

Nondeterministic density classification with diffusive probabilistic cellular automata

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We present a probabilistic cellular automaton (CA) with two absorbing states which performs classification of binary strings in a nondeterministic sense. In a system evolving under this CA rule, empty sites become occupied with a probability proportional to the number of occupied sites in the neighborhood, while occupied sites become empty with a probability proportional to the number of empty sites in the neighborhood. The probability that all sites become eventually occupied is equal to the density of occupied sites in the initial string.

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I. INTRODUCTION

Cellular automata (CA) and other spatially extended discrete dynamical systems are often used as models of complex systems with a large number of locally interacting components. One of the primary problems encountered in constructing such models is the inverse problem: the question of how to find a local CA rule which would exhibit the desired global behavior.

As a typical representative of the inverse problem, the so-called density classification task [1] has been extensively studied in recent years. The CA performing this task should converge to a fixed point of all 1's if the initial configuration contains more 1's than 0's, and to a fixed point of all 0's if the converse is true. While it has been proved [2] that the two-state rule performing this task does not exist, solutions of modified tasks are possible if one allows more than one CA rule [3], modifies specifications for the final configuration [4], or assumes different boundary condition [5]. Approximate solutions have been studied in the context of genetic algorithms in one [6] and two dimensions [7].

In what follows, we will define a probabilistic CA which solves the density classification problem in the stochastic sense, meaning that the probability that all sites become eventually occupied is equal to the density of occupied sites in the initial string.

We will assume that the dynamics takes place on a one-dimensional lattice with periodic boundary conditions. Let $s_i(k)$ denotes the state of the lattice site i at time k , where $i \in \mathbf{Z}$, $k \in \mathbf{N}$. All operations on spatial indices i are assumed to be modulo L , where L is the length of the lattice. We will further assume that $s_i(k) \in \{0,1\}$, and we will say that the site i is occupied (empty) at time k if $s_i(k) = 1$ [$s_i(k) = 0$].

The dynamics of the system can be described as follows: empty sites become occupied with a probability proportional to the number of occupied sites in the neighborhood, while occupied sites become empty with a probability proportional to the number of empty sites in the neighborhood, with all lattice sites updated simultaneously at each time step. To be more precise, let us denote by $P[s_i(k+1)|s_{i-1}(k), s_i(k), s_{i+1}(k)]$ the probability that the site

$s_i(k)$ with nearest neighbors $s_{i-1}(k), s_{i+1}(k)$ changes its state to $s_i(k+1)$ in a single time step. The following set of transition probabilities defines the aforementioned CA rule:

$$\begin{aligned} P(1|0,0,0) &= 0, & P(1|0,0,1) &= p, \\ P(1|0,1,0) &= 1-2p, & P(1|0,1,1) &= 1-p, \\ P(1|1,0,0) &= p, & P(1|1,0,1) &= 2p, \\ P(1|1,1,0) &= 1-p, & P(1|1,1,1) &= 1, \end{aligned} \quad (1)$$

where $p \in (0, 1/2)$ [the remaining eight transition probabilities can be obtained using $P(0|a,b,c) = 1 - P(1|a,b,c)$ for $a, b, c \in \{0,1\}$]. The probabilistic CA defined by Eq. (1) can be defined explicitly if we introduce a set of iid random variables $\{X_i\}_{i=0}^L$ with probability distribution $P(X_i=1) = p$, $P(X_i=0) = 1-p$, and another set $\{Y_i\}_{i=0}^L$ with probability distribution $P(Y_i=1) = 2p$, $P(Y_i=0) = 1-2p$. Dynamics of the rule (1) can then be described as

$$\begin{aligned} s_i(k+1) &= X_i(1-s_{i-1})(1-s_i)s_{i+1} + (1-Y_i)(1-s_{i-1})s_i \\ &\quad \times (1-s_{i+1}) + (1-X_i)(1-s_{i-1})s_i s_{i+1} + X_i s_{i-1} \\ &\quad \times (1-s_i)(1-s_{i+1}) + Y_i s_{i-1}(1-s_i)s_{i+1} \\ &\quad + (1-X_i)s_{i-1}s_i(1-s_{i+1}) + s_{i-1}s_i s_{i+1}. \end{aligned} \quad (2)$$

To make the above formula easier to read, we omitted the time argument, denoting $s_i(k)$ by s_i . After simplification and reordering of terms, we obtain

$$\begin{aligned} s_i(k+1) &= s_i - s_i Y_i + X_i s_{i-1} + X_i s_{i+1} + (s_{i-1} s_i + s_i s_{i+1} \\ &\quad - 2s_{i-1} s_i s_{i+1} + s_{i-1} s_{i+1})(Y_i - 2X_i). \end{aligned} \quad (3)$$

II. DIFFERENCE AND DIFFERENTIAL EQUATIONS

The state of the system at the time k is determined by the states of all lattice sites and is described by the Boolean random field $\mathbf{s}(k) = \{s_i(k) : i=0, \dots, L\}$. The Boolean field $\{\mathbf{s}(k) : k=0, 1, 2, \dots\}$ is then a Markov stochastic process. Denoting by $E_{\mathbf{s}(0)}$ the expectation of this Markov process when the initial configuration is $\mathbf{s}(0)$ we will now define the expected local density of occupied sites by $\rho_i(k)$

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$=E_{s(0)}[s_i(k)]$. The expected global density will be defined as

$$\rho(k) = L^{-1} \sum_{i=0}^L \rho_i(k). \quad (4)$$

While both $\rho_i(k)$ and $\rho(k)$ depend on the initial configuration $s(0)$, we will drop this dependence to simplify notation. We will assume that the initial configuration is exactly known (nonstochastic), hence $\rho(0) = \sum_{i=0}^L s_i(0)$ is the fraction of initially occupied sites.

Taking expectation value of both sides of Eq. (3), and taking into account that $E_{s(0)}[Y_i - 2X_i] = 0$, we obtain the following difference equation

$$\rho_i(k+1) = \rho_i(k) + p(\rho_{i+1}(k) + \rho_{i-1}(k) - 2\rho_i(k)). \quad (5)$$

After summing over all lattice sites this yields

$$\rho(k+1) = \rho(k), \quad (6)$$

which means that the expected global density should be constant, independently of the value of parameter p and independently of the initial configuration $s(0)$. We can therefore say that the probabilistic CA defined in Eq. (2) is analogous to *conservative* CA, i.e., deterministic CA which preserve the number of occupied sites [8–11].

Note that up to now we have not made any approximations, i.e., both Eqs. (5) and (6) are exact. We can, however, consider limiting behavior of Eq. (5) when the physical distance between lattice sites and the size of the time step simultaneously go to zero, using a similar procedure as described in Ref. [12]. Let $x = \epsilon i$ and $t = \tau k$. Now in Eq. (5) we can replace $\rho(i, k)$ by $\rho(x, t)$, $\rho(i \pm 1, k)$ by $\rho(x \pm \epsilon, t)$, and $\rho(i, k+1)$ by $\rho(x, t + \tau)$, which results in the following equation:

$$\rho(r, t + \tau) = \rho(x, t) + p(\rho(x + \epsilon, t) + \rho(x - \epsilon, t) - 2\rho(x, t)).$$

We will consider diffusive scaling in which time scales as a square of the spatial length, meaning that $\tau = \epsilon^2$. Taking Taylor expansion of the above equation in powers of ϵ up to the second order we obtain

$$\partial_t \rho = p \partial_x^2 \rho, \quad (7)$$

i.e., the standard diffusion equation. Due to the form of Eq. (7), in what follows we will refer to the process defined in Eq. (2) as *diffusive probabilistic cellular automaton* (DPCA).

III. ABSORPTION PROBABILITY

We will now present some simulation results illustrating dynamics of DPCA. Since main features of DPCA remain qualitatively the same for all values of the parameter p in the interval $(0, 1/2)$, we have chosen $p = 0.25$ as a representative value to perform all subsequent simulations.

Let $N(k) = \sum_{i=1}^L s_i(k)$ be the number of occupied sites at time k . If we start with $N(0) = 0$, then $N(k) = 0$ for all $k > 0$. Similarly, if $N(0) = L$, then $N(k) = L$ for all $k > 0$. The

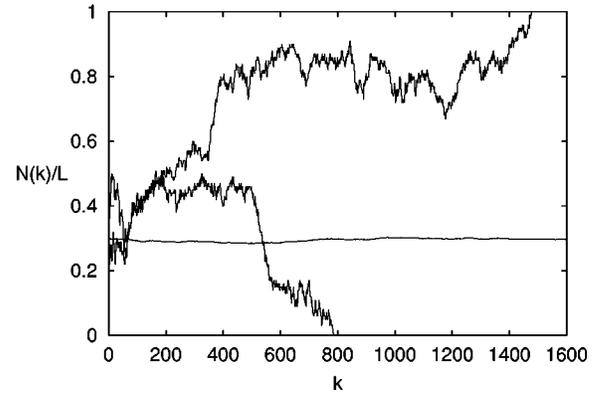


FIG. 1. Fraction of occupied sites $N(k)/L$ as a function of time k for two sample trajectories starting from identical initial configuration with $N(0) = 30$, $L = 100$, $p = 0.3$. The third, almost horizontal line represents average of 1000 such trajectories.

DPCA has thus two absorbing states, corresponding to all empty sites (to be referred to as **0**) and to all occupied sites (to be referred to as **1**). If we start with $0 < N(0) < L$, then the graph of $N(k)$ resembles a random walk, as shown in Fig. 1. Both sample trajectories shown there eventually end in the absorbing state, one of them in **0**, another one in **1**. This is a general property of the DPCA: regardless of the initial configuration, the system sooner or later ends up in one of the two absorbing states. Although the time required to reach the absorbing state can be large for a given realization of the process, the expected value of the number of time steps required to reach the absorbing state is finite, as it is the case for all finite absorbing Markov chains [13]. Figure 2 illustrates this property for $L = 100$ and the initial configuration with 30 occupied sites clustered around the center, i.e., located at $i = 35, 36, \dots, 64$. All other sites are empty. We start with an assembly of 200 of such initial configurations, all plotted as vertical lines in which black pixels represent occupied sites, while white pixels represent empty sites, as in Fig. 2(a). Each of these initial configurations evolves according to the DPCA rule, and after $k = 100$ ($k = 1000$) iterations they are again plotted as 200 vertical lines in Figs. 2(b) and 2(c). After 12 000 iterations all 200 configurations reach absorbing states, as illustrated in Fig. 2(d). Obviously, some reach the state **1**, while others **0**, yet it turns out that the fraction of configurations which ended up in the state **1** is very close to 30%, the same as the fraction of occupied sites at $k = 0$.

To explain this phenomenon, let us define $u_N(k)$ to be the probability that the number of occupied sites at time k is N . Since the Markov process $\{s(k) : k = 0, 1, 2, \dots\}$ is finite and absorbing, no matter where the process starts, the probability that after k steps it is in an absorbing state tends to 1 as k tends to infinity [13]. This implies that

$$\lim_{k \rightarrow \infty} u_N(k) = 0 \text{ if } N \neq 0 \text{ and } N \neq L, \quad (8)$$

$$\lim_{k \rightarrow \infty} (u_L(k) + u_0(k)) = 1. \quad (9)$$

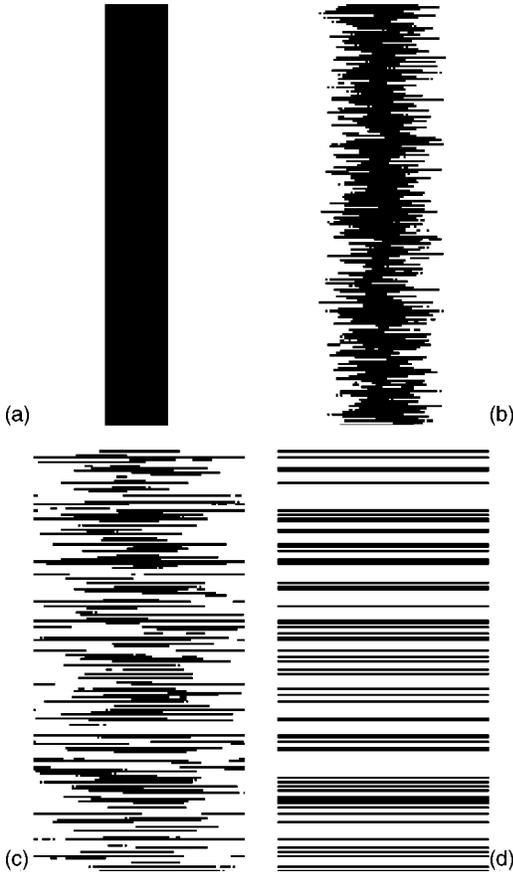


FIG. 2. Multiple realizations of the CA evolution at (a) $k=0$, (b) $k=100$, (c) $k=1000$, and (d) $k=12\,000$. Each vertical line corresponds to different realizations of the process on $L=100$ lattice. Black pixels represent occupied sites and white pixels empty sites. 200 different realizations of the process are shown.

The expected global density, as defined in Eq. (4), is independent of k , hence

$$\rho(0) = L^{-1} E_{s(0)}[N(k)] = L^{-1} \sum_{N=1}^L N u_N(k). \quad (10)$$

Taking the limit $k \rightarrow \infty$ of both sides of the above equation, and using Eqs. (8) and (9), we obtain

$$\lim_{k \rightarrow \infty} u_L(k) = \rho(0), \quad (11)$$

$$\lim_{k \rightarrow \infty} u_0(k) = 1 - \rho(0). \quad (12)$$

We have shown that *the probability that the DPCA reaches the absorbing state $\mathbf{1}$ is equal to the initial fraction of occupied sites $\rho(0)$. The probability that it reaches $\mathbf{0}$ is $1 - \rho(0)$* . This is in agreement with the behavior observed in Fig. 2.

The above can be viewed as a probabilistic generalization of the density classification process. In the standard (deterministic) version of the density classification problem we

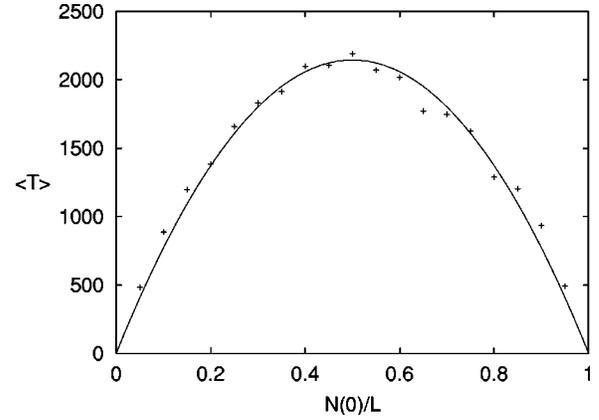


FIG. 3. The average time required to reach the absorbing state $\langle T \rangle$ as a function of the initial density $\rho(0) = N(0)/L$ for $L=100$, $p=0.3$. Each data point (+) represents the average taken over 1000 realizations of the process with identical initial configurations. The continuous line represents fitted parabola $\langle T \rangle = \alpha \rho(0)(1 - \rho(0))$.

seek a CA rule which would converge to $\mathbf{1}$ ($\mathbf{0}$) if the fraction of occupied sites in the initial string is greater (less) than $1/2$, i.e.,

$$\lim_{k \rightarrow \infty} u_L(k) = \Theta(\rho(0)), \quad (13)$$

$$\lim_{k \rightarrow \infty} u_0(k) = \Theta(1 - \rho(0)), \quad (14)$$

where $\Theta(\cdot)$ is the step function defined as $\Theta(x) = 0$ if $x \leq 1/2$, and $\Theta(x) = 1$ if $x > 1/2$. Thus the difference between the deterministic and the probabilistic density classification introduced here is the replacement of the step function in Eqs. (13)–(14) by the identity function in Eqs. (11)–(12).

As opposed to deterministically decided outcome in the standard density classification process, in DPCA it is just *more probable* that the system reaches $\mathbf{1}$ then $\mathbf{0}$ if the fraction of occupied sites in the initial string is greater than $1/2$, and it is *more probable* that it reaches $\mathbf{0}$ then $\mathbf{1}$ if the converse is true. Additionally, DPCA can in some sense *measure* concentration of occupied sites in the initial string. If we want to know what is the initial density of occupied sites, we need to run DPCA many times with the same initial condition until it reaches the absorbing state, and observe how frequently it reaches $\mathbf{1}$. This frequency will approximate $N(0)/L$, with accuracy increasing with the number of experiments.

IV. TIME TO ABSORPTION

Simulations shown in Fig. 2 indicate that for large k the system is typically in a state in which blocks of both empty and occupied sites are relatively long. We can use this observation to obtain the approximate dependence of the time required to reach the absorbing state on the density of initial configuration.

If we assume that in a given configuration all occupied sites are grouped in a few long continuous blocks, then the value of $N(k)$ cannot change too much in a single time step.

For simplicity, let us assume that the only allowed values of $\Delta N(k) = N(k+1) - N(k)$ are $\{-2, -1, 0, 1, 2\}$.

Since $\rho(k)$ is time independent, the probability that $N(k)$ increases by a given amount should be equal to the probability that it decreases by the same amount in a single time step. In the agreement with the above, let us define p_j as the probability that $\Delta N(k)$ takes the value j , so that p_j is non-zero for $j \in \{-2, -1, 0, 1, 2\}$, where $p_{-2} = p_2$, $p_{-1} = p_1$, and $2p_2 + 2p_1 + p_0 = 1$. If T_z denotes the expected time to reach the state $N(k) = 0$ or $N(k) = L$ starting from $N(0) = z$, a simple argument [14] yields the difference equation which T_z must satisfy

$$T_z = p_2 T_{z+2} + p_1 T_{z+1} + p_0 T_z + p_1 T_{z-1} + p_2 T_{z-2} + 1. \quad (15)$$

The solution of this equation satisfying boundary conditions $T_0 = 0$ and $T_L = 0$ is given by

$$T_z = \alpha z(L-z) = \alpha L^2 \rho(0)(1-\rho(0)), \quad (16)$$

where $\alpha = 1/(8p_2 + 4p_1)$, meaning that the mean time to absorption scales with lattice length as $O(L^2)$. The above result would remain valid even if we allowed further jumps than ± 2 (although the form of the coefficient α would be different).

In order to verify if this result holds for the DPCA, we performed a series of numerical experiments, computing the

average time required to reach the absorbing state for 1000 realizations of the DPCA process, for a range of initial densities. Results are shown in Fig. 3. One can clearly see that data points are aligned along a curve of parabolic shape, as expected from Eq. (16).

V. CONCLUSION

The probabilistic CA introduced in this article solves the density classification problem in a nondeterministic sense. It is interesting to note that the DPCA conserves the average number of occupied sites, similarly as deterministic rules employed in solutions of related problems mentioned in the introduction. Indeed, conservation of the number of occupied sites is a necessary condition for density classification by CA if one allows modified output configuration, as recently shown in Ref. [15]. This suggests that a wider class of probabilistic CA conserving $\rho(k)$ might be a useful paradigm in studying how locally interacting systems compute global properties, and certainly deserves further attention.

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