Propagation of linear waves in relativistic anisotropic magnetohydrodynamics

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Gedalin [Phys. Rev. E **47**, 4354 (1993)] derived a dispersion relation for linear waves in relativistic *anisotropic* Magnetohydrodynamics (MHD). This dispersion relation is used to point out the regions where the relativistic *anisotropic* MHD leads to new results that cannot be obtained using usual collisional relativistic MHD. This is highlighted by plotting a Fresnal ray surface. Conditions for the onset of firehose and mirror instabilities are also indicated. Such a study can be applied to astrophysical features such as pulsar winds, propagation of cosmic rays, etc.

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Gedalin [1] derived RAM (relativistic anisotropic magnetohydrodynamics) equations that simulate rarefied plasma embedded in a strong magnetic field. These equations complement the set of usual relativistic Magnetohydrodynamics (MHD) equations that are valid for the collision-dominated plasma. Gedalin [2] used these equations to analyze the properties of linear waves. The purpose of this Brief Report is to discuss further implications of the dispersion relation derived by Gedalin [2]; hereafter this paper is referred to as GRAM. It is convenient to write this dispersion relation as follows:

$$\omega^2 - k_\parallel^2 V_{\rm RA}^2 = 0, \tag{1}$$

$$[\omega^2 - (k_\parallel^2 V_{\rm RA}^2 + k_\perp^2 V_F^2)] \times [\omega^2 - k_\parallel^2 V_s^2] - k_\parallel^2 k_\perp^2 V_T^2 \left[V_T^2 \left(1 - \frac{V_{\rm RA}^2}{c^2} \right) \right] = 0, \tag{2}$$

where

$$V_{\rm RA}^2 = \frac{p_\perp - p_\parallel + B^2/4\pi}{1/c^2[\rho(c^2+\varepsilon) + p_\perp + B^2/4\pi]},$$

$$V_F^2 = \frac{2p_\perp + B^2/4\pi}{1/c^2[\rho(c^2+\varepsilon) + p_\perp + B^2/4\pi]}, \tag{3}$$

$$V_S^2 = \frac{3p_\parallel}{1/c^2[\rho(c^2+\varepsilon) + p_\parallel]},$$

$$V_T^2 = \frac{p_\perp}{1/c^2[\rho(c^2+\varepsilon) + p_\parallel]}, \quad \varepsilon = p_\perp/\rho + p_\parallel/2\rho.$$

The above dispersion relation and the definitions are the same as those used in GRAM except that c has been retained to facilitate reducing them to the nonrelativistic limits ($c \to \infty$). Note that all these relations utilize CGL (Chew, Goldberger, and Low [3]) expression for specific internal energy used by GRAM.

The phase speed of the modes, in dimensionless parameters, can be written as

$$C_{\rm AR}^2 = \left(\frac{1 + S_\perp^2 - S_\parallel^2}{1 + V_A^2/c^2(1 + 2S_\perp^2 + S_\parallel^2/2)} \right) \cos^2\theta, \tag{4}$$

$$C_{\rm SR,FR}^2 = 1/2 \left(\frac{(1+2S_\perp^2)\sin^2\theta + (1 + S_\perp^2 - S_\parallel^2)\cos^2\theta}{1 + V_A^2/c^2(1 + 2S_\perp^2 + S_\parallel^2/2)} + \frac{3S_\parallel^2\cos^2\theta}{1 + V_A^2/c^2(S_\perp^2 + 3S_\parallel^2/2)} \right) \pm 1/2 \left\{ \left(\frac{(1+2S_\perp^2)\sin^2\theta + (1 + S_\perp^2 - S_\parallel^2)\cos^2\theta}{1 + V_A^2/c^2(1 + 2S_\perp^2 + S_\parallel^2/2)} - \frac{3S_\parallel^2\cos^2\theta}{1 + V_A^2/c^2(S_\perp^2 + 3S_\parallel^2/2)} \right)^2 + \frac{4S_\perp^4 \sin^2\theta\cos^2\theta}{[1 + V_A^2/c^2(S_\perp^2 + 3S_\parallel^2/2)][1 + V_A^2/c^2(1 + 2S_\perp^2 + S_\parallel^2/2)]} \right\}^{1/2}, \tag{5}$$

where $V_A = (B^2/4\pi\rho)^{1/2}$ denotes the classical Alfvén speed, $S_\parallel^2 = P_\parallel/\rho V_A^2$ and $S_\perp^2 = P_\perp/\rho V_A^2$ are the analogs of sound speeds along and across the magnetic field. The dimensionless quantaties $C_{\rm AR}$, $C_{\rm FR}$, and $C_{\rm SR}$ in the above equations denote the phase speeds of the Alfvén, fast (+), and slow (−) modes in units of Alfvén speed V_A. Other symbols have the same meaning as in GRAM.

It is well known that in the usual collision-dominated MHD, Alfvén wave is the intermediate wave and the phase speeds follow the following order:

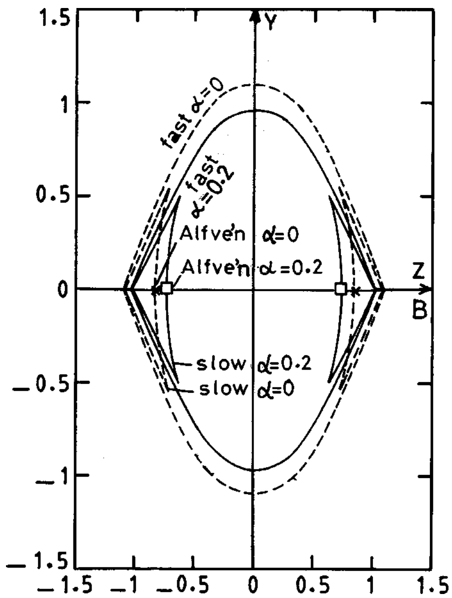

FIG. 1. Fresnel ray surfaces from a point source of the fast, slow, and Alfvén modes for the parameters $S_\parallel^2=0.4$, $S_\perp^2=0.1$, and $\alpha=V_A^2/c^2=0$ (broken line curves) and 0.2 (continuous curves). Z and Y denote components of dimensionless group velocity.

$$C_{\rm SR}\leqslant C_{\rm AR}\leqslant C_{\rm FR} \tag{6}$$

The characteristic feature of RAM is that the anisotropic pressure permits a new ordering of the phase speeds, where

$$C_{\rm AR}\leqslant C_{\rm SR}\leqslant C_{\rm FR}\,. \tag{7}$$

This indicates that the Alfvén speed can shift from the intermediate speed to the lowest speed. Although RAM model describes an entirely different physical situation, where collisions, due to rarefied nature of plasma, are infrequent, in contradistinction with the normal MHD that is collision dominated, one expects results somewhat similar to the isotropic pressure MHD when ordering given in Eq. (6) above is satisfied. Such a regime is designated as pseudo-MHD. The regime given by ordering (7) is a distinct feature of RAM, which does not have any analog in the collisional MHD; it may lead to radically different results (for example, see Fig. 1). This regime is designated as reverse-MHD.

The curve which separates the relativistic wave modes into pseudo- and reverse-MHD classes is given by

$$\begin{aligned}\{&(1+2S_\perp^2)(1+S_\perp^2-4S_\parallel^2)+S_\perp^4+V_A^2/c^2[(1+2S_\perp^2)\\ &\times[(1+S_\perp^2-S_\parallel^2)(S_\perp^2+3S_\parallel^2/2)-3S_\parallel^2(1+2S_\perp^2+S_\parallel^2/2)]\\ &+S_\perp^4(1+2S_\perp^2+S_\parallel^2/2)]\}\sin_\theta^2\cos_\theta^2 \gtrless 0,\end{aligned} \tag{8}$$

where the $>$ sign holds for the pseudo-MHD modes, the $<$ sign indicates the reverse-MHD modes, and the equality sign gives the curve that separates the two classes. The singular points corresponding to $\theta=\pi/2$ or 0 at which the phase speed of two or three modes is the same are called, respectively, a double or a triple point. There may also be a double point of degeneracy, for example, between the Alfvén and the slow mode at the origin, which indicates that neither the Alfvén nor the slow mode has a phase speed perpendicular to the magnetic field. The condition for the two subclasses of double or triple points is found from the phase speeds at $\theta=0$, $\pi/2$ and is

$$\frac{C_{\rm AR}}{V_A}=\left(\frac{1-S_\parallel^2+S_\perp^2}{1+V_A^2/c^2(S_\parallel^2/2+2S_\perp^2+1)}\right)^{1/2}, \tag{9}$$

$$\begin{aligned}\frac{C_{\rm SR,FR}}{V_A}=\Bigg[&\left(\frac{3S_\parallel^2}{1+V_A^2/c^2(3S_\parallel^2/2+S_\perp^2)}\right)^{1/2},\\ &\left(\frac{1-S_\parallel^2+S_\perp^2}{1+V_A^2/c^2(S_\parallel^2/2+2S_\perp^2+1)}\right)^{1/2}\Bigg],\end{aligned} \tag{10}$$

where the larger value in the brackets of Eq. (10) is the fast speed and the smaller is the slow speed.

The pseudo-MHD can further be divided into two subclasses. The condition for the two subclasses is

$$\begin{aligned}1+S_\perp^2-4S_\parallel^2+V_A^2/c^2[&(1+S_\perp^2-S_\parallel^2)(S_\perp^2+3S_\parallel^2/2)\\ &-3S_\parallel^2(1+2S_\perp^2+S_\parallel^2/2)]\gtrless 0,\end{aligned} \tag{11}$$

where the $>$ sign is the condition for the double point of the Alfvén and fast wave, the $<$ sign is the condition for the double point of the Alfvén and slow wave, and the equal sign indicates the triple point.

For propagation along the direction of the magnetic field, two of the three modes propagate with the phase speed $V_{\rm RA}$, which basically represents the classical transverse Alfvén wave modified by the relativistic and anisotropic effects. It is known that the inertia corresponding to the energy density of the magnetic field reduces the Alfvén speed in the model where the pressure is isotropic. In the case of anisotropic pressure one can also easily see that the relativistic Alfvén speed $V_{\rm RA}$, is reduced. In contrast to the isotropic case, the presence of anisotropy may not only reduce the phase speed to zero, but may render it complex leading to an instability. In the nonrelativistic case [4,5], this instability is known as the firehose instability; it arises where $p_\parallel>p_\perp+B^2/4\pi$. The criterion for firehose instability remains unaltered in the relativistic framework.

The third mode propagates with the phase speed V_S. This is basically an acoustic mode that cannot trigger instability in the medium.

There is another interesting case that needs to be commented upon. In the nonrelativistic framework, one finds that mirror instability appears [4] in the limiting case when $\omega\to 0$, $k_\parallel\to 0$, but $\omega/k_\parallel\neq 0$.

The criterion for instability is

$$p_\parallel<\frac{p_\perp^2}{6(p_\perp+B^2/8\pi)}.$$

The analysis of the relativistic dispersion relation in the above limit shows that the criterion for the onset of mirror instability remains the same.

Another aspect that needs to be studied is the speed of propagation of a signal. This is given by the group velocity defined as $V_g=\partial\omega/\partial\mathbf{k}=[\partial(C_n\mathbf{k})/\partial\mathbf{k}]=(\hat{k}C_n+\hat{\theta}[\partial C_n/\partial\theta])$ for each mode. Here C_n (phase speed) represents any one of the three wave modes.

Consider a spherical source which in the limit of zero radius is a point source. The direction of the magnetic field is chosen to be the direction of the z axis. The intersection of the wave front with a plane is given in terms of the curve described by (z,y)

$$\begin{pmatrix} z \\ y \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \pm C_n t+R_0 \\ \pm\dfrac{\partial C_n}{\partial\theta}t \end{pmatrix},$$

where R_0, t, and θ, respectively, denote the radius of the source, the time after excitation of the wave mode, and the angle between the normal to the wave front $\hat{k}$ and the magnetic field. The $\pm$ sign refer to wave fronts diverging away and converging toward the source, respectively. The actual wave front surface is found by rotating the curve about the z axis.

The parametric equations for the relativistic wave fronts (normalized to Alfvén speed) are

$$\frac{z}{V_A}=\frac{R_0}{V_A}\cos\theta\pm\cos\theta\frac{C_n}{V_A}t\pm\frac{\cos\theta\sin^2\theta t}{2C_nRQ/V_A}[3S_\parallel^2R-(S_\perp^2+S_\parallel^2)Q]\mp\frac{\cos\theta\sin^2\theta}{2C_nRQC'/V_A}\{[3S_\parallel^2R+(S_\perp^2+S_\parallel^2)Q]$$
$$\times[(1+2S_\perp^2-S_\perp^2\cos^2\theta-S_\parallel^2\cos^2\theta)Q-3S_\parallel^2\cos^2\theta R]+2S_\perp^4RQ(2\cos^2\theta-1)\}t, \quad (12)$$

$$\frac{y}{V_A}=\frac{R_0}{V_A}\sin\theta\pm\sin\theta\frac{C_n}{V_A}t\pm\frac{\sin\theta\cos^2\theta t}{2C_nRQ/V_A}[(S_\perp^2+S_\parallel^2)Q-3S_\parallel^2R]\pm\frac{\sin\theta\cos^2\theta}{2C_nRQC'/V_A}\{[3S_\parallel^2R+(S_\perp^2+S_\parallel^2)Q]$$
$$\times[(1+2S_\perp^2-S_\perp^2\cos^2\theta-S_\parallel^2\cos^2\theta)Q-3S_\parallel^2\cos^2\theta R]+2S_\perp^4RQ(2\cos^2\theta-1)\}t, \quad (13)$$

where

$$R=[1+V_A^2/c^2(1+2S_\perp^2+S_\parallel^2/2)],\quad Q=[1+V_A^2/c^2(S_\perp^2+3S_\parallel^2/2)],$$
$$C'=\{[(1+2S_\perp^2-S_\perp^2\cos^2\theta-S_\parallel^2\cos^2\theta)Q-3S_\parallel^2\cos^2\theta R]^2+4S_\perp^4RQ\cos^2\theta\sin^2\theta\}^{1/2}.$$

Note that each mode of phase propagation C_n (slow, fast) gives rise to the corresponding mode of group velocity. The equation for the relativistic Alfvén wave front is given by

$$\frac{V_{z_A}}{V_A}=\pm\left(\frac{1+S_\perp^2-S_\parallel^2}{R}\right)^{1/2},\quad \frac{V_{y_A}}{V_A}=0. \quad (14)$$

In order to highlight this feature which is characteristic of RAM, we plot in Fig. 1 group velocity [6] in the (z,y) plane for a point source ($R_0=0$) at $t=1$ for $S_\parallel^2=0.4=4S_\perp^2$ using parametric equations (12)–(14). For $t=1$, z and y axes of the figure are components of (dimensionless) group velocity rationalized to V_A. These values lie in the reverse-MHD regime. The curves are cross section of the wave front through magnetic field axis Z, which is the axis of symmetry. The wave fronts are the surfaces of revolution about this axis. Note that the wave front in the region $Z<0$ are the mirror images (in the y axis) of those in the region $Z>0$. For clarity, relativistic curves are shown by continuous lines in contrast to nonrelatvisitc curves $(c\to\infty,\alpha=V_A^2/c^2=0)$ which are shown by broken lines. The fast wave has the maximum group velocity while the slow mode is cusp shaped (triangular shaped), both the cusps pointing away from the origin. The pointing of cusps away from the origin is a characteristic feature of the reverse RAM, which does not exist in the isotropic pressure MHD [7] wherein both the cusps of the slow mode always point towards the origin. The Alfvén mode reduces to two points on the magnetic field axis. The relativistic effect [$\alpha=V_A^2/c^2=0.2$ (curves shown by continuous lines)] reduces the group velocity of all the three modes compared to its nonrelativistic counterparts [$c\to\infty$, $\alpha=0$ (curves shown by broken lines)]. In both, the relativistic as well as the nonrelativistic cases, two modes (slow and Alfvén) have the same velocity (double point) along the magnetic field, which is the lowest velocity. The slow and fast modes have almost the same velocity along the direction of magnetic field; this is the highest velocity. The propagation of the two or all the three modes with the same velocity depends upon the anisotropy of the pressure tensor.

In conclusion, it may be stated that for any physical situation where the magnetic field is strong such that $\omega_c T \gg 1$, $\omega_c L \gg 1$ ($\omega_c = eB/M$ is the ion-cyclotron frequency, T and L are typical temporal and spatial scales of plasma motion), the present analysis can be applied. GRAM has applied such an analysis to pulsar wind. Another astrophysical situation where this analysis can be extended is the propogation of cosmic rays through the background plasma in space. Since the cosmic rays move with relativistic speeds, these can be simulated by the present equations while the background plasma can be described by nonrelativistic equations. Such a problem is being addressed currently.

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