model, where the perturbation breaks the up-down symmetry and one has

$$\tilde{P}(q)=2\theta(q)P(q). \tag{58}$$

Even if only one-half of the states are kept in the order parameter function $\tilde{P}(q)$ obtained from dynamics, by using the symmetry $P(q)=P(-q)$ it is obviously possible to recover the full unperturbed overlap function. Indeed, this is what takes place in the spherical model, where the unperturbed overlap function (45) is given by the sum of two δ functions and $\tilde{P}(q)=2\theta(q)P_s(q)$ holds. Not so in the mean-spherical case, where the unperturbed overlap function (44) is nontrivial and $\tilde{P}(q)\neq 2\theta(q)P_{ms}(q)$ as can be seen at a glance from Fig. 1. Clearly, in the latter case $P_{ms}(q)$ cannot be reconstructed from knowledge of $\tilde{P}(q)$. Therefore, stochastic stability does not hold in the mean-spherical model. It might be interesting to investigate this point in other models treated with the spherical constraint.

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APPENDIX A

In order to determine $Y(t)$ explicitly, let us rewrite Eq. (17) by separating out of the sum the $\vec{k}=\vec{0}$ term, as we expect it to become macroscopically occupied at low temperature:

$$\frac{C(\vec{k}=0,t)}{V}+\frac{1}{V}\sum_{\vec{k}\neq\vec{0}}C(\vec{k},t)=\alpha. \tag{A1}$$

For large volume the sum may be approximated by an integral and using Eqs. (19) and (20) we find

$$\alpha Y^2(t)-\frac{2T}{V}\int_0^t dt'\,Y^2(t')-2T\int_0^t dt'\,f\left(t-t'+\frac{1}{2\Lambda^2}\right)Y^2(t')=\Delta\left[\frac{1}{V}+f\left(t+\frac{1}{2\Lambda^2}\right)\right], \tag{A2}$$

where

$$f\left(t+\frac{1}{2\Lambda^2}\right)=\int\frac{d^dk}{(2\pi)^d}e^{-2k^2(t-1/2\Lambda^2)}=\left[8\pi\left(t+\frac{1}{2\Lambda^2}\right)\right]^{-d/2}. \tag{A3}$$

The above equation can be solved by Laplace transformation, obtaining

$$\mathcal{L}(z)=\frac{\Delta[1/V+zh(z)]}{\alpha z-2T[1/V+zh(z)]}, \quad (A4)$$

where $\mathcal{L}(z)$ and $h(z)$ are the Laplace transforms of $Y^2(t)$ and $f(t+1/2\Lambda^2)$, respectively. The large time behavior of $Y^2(t)$ is controlled by the small z behavior of $\mathcal{L}(z)$. For $2<d<4$ we have

$$h(z)=K+\gamma z^{d/2-1}+O(z), \quad (A5)$$

where $K=(4\pi)^{-d/2}\Lambda^{d-2}/(d-2)$ and $\gamma=-(8\pi)^{-d/2}\Gamma(1-d/2)$ are positive constants. Inserting them in Eq. (A4) we have

$$\mathcal{L}(z)=\frac{\Delta[1/V+Kz+\gamma z^{d/2}]}{\alpha(1-T/T_C)z-2T\gamma z^{d/2}-2T/V}, \quad (A6)$$

where $T_C=\alpha/2K=(4\pi)^{d/2}\alpha(d-2)/2\Lambda^{d-2}$. Inverting the Laplace transform we have

$$Y^2(t)=Be^{t/\tau}+I(t), \quad (A7)$$

where the first contribution comes from the residue at the single pole at $x_0=1/\tau$ on the positive real axis and

$$I(t)=\frac{1}{2\pi i}\int_0^\infty dx[\mathcal{L}(xe^{-i\pi})-\mathcal{L}(xe^{i\pi})]e^{-xt} \quad (A8)$$

is the contribution from the cut along the negative real axis. Looking for the zero of the denominator of Eq. (A4) we find

$$\tau=\begin{cases}\left[\dfrac{\alpha}{2T_C\gamma}(1-T/T_C)\right]^{2/(2-d)} & \text{for } 0<\dfrac{T-T_C}{T_C}\ll 1\\ (\gamma V)^{2/d} & \text{for } T=T_C\\ \dfrac{\alpha}{2T}(1-T/T_C)V & \text{for } T<T_C,\end{cases} \quad (A9)$$

and computing the residue

$$B=\Delta\alpha\{2T\tau[\alpha(d/2-1)(T/T_C-1)+dT\tau/V]\}^{-1}. \quad (A10)$$

For $T>T_C$ the exponential dominates and the contribution from the cut can be neglected in Eq. (A7), since to leading order B and τ are independent of the volume. Not so for $T\leqslant T_C$, where taking into account the contribution from the cut we have

$$Y^2(t)=B\{e^{t/\tau}+(t/t^*)^{-\omega}[1\pm(t/\tilde{t})^{\phi}]\} \quad (A11)$$

with

$$\omega=\begin{cases}2-d/2 & \text{for } T=T_C\\ d/2 & \text{for } T<T_C,\end{cases} \quad (A12)$$

$$\phi=\begin{cases}d/2 & \text{for } T=T_C\\ 1 & \text{for } T<T_C,\end{cases} \quad (A13)$$

$$B=\begin{cases}\dfrac{\Delta V^{1-4/d}}{2T_C^2\gamma^{4/d}} & \text{for } T=T_C\\ \dfrac{\Delta}{\alpha V}(1-T/T_C)^{-2} & \text{for } T<T_C,\end{cases} \quad (A14)$$

$$\tilde{t}=\begin{cases}\dfrac{1}{8\pi}\left[\dfrac{(d/2-1)\Gamma^2(2-d/2)V}{2\cos(d\pi/2)\Gamma(2-d/2)}\right]^{2/d} & \text{for } T=T_C\\ \left[\dfrac{\alpha(1-T/T_C)\Gamma(d/2)}{4T\Gamma(d/2-1)V}\right] & \text{for } T<T_C,\end{cases} \quad (A15)$$

$$t^*=\begin{cases}[(d-2)^2\Delta^{-2}(8\pi)^{d-2}\gamma^{8/d}]^{1/(d-4)}V^{2/d} & \text{for } T=T_C\\ \dfrac{V^{2/d}}{8\pi} & \text{for } T<T_C,\end{cases} \quad (A16)$$

and where in Eq. (A11) the + and − signs apply to $T=T_C$ and $T<T_C$, respectively. Notice that in all cases $t^*\leqslant\tau$ and $\tilde{t}\geqslant t^*$; therefore the dominant contribution is given by

$$Y^2(t)\sim\begin{cases}t^{-\omega} & \text{for } t<t^*\\ e^{t/\tau} & \text{for } t>t^*.\end{cases} \quad (A17)$$

APPENDIX B

Inserting Eq. (29) into the definition (40) we obtain for the overlap function in the mean-spherical model at $T=0$,

$$P_{\mathrm{ms}}(q)=\frac{1}{2\pi\alpha V^2}\int d\varphi_0 d\varphi_0' e^{-(\varphi_0^2+\varphi_0'^2)/2\alpha V^2}\delta\left(\frac{\varphi_0\varphi_0'}{V^2}-q\right). \quad (B1)$$

The corresponding characteristic function is given by

$$\Theta_{\mathrm{ms}}(\lambda)=\frac{1}{2\pi\alpha V^2}\int d\varphi_0 d\varphi_0' \exp\left\{-\frac{1}{2\alpha V^2}(\varphi_0^2+\varphi_0'^2-2i\alpha\lambda\varphi_0\varphi_0')\right\} \quad (B2)$$

and, going over to polar integration variables, one finds

$$\Theta_{\mathrm{ms}}(\lambda)=\frac{1}{2\pi\alpha V^2}\int_0^{2\pi}d\vartheta\int_0^\infty dr r \times\exp\left\{-\frac{r^2}{2\alpha V^2}(1-2i\alpha\lambda\sin\vartheta\cos\vartheta)\right\} =\frac{1}{\pi}\int_0^{2\pi}\frac{d\vartheta}{2-\alpha\lambda e^{i\vartheta}+\alpha\lambda e^{-i\vartheta}}. \quad (B3)$$

This integral can be rewritten in the form $(1/i\pi)\oint_\gamma dz/(2z-\alpha\lambda z^2+\alpha\lambda)$ where γ is a circle of radius 1 with center at the origin of the complex plane. Since there is a simple pole at $z_0=(1-\sqrt{1+\alpha^2\lambda^2})/\alpha\lambda$ inside γ we obtain

$$\Theta_{\mathrm{ms}}(\lambda)=\frac{1}{\sqrt{1+(\lambda\alpha)^2}} \tag{B4}$$

and inverting the Fourier transform we find

$$P_{\mathrm{ms}}(q)=\frac{1}{2\pi\alpha}\int_{-\infty}^{\infty}dx\frac{e^{-i(q/\alpha)x}}{\sqrt{1+x^2}}. \tag{B5}$$

Taking into account that there are branch points at $\pm i$ and that the integration contour is closed in the negative imaginary half plane for $q>0$ and vice versa for $q<0$, eventually we obtain

$$P_{\mathrm{ms}}(q)=\frac{1}{\pi\alpha}\int_{1}^{\infty}dy\frac{e^{-(|q|/\alpha)y}}{\sqrt{y^2-1}}=\frac{1}{\pi\alpha}K_0\left(\frac{|q|}{\alpha}\right) \tag{B6}$$

where K_0 is a Bessel function of imaginary argument.

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