Exponential velocity tails in a driven inelastic Maxwell model

T. Antal,[1,2,*] Michel Droz,[1,†] and Adam Lipowski[1,3,‡]

[1]*Department of Physics, University of Geneva, CH 1211 Geneva 4, Switzerland*
[2]*Institute for Theoretical Physics, Eötvös University, Pázmány sétány 1/a, 1117 Budapest, Hungary*
[3]*Faculty of Physics, Adam Mickiewicz University, 61-614 Poznań, Poland*

The problem of the steady-state velocity distribution in a driven inelastic Maxwell model of shaken granular material is revisited. Numerical solution of the master equation and analytical arguments show that the model has bilateral exponential velocity tails $[P(v) \sim e^{-|v|/\sqrt{D}}]$, where D is the amplitude of the noise. Previous study of this model predicted Gaussian tails $[P(v) \sim e^{-av^2}]$.

Recently, granular systems have been intensively studied [1,2]. One reason of such an interest is the fact that in many respects these systems are highly unconventional and even exotic [3]. As a result, very often even standard laws of statistical physics require certain modification when applied to granular systems.

As an example, let us consider the Maxwell-Boltzmann law for velocity distribution $P(v)$ of atoms or molecules in a gaseous state, which states that $P(v) \sim e^{-av^2}$. Under certain experimental conditions, a granular system can be considered as a gas and a natural question is what is its velocity distribution. Numerous theoretical and experimental studies do not provide a simple answer to this question. On the contrary, they show that $P(v)$ depends on certain details of the system as, for example, how energy is transferred by a thermostat into the system, in order to balance the energy loss during inelastic collisions. Theoretical works have shown that $P(v)$ might be of the form e^{-av^2} [4,5], $e^{-av^{3/2}}$ [6,7], or e^{-av} [8]. Under certain conditions experiments show clear deviations from the Maxwell-Boltzmann law, but it is still rather difficult to decide what is the form of $P(v)$ in real granular systems [9].

Very often granular systems are described using the so-called inelastic hard-sphere models, which could be then analyzed using corresponding Boltzmann equations. These off-lattice two- or three-dimensional systems are, however, very difficult to study, especially when we want to explore large-velocity regions of the phase space. A possible alternative is to construct simplified models for which more accurate calculations are possible. One class of such models are Maxwell models for which the collision term in the corresponding Boltzmann equation is velocity independent. Recent studies show that for Maxwell models $P(v)$ might take either exponential (e^{-av}) [10] or Gaussian (e^{-av^2}) form [11,12].

A particularly interesting Maxwell model was proposed by Ben-Naim and Krapivsky (BNK) [12]. These authors presented an elegant solution of the master equation of their model using a Fourier transform method. However, their conclusion that $P(v)$ has a Gaussian decay is based on certain approximation whose validity is difficult to assess. Namely, they infer the large-velocity behavior of $P(v)$ from the low-k behavior of the Fourier transform $\hat{P}(k)$. Since BNK model is one of the very few models for which exact or numerical but very precise calculations can be made, it would be desirable to clarify the validity of this approach.

In the present paper, we reexamine the BNK model. Analyzing numerically the solution of the master equation of the model, we obtain the velocity distribution $P(v)$. Asymptotically ($v \to \infty$), this quantity shows bilateral exponential tails $[P(v) \sim e^{-a|v|}]$ and such a behavior is seen over more than ten decades. We also present analytical arguments that support existence of bilateral exponential tails in this model. From our analysis, it follows that $a = 1/\sqrt{D}$, where D is the amplitude of noise which simulates the input of energy into the system.

To introduce the model, we consider a collection of particles that are characterized by a single parameter, their velocity v. In this model, positions of particles are not specified hence the model neglects any spatial correlations. Particles undergo two-body inelastic collisions that change their velocities according to $(v_1, v_2) \to (v_1', v_2')$, where

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ 1-\gamma & \gamma \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{1}$$

and γ is the inelasticity parameter ($0 < \gamma < 1$). Particles that participate in a collision are chosen at random. In addition to the collision rule (1), particles are subjected to the uncorrelated white noise of strength D. Existence of noise ensures that the model has a well-defined nontrivial steady state. In the steady state, the velocity distribution $P(v)$ satisfies the following equation [12]:

$$-DP''(v) = -P(v) + \frac{1}{1-\gamma} \int_{-\infty}^{+\infty} du\, P(u) P\left(\frac{v-\gamma u}{1-\gamma}\right), \tag{2}$$

where double prime $''$ denotes the second derivative with respect to velocity. To solve Eq. (2), BNK introduced the

*Electronic address: Tibor.Antal@physics.unige.ch
†Electronic address: Michel.Droz@physics.unige.ch
‡Electronic address: lipowski@amu.edu.pl

Fourier transform of the velocity distribution $\hat{P}(k) = \int dv\, e^{ikv} P(v)$, which in the steady state satisfies

$$(1 + Dk^2)\hat{P}(k) = \hat{P}[\gamma k]\hat{P}[(1 - \gamma)k]. \qquad (3)$$

This equation admits the following solution:

$$\hat{P}(k) = \prod_{i=0}^{\infty} \prod_{j=0}^{i} [1 + \gamma^{2j}(1 - \gamma)^{2(i-j)} Dk^2]^{-\binom{i}{j}}. \qquad (4)$$

After some transformations, Eq. (4) can be written as

$$\hat{P}(k) = \exp\left[\sum_{n=1}^{\infty} \frac{(-Dk^2)^n}{n a_{2n}(\gamma)}\right], \qquad (5)$$

where $a_n(\gamma) = 1 - (1 - \gamma)^n - \gamma^n$. To obtain a large-velocity behavior of $P(v)$, BNK truncate the series in Eq. (5) keeping only the first, quadratic in k, term

$$\hat{P}(k) \approx \exp\left[\frac{-Dk^2}{a_2(\gamma)}\right]. \qquad (6)$$

Subsequently, taking the inverse Fourier transform of this Gaussian function they obtain $P(v) \sim \exp(-v^2/2v_0^2)$, where $v_0^2 = 2D/a_2(\gamma)$.

In our opinion, the above procedure of truncating the series and inverting the resulting function is not well justified. The approximation (6) agrees with the exact expression (5) only up to the k^2 term of the Taylor expansion. Equally consistent approximation can be given as

$$\hat{P}(k) \approx \frac{c^2}{c^2 + k^2}, \qquad (7)$$

where $c = \sqrt{a_2(\gamma)/D}$. But the inverse Fourier transform of Eq. (7) is $(c/2)\exp(-c|v|)$, which is qualitatively different than the Gaussian decay obtained by BNK. Both Eqs. (6) and (7) are consistent with the exact solution (5) up to the k^2 order, but differ at the higher-orders. Apparently, these higher-order terms qualitatively affect the large-velocity limit of $P(v)$. Consequently, without more detailed arguments such approximations are not justified. Indeed, as we will show below, $P(v)$ has a bilateral exponential decay but with a different coefficient c.

To check the validity of the BNK approach, we calculate $P(v)$ numerically as an inverse Fourier transform of $\hat{P}(k)$. One way to compute $\hat{P}(k)$ is to evaluate the infinite sums in the logarithm of Eq. (4). However, it turns out that using Eq. (5) leads to a better precision. Let us notice, however, that the series in Eq. (5) converges but only for $|k| < k_c = 1/\sqrt{D}$. To calculate $\hat{P}(k)$ for $|k| > k_c$, we can then use Eq. (3), provided that γk and $(1 - \gamma)k$ fall within the range of convergence. If not, we have to calculate $\hat{P}(\gamma k)$ and $\hat{P}((1 - \gamma)k)$ referring again to Eq. (3). Implementing this recursive procedure, we calculated $\hat{P}(k)$ and then using the fast Fourier transformation algorithm we obtained $P(v)$. Our results for $D = 1$ are shown in Fig. 1. This figure clearly shows
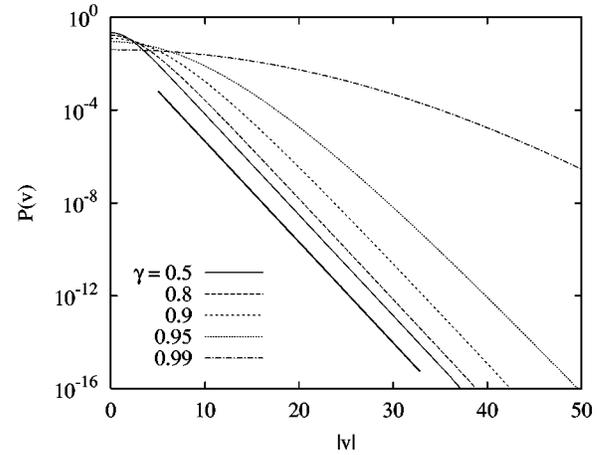
FIG. 1. The velocity distribution $P(v)$ as a function of v calculated for $D = 1$. Due to symmetry only $v > 0$ part is shown. The thick line represents the exponential function $\exp(-v)$.

that $P(v)$ decays exponentially. For $\gamma = 0.5$ and 0.8, such a behavior is seen for more than ten decades. For γ close to unity there is an $\exp(-av^2)$ decay for small velocities which asymptotically is replaced by the bilateral exponential decay $[\exp(-c|v|)]$. Moreover, as shown in Fig. 1, the asymptotic slope c is independent of γ and is approximately equal to 1. As we will show below this slope only depends on the amplitude of noise D. Before doing that let us notice that the D dependence of the velocity distribution $P(v)$ can be easily inferred from the fact that D and k enter its Fourier transform only through Dk^2 terms [see Eqs. (4) and (5)]. From this property, one can easily obtain that

$$P(v, D) = \frac{1}{\sqrt{D}} P\left(\frac{v}{\sqrt{D}}, D = 1\right), \qquad (8)$$

where we explicitly indicated the dependence on the noise D.

In the following, we provide some analytical arguments showing that $P(v)$ decays exponentially. Let us assume that in Eq. (2) the second term (gain) can be neglected. Then Eq. (2) simplifies to $DP''(v) = P(v)$ and the normalizable solution reads

$$P(v) \sim e^{-|v|/\sqrt{D}}. \qquad (9)$$

Let us note that the exponential part is γ independent (since γ enters the master equation only through the neglected gain term). Moreover, such a solution is in a very good agreement with the numerical calculations (see Fig. 1).

For more general models it is known that neglecting the gain term is justified for $v \to \infty$ when the resulting solution decays faster than exponentially [6,7]. Our solution (9) is thus a marginal case. However, we can show that for our model the gain term in the limit $v \to \infty$ indeed can be neglected. First, let us evaluate the gain term for the solution (9). Elementary integration for $D = 1$ and $v > 0$ gives

$$\frac{1}{1-\gamma}\int_{-\infty}^{+\infty}du\,\exp(-|u|)\exp\left[-\left|\frac{v-\gamma u}{1-\gamma}\right|\right]$$

$$=\frac{2}{2\gamma-1}\left[\gamma\exp\left(\frac{-v}{\gamma}\right)+(\gamma-1)\exp\left(\frac{-v}{1-\gamma}\right)\right]. \quad (10)$$

One can see that since $0<\gamma<1$, the gain term decays exponentially with v but faster than the solution (9) [i.e., a non-neglected loss term in Eq. (2)]. We expect that Eq. (9) is only an asymptotic ($v\to\infty$) solution of the master equation (2). Thus, for velocities $u\sim O(1)$ the distribution $P(u)$ deviates from the asymptotic form (9). This will modify the integral (10), but only in the vicinity of $u=0$ and $u=v/\gamma$. As we argue below, such a modification of $P(u)$ still leads to the gain term decaying faster than the loss term.

Indeed, the contribution around $u=0$ is a product of $P(u)$ and of the exponential term $\exp[(-v-\gamma u)/(1-\gamma)]$. Thus, for large v, a modification of $P(u)$ around $u=0$ will change our estimation (10), but only at the order of $\exp[-v/(1-\gamma)]$. Similarly, one can show that a modification of $P(u)$ around $u=v/\gamma$ will change Eq. (10) by a factor of the order of $\exp(-v/\gamma)$. Consequently, the gain term again decays faster than the solution $P(v)$, which justifies its neglect. Let us also notice that when γ approaches 0 or 1 the model becomes energy conserving and the bilateral exponential distribution (9) is no longer expected to hold. But it is easy to notice that in such cases the decay of the gain term matches the decay of the solution $P(v)$, and it cannot be neglected.

As a comment let us notice that knowing the second moment of $P(v)$ [12] enables us to calculate the effective temperature T of our systems defined as an averaged square velocity,

$$T=\langle v^2\rangle=\frac{D}{\gamma(1-\gamma)}. \quad (11)$$

To check the validity of our calculations, we also made Monte Carlo simulations of this model. Results, which are shown in Fig. 2, confirm the bilateral exponential decay of $P(v)$ although the accuracy is this time much lower. Our Monte Carlo data are rescaled in such a way that their variance matches that obtained using the Fourier transform method, and a very good agreement is seen even on the logarithmic scale.
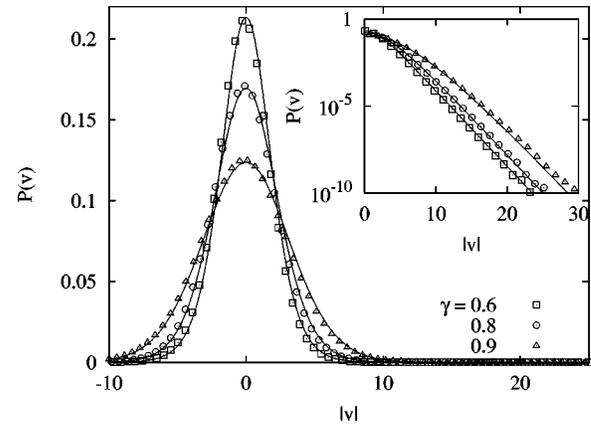
FIG. 2. Velocity distribution $P(v)$ calculated using Monte Carlo simulations. Simulations were made for $N=10^5$ particles. Continuous lines are the results obtained from Fourier inversion. The inset shows our data in the semilogarithmic scale.

In conclusion, we have shown that the Maxwell model proposed by Ben-Naim and Krapivsky has velocity distribution decaying as a bilateral exponential. Together with the recent results by Ernst and Brito [10], it indicates that such a decay might be of more generic nature for this class of models. As a possible extension, it would be desirable to examine some other Maxwell models where velocities are not scalars but rather d-dimensional vectors. Actually, such models were already studied and the analysis indicates that, for increasing d, correlations between the velocities and the deviation from the pure Gaussian distribution decrease [13]. One possibility is that there might be a critical dimension d_c and such that for $d<d_c$ the velocity distribution has a bilateral exponential decay (as in the present model), while it has Gaussian decay for $d\geq d_c$. Analysis of such models is, however, left for the future.

Note added. After our paper was completed, U. M. B. Marconi brought to our attention a Maxwell model of a granular mixture for which an exponential velocity distribution was also obtained [14]. Morever, P. Krapivsky and E. Ben-Naim sent us another analytical derivation of an exponential velocity distribution for the BNK model.

[1] P.B. Umbanhowar, F. Melo, and H.L. Swinney, Nature (London) **382**, 793 (1996).

[2] T. Shinbrot and F.J. Muzzio, Nature (London) **410**, 251 (2001).

[3] L. Kadanoff, Rev. Mod. Phys. **71**, 435 (1999).

[4] C. Bizon, M.D. Shattuck, J.B. Swift, and H.L. Swinney, Phys. Rev. E **60**, 4340 (1999).

[5] T. Biben, Ph.A. Martin, and J. Piasecki, Physica A **310**, 308 (2002).

[6] T.P.C. Noije and M.H. Ernst, Granular Matter **1**, 57 (1998).

[7] A. Barrat, T. Biben, Z. Rácz, E. Trizac, and F. van Wijland, J. Phys. A **35**, 463 (2002).

[8] J.M. Montanero and A. Santos, Granular Matter **2**, 53 (2000).

[9] J.S. Olafsen and J.S. Urbach, Phys. Rev. Lett. **81**, 4369 (1998); J.S. Olafsen and J.S. Urbach, Phys. Rev. E **60**, R2468 (1999).

[10] M.H. Ernst and R. Brito, Phys. Rev. E **65**, 040301 (2002).

[11] J.A. Carrillo, C. Cercignani, and I.M. Gamba, Phys. Rev. E **62**, 7700 (2000).

[12] E. Ben-Naim and P.L. Krapivsky, Phys. Rev. E **61**, R5 (2000).

[13] E. Ben-Naim and P.L. Krapivsky, Phys. Rev. E **66**, 011309 (2002).

[14] U.M.B. Marconi and A. Puglisi, Phys. Rev. E **66**, 011301 (2002).