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Viscous dissipation for Euler's disk

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It is shown that Moffatt's recent calculation regarding the viscous dissipation of circulating air underneath a spinning coin overlooked the importance of the finite width of the viscous boundary layer. Including the enhanced dissipation in the boundary layer gives a larger dissipation from the moving air, and a scaling law of the decay of the coin's angle that is in much better accord with that observed. However, rolling frictional drag with the surface is an additional damping mechanism that could well dominate that from circulating air.

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It is natural to wonder what causes the dissipation of energy that allows a spinning, wobbling, coin to eventually come to rest. Moffatt [1] showed that a source of damping for "Euler's disk" [2] is viscous dissipation in the air flowing between the disk (of radius $a = 3.75$ cm) and the base it is oscillating upon at frequency Ω (see Fig. 1 of Ref. [1]). I show here that Moffatt underestimated this dissipation by neglecting the finite time it takes for viscosity to act across this thin layer.

For small angles α between the base and disk, the frequency of wobble is $\Omega^2 = 4g/a\alpha$, where $g = 980$ cm^2 s^{-1} is the Earth's gravitational acceleration. Moffatt assumed that the viscous boundary layer extended completely across the vertical gap (of width $\approx a\alpha$) between the disk and the base. However, for this to be true, the time for the viscous flow (with kinematic viscosity [3] $\nu \equiv \mu/\rho \approx 0.15$ cm^2 s^{-1} at 20°) to be established across the gap, $t_v \approx (a\alpha)^2/\nu$, must be shorter than the oscillation period; $\Omega t_v \ll 1$. Hence, Moffatt's calculation is only valid for angles smaller than

$$\alpha_c \approx \left(\frac{\nu^2}{4ga^3} \right)^{1/3} \approx 5 \times 10^{-3} \text{ rad} \approx 0.3°, \tag{1}$$

which corresponds to oscillating frequencies higher than ≈ 70 Hz, or just the very end of the collapse.

For most larger angles, $\alpha > \alpha_c$, the viscous dissipation from air occurs in the oscillating thin boundary layers [4] at the disk and base surfaces of width $\delta \approx (2\nu/\Omega)^{1/2} \ll \alpha a$. The dissipation rate of the air flowing at speed $u \approx \Omega a$ is then roughly given by

$$\Phi \sim \mu a^2 \delta \left(\frac{du}{dz} \right)^2 \sim \frac{4g\mu a^3}{\alpha \delta} \propto \alpha^{-5/4}, \tag{2}$$

which is *larger* than Moffatt's value by $\alpha a/\delta \approx (\alpha/\alpha_c)^{3/4} \gg 1$. Just as viscous flow is more difficult in a smaller pipe, forcing the viscous boundary layer into a thin layer increases the dissipation. Accurately calculating the dimensionless prefactors for Eq. (2) and the transition to Moffatt's scalings for Φ, when $\alpha < \alpha_c$ requires a full solution of the oscillating viscous boundary layer flow in the disk geometry. Such a calculation would also make clearer whether or not such a flow is demanded, namely, one where the viscous boundary layer is reformed for every oscillation period. It is what I have assumed for this paper.

The parameter dependences of Φ are accurate when $\alpha > \alpha_c$ and allow us to find the scalings of Ω and α with time as the toy "spins-up" and falls to the base under this sole dissipation mechanism (see recent discussions [5–7] of surface friction). The energy of the disk (of mass M) is $E = 3Mga\alpha/2$, and using Φ from Eq. (2) in $-\Phi = dE/dt$, the equation that relates the final angle (α_f) to the initial (α_i) is

$$\alpha_f^{9/4} = \alpha_i^{9/4} - \frac{t_f - t_i}{t_{\rm BL}}, \tag{3}$$

where t_i and t_f are the initial and final times, and

$$t_{\rm BL} = \frac{M}{6\mu a} \left(\frac{\nu^2}{a^3 g} \right)^{1/4} = \frac{M}{6\mu a} (4\alpha_c^3)^{1/4} \approx 42 \text{ min}, \tag{4}$$

for the designed toy with $M = 400$ g. The time to collapse to $\alpha \ll \alpha_i$ is $t_c = t_{\rm BL} \alpha_i^{9/4}$, which gives 50 sec from an initial angle of 10°. This roughly agrees with that observed, though, as earlier emphasized, the parameter dependences of Eq. (2) are more reliable than the numerical prefactors. An immediate experimental check is the predicted scaling with time; $\alpha(t) = [(t_c - t)/t_{\rm BL}]^{4/9}$. For $\alpha > \alpha_c$ the frequency decay should scale with time

$$\Omega \propto (t_c - t)^{-2/9}. \tag{5}$$

Moffatt's scaling has an exponent of −1/6.

McDonald and McDonald [7] measured this scaling law for Euler's disk [2] and found a best fit slope of −1/4 over a large frequency range. Though this clearly disagrees with Moffatt's scaling, the data do allow for a scaling of −2/9 over the frequency range of 20–70 Hz. More accurate measurements both in and out of vacuum (such as carried out already in Ref. [5]) would resolve these issues once and for all. The recent work of Petrie *et al.* [8] ruled out sliding friction due to center of mass motion, leaving rolling friction from the changing contact point as the sole dissipation mechanism other than air.

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Turbulent flow could ensue if the Reynolds number (as written for an oscillating viscous boundary later [9], where the speed is $u \approx a\Omega$)

$$\mathrm{Re} \approx \frac{au}{\nu} \approx \frac{1}{\alpha_c^2}\left(\frac{\alpha_c}{\alpha}\right)^{1/2} \approx 4 \times 10^4 \left(\frac{\alpha_c}{\alpha}\right)^{1/2}, \qquad (6)$$

becomes too large. The onset of turbulence occurs when $\mathrm{Re} > 10^5$ in such an oscillating viscous boundary layers [9]. The viscous flow for this toy is close to the onset of turbulence when $\alpha \sim \alpha_c$, in which case neither Moffatt's nor my scaling would apply. Presuming turbulent drag is $\propto u^2$, the resulting scaling exponent in Eq. (5) would become $-1/5$ if the dissipation is dominated by turbulent air flow.

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