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Bifurcation diversity of dynamic thermocapillary liquid layers

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We study, from the theoretical point of view, instabilities appearing in a liquid layer, where a dynamic flow is imposed through a nonzero temperature gradient at the bottom. Experimentally many interesting dynamical behaviors have been discovered in this system. In this Brief Report we prove that the basic solution can display great richness of bifurcations which are controlled by heat related parameters. Different kinds of spatially extended and localized structures appear, which are both stationary or oscillatory. These last ones can present amazing patterns such as squares or spirals. Also competing solutions at codimension-two bifurcation points have been found: stationary radial rolls with different wave numbers, radial rolls with hydrothermal waves, and hydrothermal waves with different wave numbers. Remarkably our results recover many features of numerous reported experiments, predict new instabilities, and by giving a deeper insight into how physical parameters contribute to bifurcations, open a gateway to control those instabilities.

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Instabilities and pattern formation in buoyant-thermocapillary flows have been extensively studied in the last years. Classically, heat is applied uniformly from below [1] where the conductive solution becomes unstable for temperature gradients beyond a certain threshold. A more general setup considers thermoconvective instabilities when a basic dynamic flow is imposed through nonzero temperature gradients [2–4]. Recently a lot of attention has been paid to the case where temperature gradients are constant [2,5–10]. This process, frequently referred to as lateral heating convection, displays many interesting instabilities [5,6,11–13]. This problem has been treated from different points of view: experimental [5–8,11] and theoretical both with semiexact [2,9] and numerical solutions [10,14,15]. In experiments reported in [5–8] several parameters are involved: Prandtl, Rayleigh, Marangoni numbers, and depth of the fluid layer, whose changing values lead to different transitions. A parameter also present in those experiments, perhaps not sufficiently enhanced since it is difficult to tune, is the heat exchange with the atmosphere, quantified by the Biot number. In semiexact theoretical works [2,9], heat exchange conditions with the atmosphere are fitted in order to find explicit solutions within the parallel flow approach. In numerical studies [14,15] beyond this approach, insulated boundaries at the top and at the bottom are considered, i.e., $B=0$. In [10,16] results are obtained where the importance of heat related parameters to develop the instabilities is adressed. In this Brief Report we exploit this idea to prove that a great bifurcation diversity can be achieved by manipulating heat related parameters, in particular the Biot number. Our bifurcations explain many experimental results described in [5–8] and anticipate new instabilities.

The physical setup is shown in Fig. 1. A horizontal fluid layer of depth d (z coordinate) is in a container limited by two concentric cylinders of radii a and $a+\delta$ (r coordinate). The bottom plate is rigid and the top is open to the atmosphere. The inner cylinder has a temperature $T_{\max}$ whereas the outer one is at $T_{\min}$, and the environmental temperature is T_0. In the equations governing the system u_r, u_ϕ, and u_z are the components of the velocity field $\mathbf{u}$, Θ is the temperature, p is the pressure, $\mathbf{r}$ is the radio vector, and t is the time. The system evolves according to momentum and mass balance equations and to the energy conservation principle, which in dimensionless form are (see Ref. [16])

$$\nabla\cdot\mathbf{u}=0, \tag{1}$$

$$\partial_t\Theta+\mathbf{u}\cdot\nabla\Theta=\nabla^2\Theta, \tag{2}$$

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathrm{Pr}\left(-\nabla p+\nabla^2\mathbf{u}+\frac{\mathrm{Ra}\,\rho}{\alpha\rho_0\Delta T}\mathbf{e}_z\right), \tag{3}$$

where the operators and fields are expressed in cylindrical coordinates and the Oberbeck-Bousinesq approximation has been used. Here $\mathbf{e}_z$ is the unit vector in the z direction, ρ is the density, α is the thermal expansion coefficient and ρ_0 is the mean density. The following dimensionless numbers have been introduced: the Prandtl number $\mathrm{Pr}=\nu/\kappa$ and the Rayleigh number $\mathrm{Ra}=g\alpha\Delta T d^3/\kappa\nu$, which represents the

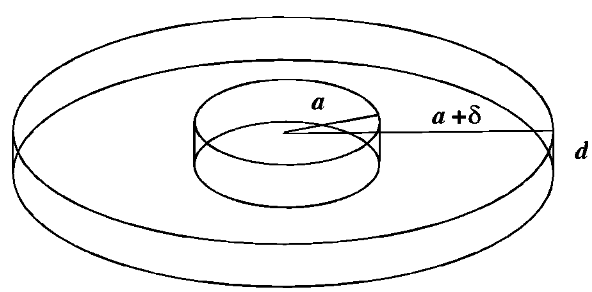

FIG. 1. Problem setup ($a=0.01$ m, $\delta=0.02$ m).

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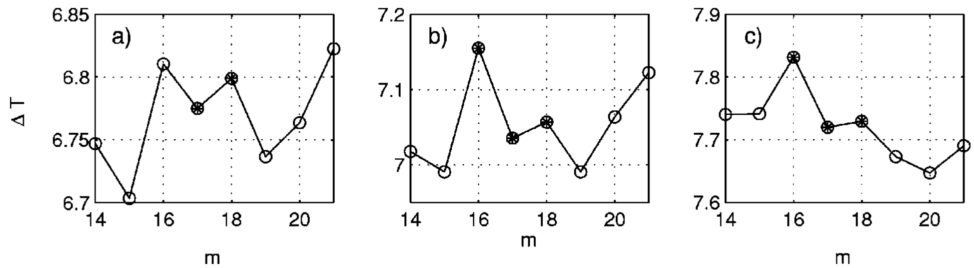

FIG. 2. Marginal stability diagrams which show ΔT_c vs the wave number m for a decreasing sequence of Biot numbers and $\Delta T_h = 6\ ^\circ\mathrm{C}$. Filled black circles indicate a complex eigenvalue while nonfilled circles correspond to a real eigenvalue. (a) Stationary bifurcation at $B=0.9$, $m_c=15$, and $\Delta T_c=6.70\ ^\circ\mathrm{C}$. (b) Codimension-two bifurcation of stationary modes with wave numbers $m_c=15$ and $m_c=19$ at $\Delta T_c=6.99\ ^\circ\mathrm{C}$ and $B=0.83$. (c) Stationary bifurcation at $B=0.8$, $m_c=20$, and $\Delta T_c=7.14\ ^\circ\mathrm{C}$.

buoyant effect. In these definitions ν is the kinematic viscosity of the liquid, κ is the thermal diffusivity, g is the gravity constant, and $\Delta T = T_{\max} - T_0$.

The boundary conditions (bc) are (on $z=1$),

$$u_z = \frac{\partial u_r}{\partial z} + M\frac{\partial \Theta}{\partial r} = \frac{\partial u_\phi}{\partial z} + \frac{M}{r}\frac{\partial \Theta}{\partial \phi} = \frac{\partial \Theta}{\partial z} + B\Theta = 0 \qquad (4)$$

on $z=0$,

$$u_r = u_\phi = u_z = 0, \quad \Theta = \left(-\frac{r}{\delta^*} + \frac{a}{\delta} \right) \frac{\Delta T_h}{\Delta T} + 1, \qquad (5)$$

on $r=a^*$, $r=a^*+\delta^*$,

$$u_r = u_\phi = u_z = 0, \quad \partial_r \Theta = 0. \qquad (6)$$

Here B is the Biot number which quantifies the heat exchange with the atmosphere, $a^* = a/d$, $\delta^* = \delta/d$, $\Delta T_h = T_{\max} - T_{\min}$, and $M = \gamma \Delta T d/(\kappa \nu \rho_0)$ is the Marangoni number which includes the surface tension coefficient γ. The only control parameter mentioned in Refs. [2,5–9,11,14,15] is ΔT_h, however as discussed in [10,16] we find a new one, ΔT, also related to temperature. We prove in this Brief Report that both these physical quantities together with the Biot number are able to control a great variety of bifurcations, some of them appearing in the experiments reported in [5–8,11].

Basic state and linear stability analysis. The horizontal temperature gradient at the bottom settles in a stationary convective motion, which is computated as in Ref. [16]. Increasing the control parameter ΔT makes the basic flow unstable and different bifurcations arise. The linear stability analysis supplies information on the critical values of ΔT at which this happens and on the shape of growing instabilities. We study the stability by perturbing the basic solutions with fields depending on r, ϕ, and z coordinates in a fully three-dimensional (3D) analysis, following the numerical scheme of Refs. [16,17]. Due to the periodical boundary conditions in the azimuthal coordinate the perturbations of any physical function X can be factorized, and along ϕ is expanded by Fourier modes,

$$X(r,\phi,z,t) = X(r,z)e^{im\phi + \lambda t}, \qquad (7)$$

where m is the wave number. The eigenvalue λ characterizes the instability, when its real part is negative the basic state is stable but if it is positive the basic solution is unstable. In this case the imaginary part of λ can be zero and then the bifurcation is stationary while if it is nonzero the bifurcation is oscillatory.

The bifurcations obtained when ΔT is increased can be very diverse, depending on the values of the heat related quantities ΔT_h and B. We notice that in this Brief Report all the parameter values that in the experimental Refs. [2,6–8] are considered to influence the type of transition are kept constant. In particular we fix both the size of the container and the depth of the fluid layer, i.e., the aspect ratio is $\delta^* = 10$, and the Prandtl number ($\mathrm{Pr} = \infty$). These asumptions are quite standard and accurately represent any fluid with a Prandtl number above 10 in a large cell where size and geometry effects are minimized.

To explore the bifurcations we first set $\Delta T_h = 6\ ^\circ\mathrm{C}$. Under this condition Fig. 2 shows the marginal stability curve (critical values of ΔT versus the wave number m) for different Biot numbers. A codimension-two bifurcation is proved to be at $\Delta T_c = 6.99\ ^\circ\mathrm{C}$ and $B=0.83$, where two stationary modes

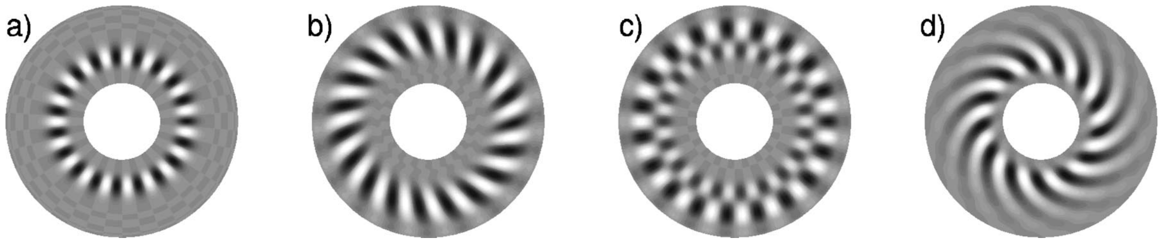

FIG. 3. Unstable modes of the temperature fields at different physical conditions. (a) Stationary mode with $m_c=19$ at $\Delta T_h = 6\,^\circ\mathrm{C}$, $\Delta T = 7\ ^\circ\mathrm{C}$ and $B=0.83$. (b) Travelling wave with $m_c=19$ at $\Delta T_h = 6\ ^\circ\mathrm{C}$, $\Delta T = 15.04\ ^\circ\mathrm{C}$ and $B=0.25$. (c) Square patterns at same conditions of (b) obtained by overlapping of left and right travelling waves. (d) Spiral-like travelling wave with $m_c=13$ at $\Delta T_h = 1\ ^\circ\mathrm{C}$, $\Delta T = 7.69\ ^\circ\mathrm{C}$ and $B=0.33$.

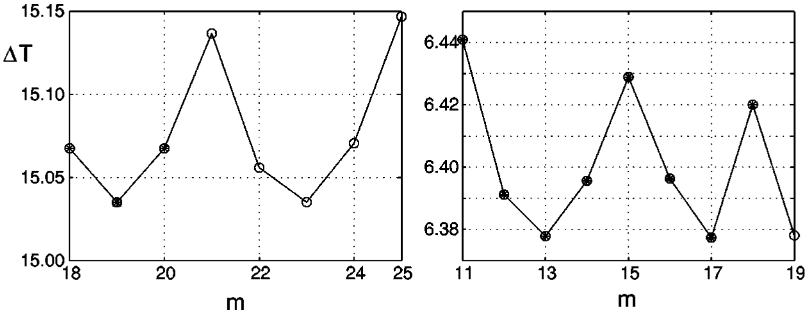

FIG. 4. Marginal stability diagrams which show ΔT_c vs the wave number m at different physical conditions. Filled black circles indicate complex eigenvalue while nonfilled circles correspond to a real eigenvalue. (a) A codimension two bifurcation of stationary and oscillatory modes at $\Delta T_h = 6$ ºC, $B = 0.25$ and $\Delta T_c = 15.04$ ºC. (b) Competition of one stationary and two oscillatory modes at $\Delta T_h = 5$ ºC, $B = 0.85$, and $\Delta T_c = 6.36$ ºC.

of wave numbers $m_c = 15$ and $m_c = 19$ become unstable [see Fig. 2(b)]. Slightly increasing the Biot number ($B = 0.9$) favors the mode $m_c = 15$, which grows at $\Delta T_c = 6.70$ °C [see Fig. 2(a)], while if decreased ($B = 0.8$) it is the mode $m_c = 19$ that grows at $\Delta T_c = 7.14$ °C [see Fig. 2(c)]. Figure 3(a) shows the spatial structure in the x-y plane of the mode $m_c = 19$ at $\Delta T = 7$ °C and $B = 0.83$. It is localized near the hot side, which is a feature that coincides with those reported in experiments [18] and also in previous theoretical works [10].

For $\Delta T_h = 6$ °C in quite different Biot values ($B = 0.25$) a different codimension two point exists at $\Delta T_c = 15.04$ °C, as Fig. 4(a) shows. In this case an oscillatory ($m_c = 19$) and a stationary mode ($m_c = 23$) become unstable. If the Biot number increases ($B = 0.3$) the stationary mode is favored, while if decreased ($B = 0.25$) it is the oscillatory mode that grows at $\Delta T_h = 17.94$ °C. Figure 3(b) shows the oscillatory structure at the codimension two point, which is localized near the cold side. The appearance of structures near the cold side is also reported in experiments [6,8], and now is achieved in a 3D theoretical study. At the critical point the oscillatory mode appears with its complex conjugate, and both solutions are overlapped to get a real solution. Complex amplitudes of these modes lead to the square pattern or multicellular state shown in Fig. 3(c). The stability of this solution is suggested by the fact that it has been experimentally observed [6,11].

Second we explore bifurcations for $\Delta T_h = 5$ °C. Figure 4(b) shows the marginal stability curve obtained at $B = 0.85$. At $\Delta T_c \sim 6.36$ °C there are three relative minima corresponding to one stationary and two oscillatory modes, whose wave numbers are $m_c = 19$, $m_c = 13$, and $m_c = 17$, respectively. We have not proved that these structures have the same threshold, as it has been done for the codimension two points explained above. However, these modes have a very close critical value and beyond it could be expected competition between them. Therefore we have shown that a great variety of bifurcations can be obtained: stationary, oscillatory, stationary-stationary, oscillatory-stationary, etc. They are obtained for any ΔT_h and are controlled by the Biot number B, whose values can be tuned to achieve one or another kind of bifurcation or codimension two points. This fact could provide a justification of the experimental control of hydrothermal waves reported in [19], where to supress the hydrothermal waves they use a laser beam that is modifying the heat exchange at the surface. Figure 5 shows for each ΔT_h the Biot numbers at which codimension-two bifurcations occur. It can be seen that the dependence is not monotonous and has a maximum at $\Delta T_h = 4$ °C. These codimension two points can be any of the types described above. For instance at $\Delta T_h = 1$ °C and $B = 0.33$ stationary and oscillatory modes compete at $\Delta T_c = 7.69$ °C. This oscillatory mode, whose wave number is $m_c = 13$, has the spiral-like appearance depicted in Fig. 3(d). This result, which is novel in this kind of problem, suggests that in the origin of spiral patterns in convection heat transport plays an important role. The existence and dynamics of spiral patterns in non-Boussinesq Rayleigh-Bénard convection has been a matter discussed in [20,21], but in our context its existence is predicted for the first time.

In conclusion we can say that a great diversity of transitions have been found in a thermoconvective problem with an imposed constant temperature gradient at the bottom. Stationary or oscillatory instabilities have been previously reported in experiments, and several features have been recovered in our results, as their appearance near the cold side or the multicellular states. Other results such as the existence of several codimension two points with competition between different modes or spiral-like structures are predicted. These bifurcations are controlled by heat related parameters (ΔT_h, ΔT, and B). This fact could provide a theoretical justification of the control mechanism described in [19].

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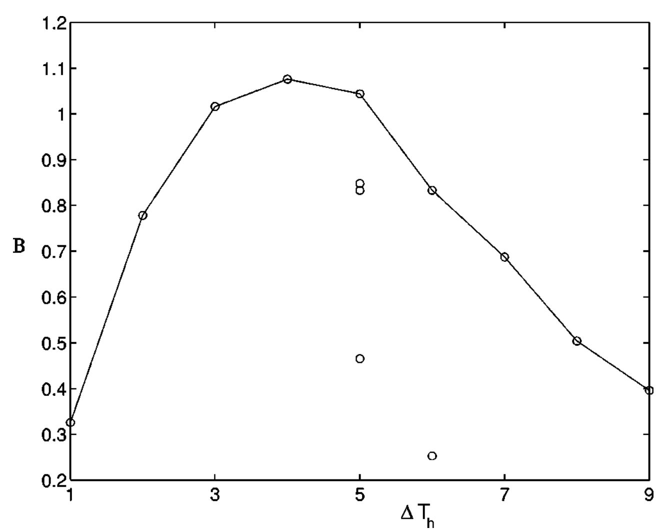

FIG. 5. Biot number at which a codimension-two bifurcation point appears for different ΔT_h values.

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