Statistical mechanics in the context of special relativity

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In Ref. [Physica A **296**, 405 (2001)], starting from the one parameter deformation of the exponential function $\exp_{\{\kappa\}}(x)=(\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}$, a statistical mechanics has been constructed which reduces to the ordinary Boltzmann-Gibbs statistical mechanics as the deformation parameter κ approaches to zero. The distribution $f=\exp_{\{\kappa\}}(-\beta E+\beta\mu)$ obtained within this statistical mechanics shows a power law tail and depends on the nonspecified parameter β, containing all the information about the temperature of the system. On the other hand, the entropic form $S_\kappa=\int d^3p(c_\kappa f^{1+\kappa}+c_{-\kappa}f^{1-\kappa})$, which after maximization produces the distribution f and reduces to the standard Boltzmann-Shannon entropy S_0 as $\kappa\to 0$, contains the coefficient c_κ whose expression involves, beside the Boltzmann constant, another nonspecified parameter α. In the present effort we show that S_κ is the unique existing entropy obtained by a continuous deformation of S_0 and preserving unaltered its fundamental properties of concavity, additivity, and extensivity. These properties of S_κ permit to determine unequivocally the values of the above mentioned parameters β and α. Subsequently, we explain the origin of the deformation mechanism introduced by κ and show that this deformation emerges naturally within the Einstein special relativity. Furthermore, we extend the theory in order to treat statistical systems in a time dependent and relativistic context. Then, we show that it is possible to determine in a self consistent scheme within the special relativity the values of the free parameter κ which results to depend on the light speed c and reduces to zero as $c\to\infty$ recovering in this way the ordinary statistical mechanics and thermodynamics. The statistical mechanics here presented, does not contain free parameters, preserves unaltered the mathematical and epistemological structure of the ordinary statistical mechanics and is suitable to describe a very large class of experimentally observed phenomena in low and high energy physics and in natural, economic, and social sciences. Finally, in order to test the correctness and predictability of the theory, as working example we consider the cosmic rays spectrum, which spans 13 decades in energy and 33 decades in flux, finding a high quality agreement between our predictions and observed data.

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I. INTRODUCTION

The following one-parameter deformations of the exponential and logarithm functions

$$\exp_{\{\kappa\}}(x)=(\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}, \tag{1.1}$$

$$\ln_{\{\kappa\}}(x)=\frac{x^{\kappa}-x^{-\kappa}}{2\kappa}, \tag{1.2}$$

which reduce to the standard exponential and logarithm, respectively, as the real deformation parameter κ approaches zero, have been introduced recently in Ref. [1]. The above functions have many very interesting properties [1–4] (some being identical to the ones of the undeformed functions) that permit one to construct a statistical mechanics (and thermodynamics) which generalizes the standard Boltzmann-Gibbs one. This κ-deformed statistical mechanics preserves unaltered the structure of the ordinary one and can be used to explain a very large class of experimentally observed phenomena described by distribution functions exhibiting power law tails. The areas where this formalism can be applied include among others, low and high energy physics, astrophysics, econophysics, geology, biology, mathematics, information theory, linguistics, etc. [5–9].

In Ref. [1] it has been shown that the statistical distribution

$$f=\exp_{\{\kappa\}}(-\beta[E-\mu]), \tag{1.3}$$

which generalizes the Boltzmann-Gibbs distribution, can be obtained also by maximizing, after properly constrained, the entropy

$$S_\kappa=\int d^n v(c_\kappa f^{1+\kappa}+c_{-\kappa}f^{1-\kappa}), \tag{1.4}$$

which reduces to the standard entropy S_0 as the deformation parameter approaches to zero. The coefficient c_κ, which also absorbs the Boltzmann constant k_B, depends on a free parameter α [see Eq. (65) of Ref. [1]] which remains to be determined together with the parameter β which contains the information about the temperature T of the system.

A first question which arises naturally is if it is possible and how to find any criterion which allows us to fix the parameters β and α or at least express these in terms of the deformation parameter κ, in order to reduce the free parameters of the theory.

A second question regards the properties of the entropy S_κ. It is well known that the Boltzmann-Shannon entropy S_0

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is concave, additive, and extensive. We know that S_κ is concave with respect to the variable *f*, but what happens about its additivity and extensivity? More in general, beside the Boltzmann-Shannon entropy other concave, additive, and extensive entropies do exist?

A third question is related to the physical mechanism which originates the deformation introduced by the parameter κ. In other words, does a more fundamental theory exist where this deformation emerges, or is it simply a purely mathematical tool?

A fourth question is if it is possible to extend the theory originally developed in the framework of a classical kinetics to treat statistical systems in the context of a relativistic kinetics.

A fifth question regards the deformation parameter κ. This parameter will continue to remain free or is it possible to determine its value self consistently within the theory?

The present paper deals with the above questions and its purpose is double. First, we will show that S_κ is the unique existing, concave, additive, and extensive entropy, beside the Boltzmann-Shannon entropy. As we will see these properties of S_κ are sufficient to determine unequivocally the values of parameters β and α. Second, we will show that the deformation introduced by κ is a purely relativistic effect and then we will explain the deformation mechanism within the Einstein special relativity. Then, we will formulate a relativistic κ kinetics and we will calculate the value of κ.

Finally, in order to test the predictability and correctness of theory here proposed we will consider two sets of experimental data. In particular we will analyze the cosmic rays spectrum which spans 13 decades in energy and 33 decades in particle flux that, as it is widely known, violates the Boltzmann-Gibbs statistics. As we will see, we have a high quality agreement between the theory and the observed data.

The paper is organized as it follows. In Sec. II, generalizing the approach proposed in Refs. [1,2], we introduce a class of one parameter deformed structures and study their mathematical properties.

In Sec. III, within this context, starting from the Jaynes maximum entropy principle we consider the most general class of deformed statistical mechanics preserving the main features of the standard Boltzmann-Gibbs one.

In Sec. IV, we show that the entropy S_κ introduced in Ref. [1] is the only one existing beside the Boltzmann-Shannon entropy S_0 which is simultaneously concave, additive, and extensive. Then the statistical mechanics and thermodynamics based on S_κ can be viewed as a natural extension of the Boltzmann-Gibbs one, recovering this last as the deformation parameter κ approaches to zero.

In Sec. V, we consider the mean properties of κ exponential and κ logarithm which have a fundamental role in the formulation of the statistical mechanics.

In Sec. VI, we extend the formalism to a time dependent and relativistic context. In particular after introducing the relativistic κ-kinetic evolution equation we study its stationary state and prove the *H* theorem.

In Sec. VII, we explain the origin of the κ deformation and show that it emerges naturally within the Einstein special relativity.

In Sec. VIII, we propose an approach which permits to determine within the special relativity the value of the parameter κ.

In Sec. IX, we compare two sets of experimental data with the predictions of the present theory.

Finally in Sec. X, some concluding remarks are reported.

II. DEFORMED MATHEMATICS

A. Generator of the deformation

Let $g(x)$ be an arbitrary real function of the real variable *x*, that we call generator of the deformation, having the following properties:

(i) $g(x)\in C^\infty(\mathbf{R})$;
(ii) $g(-x)=-g(x)$;
(iii) $dg(x)/dx>0$;
(iv) $g(\pm\infty)=\pm\infty$; and
(v) $g(x)\approx x$, for $x\to 0$.

Starting from the generator $g(x)$, we construct the real function $x_{\{\kappa\}}$ of the real variable *x* and depending on the real parameter κ, as follows:

$$x_{\{\kappa\}}=\frac{1}{\kappa}\operatorname{arcsinh} g(\kappa x). \tag{2.1}$$

Its properties descend directly from the ones of the generator $g(x)$:

(i) $x_{\{\kappa\}}\in C^\infty(\mathbf{R})$;
(ii) $(-x)_{\{\kappa\}}=-x_{\{\kappa\}}$;
(iii) $dx_{\{\kappa\}}/dx>0$;
(iv) $(\pm\infty)_{\{\kappa\}}=\pm\infty$;
(v) $x_{\{\kappa\}}\approx x$, for $x\to 0$ and then $0_{\{\kappa\}}=0$;
(vi) $x_{\{\kappa\}}\approx x$, for $\kappa\to 0$ and then $x_{\{0\}}=x$; and
(vii) $x_{\{-\kappa\}}=x_{\{\kappa\}}$.

Together with the function $x_{\{\kappa\}}$ one can introduce the inverse function $x^{\{\kappa\}}$, through $(x^{\{\kappa\}})_{\{\kappa\}}=(x_{\{\kappa\}})^{\{\kappa\}}=x$, which assumes the form

$$x^{\{\kappa\}}=\frac{1}{\kappa}g^{-1}(\sinh \kappa x). \tag{2.2}$$

B. Deformed algebra

Proposition 1. The composition law $\overset{\kappa}{\oplus}$ defined through

$$(x\overset{\kappa}{\oplus}y)_{\{\kappa\}}=x_{\{\kappa\}}+y_{\{\kappa\}}, \tag{2.3}$$

which reduces to the ordinary sum as $\kappa\to 0$, namely $x\overset{0}{\oplus}y=x+y$, is a deformed sum and the algebraic structure $(\mathbf{R},\overset{\kappa}{\oplus})$ forms an Abelian group.

Proof. Indeed, from the definition of $x_{\{\kappa\}}$, the following properties of $\overset{\kappa}{\oplus}$ follow:

(i) associativity property, $(x\overset{\kappa}{\oplus}y)\overset{\kappa}{\oplus}z=x\overset{\kappa}{\oplus}(y\overset{\kappa}{\oplus}z)$;

(ii) neutral element, $x\overset{\kappa}{\oplus}0=0\overset{\kappa}{\oplus}x=x$;

(iii) opposite element, $x\overset{\kappa}{\oplus}(-x)=(-x)\overset{\kappa}{\oplus}x=0$; and

(iv) commutativity property, $x\overset{\kappa}{\oplus}y=y\overset{\kappa}{\oplus}x$.

Of course the κ-difference indicated with $\overset{\kappa}{\ominus}$ is defined as $x\overset{\kappa}{\ominus}y=x\overset{\kappa}{\oplus}(-y)$.

Proposition 2. The composition law $\overset{\kappa}{\otimes}$ defined through

$$(x\overset{\kappa}{\otimes}y)_{\{\kappa\}}=x_{\{\kappa\}}\cdot y_{\{\kappa\}}, \tag{2.4}$$

which reduces to the ordinary product as $\kappa\to 0$, namely $x\overset{0}{\otimes}y=xy$, is a deformed product and the algebraic structure $(\mathbf{R}-\{0\},\overset{\kappa}{\otimes})$ forms an Abelian group.

Proof. Indeed from the definition of $x_{\{\kappa\}}$ we have the following properties:

(i) associative law, $(x\overset{\kappa}{\otimes}y)\overset{\kappa}{\otimes}z=x\overset{\kappa}{\otimes}(y\overset{\kappa}{\otimes}z)$;

(ii) neutral element, $x\overset{\kappa}{\otimes}I=I\overset{\kappa}{\otimes}x=x$;

(iii) inverse element, $x\overset{\kappa}{\otimes}\overline{x}=\overline{x}\overset{\kappa}{\otimes}x=I$; and

(iv) commutative law: $x\overset{\kappa}{\otimes}y=y\overset{\kappa}{\otimes}x$; $I=1^{\{\kappa\}}$ being the neutral element while the inverse element of x is $\overline{x}=(1/x_{\{\kappa\}})^{\{\kappa\}}$. Of course the κ division $\overset{\kappa}{\oslash}$ is defined as $x\overset{\kappa}{\oslash}y=x\overset{\kappa}{\otimes}\overline{y}$.

Proposition 3. The deformed sum $\overset{\kappa}{\oplus}$ and product $\overset{\kappa}{\otimes}$ obey the distributive law

$$z\overset{\kappa}{\otimes}(x\overset{\kappa}{\oplus}y)=(z\overset{\kappa}{\otimes}x)\overset{\kappa}{\oplus}(z\overset{\kappa}{\otimes}y), \tag{2.5}$$

and then the algebraic structure $(\mathbf{R},\overset{\kappa}{\oplus},\overset{\kappa}{\otimes})$ forms an Abelian field.

Proof. This proposition follows from the definitions (2.3) and (2.4).

We remark that the field $(\mathbf{R},\overset{\kappa}{\oplus},\overset{\kappa}{\otimes})$ is isomorphic with the field $(\mathbf{R},+,\cdot)$. Moreover, $z\cdot(x\overset{\kappa}{\oplus}y)\neq(z\cdot x)\overset{\kappa}{\oplus}(z\cdot y)$ and then the structure $(\mathbf{R},\overset{\kappa}{\oplus},\cdot)$ it is not a field.

Proposition 4. The function $x^{\{\kappa\}}$ has the two following properties:

$$x^{\{\kappa\}}\overset{\kappa}{\oplus}y^{\{\kappa\}}=(x+y)^{\{\kappa\}}, \tag{2.6}$$

$$x^{\{\kappa\}}\overset{\kappa}{\otimes}y^{\{\kappa\}}=(xy)^{\{\kappa\}}. \tag{2.7}$$

Proof. These properties follow directly from the definitions given by Eqs. (2.3) and (2.4).

Proposition 5. The function $x_{\{\kappa\}}$ and its inverse $x^{\{\kappa\}}$ obey the following scaling laws:

$$x'_{\{\kappa'\}}=zx_{\{\kappa\}}, \tag{2.8}$$

$$x'^{\{\kappa'\}}=zx^{\{\kappa\}}, \tag{2.9}$$

with

$$x'=zx, \tag{2.10}$$

$$\kappa'=\kappa/z. \tag{2.11}$$

Proof. These laws follow from the definitions of the functions $x_{\{\kappa\}}$ and $x^{\{\kappa\}}$.

Proposition 6. The pseudodistributive law

$$z\cdot(x\overset{\kappa}{\oplus}y)=(z\cdot x)\overset{\kappa/z}{\oplus}(z\cdot y) \tag{2.12}$$

holds and then the structure $(\mathbf{R},\overset{\kappa}{\oplus},\cdot)$ is a pseudofield.

Proof. Indeed, by using the propositions 1 and 5 one obtains

$$\begin{aligned} z\cdot(x\overset{\kappa}{\oplus}y)&=z\cdot[(x\overset{\kappa}{\oplus}y)_{\{\kappa\}}]^{\{\kappa\}}=z\cdot(x_{\{\kappa\}}+y_{\{\kappa\}})^{\{\kappa\}}\\ &=z\cdot\left(\frac{1}{z}x'_{\{\kappa'\}}+\frac{1}{z}y'_{\{\kappa'\}}\right)^{\{\kappa\}}\\ &=z\cdot\left(\frac{1}{z}(x'\overset{\kappa'}{\oplus}y')_{\{\kappa'\}}\right)^{\{\kappa\}}\\ &=z\cdot\left[\left(\frac{1}{z}(x'\overset{\kappa'}{\oplus}y')\right)_{\{\kappa\}}\right]^{\{\kappa\}}\\ &=x'\overset{\kappa'}{\oplus}y'=(z\cdot x)\overset{\kappa/z}{\oplus}(z\cdot y). \end{aligned}$$

C. Deformed derivative

Consider the two algebraic structures $(X,\overset{\kappa}{\oplus},\cdot)$ and $(Y,+,\cdot)$ with $X\equiv\mathbf{R}$ and $Y\equiv\mathbf{R}$. Let us introduce the set of the functions $\mathcal{F}=\{f:X\overset{f}{\to}Y\}$ with $\mathcal{F}\subseteq C^{\infty}(X)$.

The κ differential $d_{\{\kappa\}}x$ is defined as

$$d_{\{\kappa\}}x=\lim_{z\to x}x\overset{\kappa}{\ominus}z. \tag{2.13}$$

and results in being

$$d_{\{\kappa\}}x=dx_{\{\kappa\}}. \tag{2.14}$$

We define the κ derivative for the functions of the set $\mathcal{F}$ through

$$\frac{df(x)}{d_{\{\kappa\}}x}=\lim_{z\to x}\frac{f(x)-f(z)}{x\overset{\kappa}{\ominus}z}, \tag{2.15}$$

with $x,z\in X$ and $f(x),f(z)\in Y$. We observe that the κ derivative, which reduces to the usual one as the deformation parameter $\kappa\to 0$, can be written in the form

$$\frac{d\,f(x)}{d_{\{\kappa\}}x}=\frac{d\,f(x)}{d\,x_{\{\kappa\}}}=\frac{1}{dx_{\{\kappa\}}/dx}\,\frac{d\,f(x)}{dx}, \tag{2.16}$$

from which clearly it appears that the κ derivative is governed by the same rules of the ordinary one.

D. Deformed exponential

The κ exponential $\exp_{\{\kappa\}}(x)\in\mathcal{F}$ is defined as eigenstate of the κ derivative

$$\frac{d\,\exp_{\{\kappa\}}(x)}{dx_{\{\kappa\}}}=\exp_{\{\kappa\}}(x), \tag{2.17}$$

and is given by

$$\exp_{\{\kappa\}}(x)=\exp(x_{\{\kappa\}}). \tag{2.18}$$

It results in

$$\exp_{\{0\}}(x)=\exp(x), \tag{2.19}$$

$$\exp_{\{-\kappa\}}(x)=\exp_{\{\kappa\}}(x). \tag{2.20}$$

The κ exponential, just as the ordinary exponential, has the properties

$$\exp_{\{\kappa\}}(x)\in C^{\infty}(\mathbf{R}), \tag{2.21}$$

$$\frac{d}{dx}\exp_{\{\kappa\}}(x)>0, \tag{2.22}$$

$$\exp_{\{\kappa\}}(-\infty)=0^{+}, \tag{2.23}$$

$$\exp_{\{\kappa\}}(0)=1, \tag{2.24}$$

$$\exp_{\{\kappa\}}(+\infty)=+\infty, \tag{2.25}$$

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(-x)=1. \tag{2.26}$$

Furthermore, the κ exponential has the two properties

$$[\exp_{\{\kappa\}}(x)]^{r}=\exp_{\{\kappa/r\}}(rx), \tag{2.27}$$

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(y)=\exp_{\{\kappa\}}(x\overset{\kappa}{\oplus}y), \tag{2.28}$$

with $r\in\mathbf{R}$, and can be expressed in terms of the generator $g(x)$ as

$$\exp_{\{\kappa\}}(x)=[\sqrt{1+g(\kappa x)^{2}}+g(\kappa x)]^{1/\kappa}. \tag{2.29}$$

E. Deformed logarithm

The κ logarithm $\ln_{\{\kappa\}}(x)$ is defined as the inverse function of the of κ exponential, namely $\ln_{\{\kappa\}}(\exp_{\{\kappa\}}x)=\exp_{\{\kappa\}}(\ln_{\{\kappa\}}x)=x$, and is given by

$$\ln_{\{\kappa\}}(x)=(\ln x)^{\{\kappa\}}. \tag{2.30}$$

It results in

$$\ln_{\{0\}}(x)=\ln(x), \tag{2.31}$$

$$\ln_{\{-\kappa\}}(x)=\ln_{\{\kappa\}}(x). \tag{2.32}$$

The κ logarithm, just as the ordinary logarithm, has the properties

$$\ln_{\{\kappa\}}(x)\in C^{\infty}(\mathbf{R}^{+}), \tag{2.33}$$

$$\frac{d}{dx}\ln_{\{\kappa\}}(x)>0, \tag{2.34}$$

$$\ln_{\{\kappa\}}(0^{+})=-\infty, \tag{2.35}$$

$$\ln_{\{\kappa\}}(1)=0, \tag{2.36}$$

$$\ln_{\{\kappa\}}(+\infty)=+\infty, \tag{2.37}$$

$$\ln_{\{\kappa\}}(1/x)=-\ln_{\{\kappa\}}(x). \tag{2.38}$$

Furthermore, the κ logarithm has the two properties

$$\ln_{\{\kappa\}}(x^{r})=r\,\ln_{\{r\kappa\}}(x), \tag{2.39}$$

$$\ln_{\{\kappa\}}(xy)=\ln_{\{\kappa\}}(x)\overset{\kappa}{\oplus}\ln_{\{\kappa\}}(y), \tag{2.40}$$

with $r\in\mathbf{R}$, and can be expressed in terms of the generator $g(x)$ as

$$\ln_{\{\kappa\}}(x)=\frac{1}{\kappa}g^{-1}\left(\frac{x^{\kappa}-x^{-\kappa}}{2}\right). \tag{2.41}$$

Equation (2.41) defines a very large class of deformed logarithms varying the arbitrary function $g(x)$. These deformed logarithms can depend on many other parameter [through the generator $g(x)$] besides the parameter κ. We recall briefly that in literature one can find other one [10,11] or two [12] parameter deformations of the exponential and logarithm functions. Anyway in the following we will consider the deformed logarithms defined through Eq. (2.41) and depending only on the parameter κ.

F. Deformed trigonometry

We define the κ-hyperbolic sine and cosine

$$\sinh_{\{\kappa\}}(x)=\frac{1}{2}[\exp_{\{\kappa\}}(x)-\exp_{\{\kappa\}}(-x)], \tag{2.42}$$

$$\cosh_{\{\kappa\}}(x)=\frac{1}{2}[\exp_{\{\kappa\}}(x)+\exp_{\{\kappa\}}(-x)], \tag{2.43}$$

starting from the κ-Euler formula

$$\exp_{\{\kappa\}}(\pm x)=\cosh_{\{\kappa\}}(x)\pm\sinh_{\{\kappa\}}(x). \tag{2.44}$$

It is straightforward to introduce the κ-hyperbolic trigonometry, which reduces to the ordinary one as $\kappa\rightarrow 0$. For instance, the formulas

$$\cosh^2_{\{\kappa\}}(x)-\sinh^2_{\{\kappa\}}(x)=1, \tag{2.45}$$

$$\tanh_{\{\kappa\}}(x)=\frac{\sinh_{\{\kappa\}}(x)}{\cosh_{\{\kappa\}}(x)}, \tag{2.46}$$

$$\coth_{\{\kappa\}}(x)=\frac{\cosh_{\{\kappa\}}(x)}{\sinh_{\{\kappa\}}(x)} \tag{2.47}$$

still hold true. All the formulas of the ordinary hyperbolic trigonometry still hold true after properly deformed. The deformation of a given formula can be obtained starting from the corresponding undeformed formula, and then by making in the argument of the hyperbolic trigonometric functions the substitutions $x+y\rightarrow x\overset{\kappa}{\oplus}y$, and obviously $nx\rightarrow x\overset{\kappa}{\oplus}x\ldots\overset{\kappa}{\oplus}x$ (n times). For instance, it results in

$$\sinh_{\{\kappa\}}(x\overset{\kappa}{\oplus}y)+\sinh_{\{\kappa\}}(x\overset{\kappa}{\ominus}y)=2\,\sinh_{\{\kappa\}}(x)\cosh_{\{\kappa\}}(y), \tag{2.48}$$

$$\tanh_{\{\kappa\}}(x)+\tanh_{\{\kappa\}}(y)=\frac{\sinh_{\{\kappa\}}(x\overset{\kappa}{\oplus}y)}{\cosh_{\{\kappa\}}(x)\cosh_{\{\kappa\}}(y)}, \tag{2.49}$$

and so on.

The κ–De Moivre formula involving hyperbolic trigonometric functions having arguments of the type rx with $r\in\mathbf{R}$, assumes the form

$$[\cosh_{\{\kappa\}}(x)\pm\sinh_{\{\kappa\}}(x)]^r=\cosh_{\{\kappa/r\}}(rx)\pm\sinh_{\{\kappa/r\}}(rx). \tag{2.50}$$

Also the formulas involving the derivatives of the hyperbolic trigonometric function still hold, after properly deformed. For instance, we have

$$\frac{d\,\sinh_{\{\kappa\}}(x)}{dx_{\{\kappa\}}}=\cosh_{\{\kappa\}}(x), \tag{2.51}$$

$$\frac{d\,\tanh_{\{\kappa\}}(x)}{dx_{\{\kappa\}}}=\frac{1}{[\cosh_{\{\kappa\}}(x)]^2}, \tag{2.52}$$

and so on.

The κ-cyclic trigonometry can be constructed analogously. The κ sine and κ cosine are defined as

$$\sin_{\{\kappa\}}(x)=-i\,\sinh_{\{\kappa\}}(ix), \tag{2.53}$$

$$\cos_{\{\kappa\}}(x)=\cosh_{\{\kappa\}}(ix). \tag{2.54}$$

We remark that it results in: $\sin_{\{\kappa\}}(x)=\sin(x_{\{i\kappa\}})$ and $\cos_{\{\kappa\}}(x)=\cos(x_{\{i\kappa\}})$.

G. Deformed inverse functions

The κ-inverse hyperbolic or cyclic trigonometric functions can be introduced starting from the corresponding direct functions just as in the case of the undeformed mathematics. It is trivial to verify that κ-inverse functions are related to the κ logarithm by the usual formulas of standard mathematics. For instance, we have

$$\operatorname{arcsin}_{\{\kappa\}}(x)=-i\ln_{\{\kappa\}}(ix+\sqrt{1-x^2}), \tag{2.55}$$

$$\operatorname{arctanh}_{\{\kappa\}}(x)=\frac{1}{2}\ln_{\{\kappa\}}\frac{1+x}{1-x}, \tag{2.56}$$

and so on.

H. Deformed product and sum of functions

Let us consider the set of the non-negative real functions $\mathcal{D}=\{f,h,w,\ldots\}$.

Proposition 7. The composition law $\underset{\kappa}{\otimes}$ defined through

$$f\underset{\kappa}{\otimes}h=\exp_{\{\kappa\}}(\ln_{\{\kappa\}}f+\ln_{\{\kappa\}}h), \tag{2.57}$$

which reduces to the ordinary product as $\kappa\rightarrow 0$, namely $f\underset{0}{\otimes}h=f\cdot h$, is a deformed product and the algebraic structure $(\mathcal{D}-\{0\},\underset{\kappa}{\otimes})$ forms an Abelian group.

Proof. Indeed, this product has the following properties: (i) associative law, $(f\underset{\kappa}{\otimes}h)\underset{\kappa}{\otimes}w=f\underset{\kappa}{\otimes}(h\underset{\kappa}{\otimes}w)$;

(ii) neutral element, $f\underset{\kappa}{\otimes}1=1\underset{\kappa}{\otimes}f=f$;

(iii) inverse element, $f\underset{\kappa}{\otimes}(1/f)=(1/f)\underset{\kappa}{\otimes}f=1$; and

(iv) commutative law, $f\underset{\kappa}{\otimes}h=h\underset{\kappa}{\otimes}f$.

Of course the division $\underset{\kappa}{\oslash}$ can be defined through $f\underset{\kappa}{\oslash}h=f\underset{\kappa}{\otimes}(1/h)$. The deformed κ power $f^{\otimes r}$ is defined through

$$f^{\otimes r}=\exp_{\{\kappa\}}(r\ln_{\{\kappa\}}f), \tag{2.58}$$

and generalizes the ordinary power f^r. In particular, when r is integer one has $f^{\otimes r}=f\underset{\kappa}{\otimes}f\ldots\underset{\kappa}{\otimes}f$, (r times).

Proposition 8. The algebraic structure $(\mathcal{D},\underset{\kappa}{\otimes})$ forms an Abelian monoid.

Proof. Indeed the element 0 does not admit an inverse element.

Furthermore, just as in the case of the ordinary product, it results in $f\underset{\kappa}{\otimes}0=0\underset{\kappa}{\otimes}f=0$.

Proposition 9. The composition law $\underset{\kappa}{\oplus}$ defined through

$$f \underset{\kappa}{\oplus} h = \exp_{\{\kappa\}}\{\ln[\exp(\ln_{\{\kappa\}} f) + \exp(\ln_{\{\kappa\}} h)]\}, \quad (2.59)$$

which reduces to the ordinary sum as the deformation parameter approaches to zero, namely $f \underset{0}{\oplus} h = f + h$, is a deformed sum and the algebraic structure $(\mathcal{D}, \underset{\kappa}{\oplus})$ forms an Abelian monoid.

Proof. Indeed, this sum has the following properties:
(i) associative law, $(f \underset{\kappa}{\oplus} h) \underset{\kappa}{\oplus} w = f \underset{\kappa}{\oplus} (h \underset{\kappa}{\oplus} w)$;
(ii) neutral element, $f \underset{\kappa}{\oplus} 0 = 0 \underset{\kappa}{\oplus} f = f$; and
(iii) commutative law, $f \underset{\kappa}{\oplus} h = h \underset{\kappa}{\oplus} f$.

We remark that the product $\underset{\kappa}{\otimes}$ and sum $\underset{\kappa}{\oplus}$ are distributive operations $w \underset{\kappa}{\otimes} (f \underset{\kappa}{\oplus} h) = (w \underset{\kappa}{\otimes} f) \underset{\kappa}{\oplus} (w \underset{\kappa}{\otimes} h)$. The product $\underset{\kappa}{\otimes}$ allows us to write the following property of the κ exponential

$$\exp_{\{\kappa\}}(x) \underset{\kappa}{\otimes} \exp_{\{\kappa\}}(y) = \exp_{\{\kappa\}}(x+y). \quad (2.60)$$

Equivalently Eq. (2.60) can be written also in the form

$$\ln_{\{\kappa\}}(f \underset{\kappa}{\otimes} h) = \ln_{\{\kappa\}}(f) + \ln_{\{\kappa\}}(h). \quad (2.61)$$

Equation (2.61) gives a relevant property for the κ logarithm.

Finally, starting from the definition of the κ power $f^{\otimes r}$, we obtain the following relation:

$$r \ln_{\{\kappa\}}(f) = \ln_{\{\kappa\}}(f^{\otimes r}). \quad (2.62)$$

The relations given by Eqs. (2.61) and (2.62), which express two mathematical properties of the κ logarithm, will be very useful in the following section in defining a new additive and extensive entropy.

III. THE JAYNES MAXIMUM ENTROPY PRINCIPLE

Let us consider the following non-normalized statistical distribution involving the κ exponential,

$$f = \exp_{\{\kappa\}}[-\beta(E-\mu)]. \quad (3.1)$$

We write the real nonspecified parameter β as

$$\beta = \frac{1}{\lambda k_B T}, \quad (3.2)$$

λ being a new real parameter, k_B the Boltzmann constant, and T the temperature of the system.

In the following it will be useful to introduce the distribution

$$n = \alpha \exp_{\{\kappa\}}\left(-\frac{E-\mu}{\lambda k_B T}\right), \quad (3.3)$$

which is related with f through $n = \alpha f$, α being another new real parameter which will be determined together with λ in the following.

We recall that in the ordinary statistical mechanics the mean value of a given physical quantity $A(p,n)$, depending on the variable p and the distribution $n = n(p)$, is defined as

$$\langle A(p,n)\rangle = \frac{\int d^3p\, A(p,n) n(p)}{\int d^3p\, n(p)}. \quad (3.4)$$

Analogously, in the case where $A = A(p_1, p_2, n_1, n_2)$ depends on two independent variables p_1, p_2 and on the two independent distribution functions $n_1(p_1)$ and $n_2(p_2)$, we have that the mean value is given by

$$\langle A\rangle = \frac{\int d^3p_1 d^3p_2 A\, n_1(p_1) n_2(p_2)}{\int d^3p_1 d^3p_2 n_1(p_1) n_2(p_2)}. \quad (3.5)$$

It is easy to verify that the stationary distribution n can be obtained as a solution of the following variational equation:

$$\frac{\delta}{\delta n}\int d^3p\left[-k_B\lambda \int \ln_{\{\kappa\}}(n/\alpha)dn - \frac{1}{T}En + \frac{\mu}{T}n\right] = 0. \quad (3.6)$$

Then the distribution n can be viewed as maximizing the information content I_κ,

$$I_\kappa = \int d^3p\, J_\kappa(n), \quad J_\kappa(n) = \lambda \int \ln_{\{\kappa\}}(n/\alpha)dn, \quad (3.7)$$

under the constraints

$$\int d^3p\, En = U, \quad (3.8)$$

$$\int d^3p\, n = N, \quad (3.9)$$

imposing the conservation of the mean energy and of the particle number, respectively. Note that the chemical potential μ should be chosen in such a way to set the particle number equal to unity, namely $N = 1$[4].

We observe that when $\kappa = 0$ it results $J_0(n) = n \ln n$ and the information content I_0 is the mean value of the ordinary logarithm. In this case the above variational equation expresses the Jaynes maximum entropy principle which conducts to the Boltzmann-Gibbs statitical mechanics. In the following, in analogy with the standard statistical mechanics, we require that I_κ must be expressed as the mean value of $\ln_{\{\kappa\}} n$. To do so we must consider the subclass of the deformed logarithms obeying the condition

$$\lambda \int \ln_{\{\kappa\}}(n/\alpha)\, dn = n \ln_{\{\kappa\}} n. \tag{3.10}$$

The above condition, as we will see in the following, allows the simultaneous determination of both the form of κ logarithm and the values of the free parameters α and λ. First, we observe that this class contains the standard logarithm $\ln n$, for which results $\kappa=0$, $\alpha=1/e$, $\lambda=1$. It will be the task of the following section to investigate on the existence of new additional solutions of Eq. (3.10), beside the standard logarithm. Taking into account this condition we can write the variational equation (3.6) in the form

$$\frac{\delta}{\delta n}\int d^3p\left(-k_B n \ln_{\{\kappa\}} n - \frac{1}{T}E\, n + \frac{\mu}{T} n\right)=0. \tag{3.11}$$

We define the κ entropy through

$$S_\kappa = -k_B \int d^3p\, n \ln_{\{\kappa\}} n, \tag{3.12}$$

so that S_κ can be viewed as proportional to the mean value of the $\ln_{\{\kappa\}} n$, namely

$$S_\kappa = -k_B \langle \ln_{\{\kappa\}} n\rangle. \tag{3.13}$$

In this definition of S_κ we have a perfect analogy with the Shannon entropy S_0 which is the proportional to the mean value of the $\ln n$. It is remarkable that in both the definitions of S_κ and S_0 appears the standard mean value given by Eq. (3.4).

Equation (3.11) assumes the form

$$\frac{\delta}{\delta n}\left(-k_B\langle \ln_{\{\kappa\}} n\rangle - \frac{1}{T}\langle E\rangle + \frac{\mu}{T}\right)=0, \tag{3.14}$$

and then

$$\frac{\delta}{\delta n}\left(S_\kappa - \frac{1}{T}U + \frac{\mu}{T}\right)=0. \tag{3.15}$$

The above variational equation can be viewed as defining a maximum entropy principle analogous of the Jaynes principle of the standard Boltzmann-Gibbs statistical mechanics [13]. We remark that this maximum entropy principle, in the form given by Eq. (3.11), holds only and exclusively for the subclass of κ logarithms, which are solutions of the integral equation (3.10).

We show now that the families of entropies, defined through (3.12) and involving the κ-logarithms which are solutions of Eq. (3.10), have two important properties typical of the Shannon entropy. To do so we consider the properties (2.61) and (2.62) of κ logarithm, which rearrange as

$$\ln_{\{\kappa\}} n_1 + \ln_{\{\kappa\}} n_2 = \ln_{\{\kappa\}} n_{12}, \tag{3.16}$$

$$r \ln_{\{\kappa\}} n = \ln_{\{\kappa\}} n^*, \tag{3.17}$$

with $n_{12} = n_1 \underset{\kappa}{\otimes} n_2$ and $n^* = n^{\otimes r}$. When the systems 1 and 2 described through n_1 and n_2, respectively, are statistically independent, and after taking into account the definitions of the mean values (3.4), (3.5) and of the κ entropy (3.13), the two above properties of the κ logarithm transform into the following properties for the κ entropy:

$$S_\kappa[n_1] + S_\kappa[n_2] = S_\kappa[n_{12}], \tag{3.18}$$

$$r S_\kappa[n] = S_\kappa[n^*]. \tag{3.19}$$

Equations (3.18) and (3.19) say that the entropies S_κ defined starting from the κ logarithms which are solutions of Eq. (3.10) are additive and extensive just as the Shannon entropy. The distribution n_{12} describes the composite system obtained starting from the systems 1 and 2 while n^* describes the scaled system related to the system described through n. Note that the state described through the distribution n_{12} is different with respect to the state described through the distribution $n_1 n_2$ resulting $S_\kappa[n_{12}] \leqslant S_\kappa[n_1 n_2]$.

Finally, from the concavity property of the deformed κ logarithm the concavity of S_κ, follows

$$S_\kappa[t n_1 + (1-t) n_2] \geqslant t S_\kappa[n_1] + (1-t) S_\kappa[n_2], \tag{3.20}$$

with $0 \leqslant t \leqslant 1$.

As we have already noted, the ordinary logarithm is solution of Eq. (3.10) and then the Shannon entropy

$$S_0[n] = -k_B \int d^3p\, n \ln n, \tag{3.21}$$

which is additive and extensive ($n_{12} = n_1 n_2$ and $n^* = n^r$), is admitted within the present formalism.

In the following section we will show that Eq. (3.10) admits a new (only one) less evident solution. Then beside the Shannon entropy we have a new concave, additive, and extensive entropy which is the κ-entropy proposed in Ref. [1].

IV. THE NEW ADDITIVE AND EXTENSIVE ENTROPY

We consider Eq. (3.10) which, after performing a derivation with respect to n, assumes the form

$$n \frac{d}{dn}\ln_{\{\kappa\}} n + \ln_{\{\kappa\}} n - \lambda \ln_{\{\kappa\}}(n/\alpha) = 0. \tag{4.1}$$

In the following we will determine the explicit form of $\ln_{\{\kappa\}} n$ by solving this differential-functional equation. We recall that $\ln_{\{\kappa\}} n$ can be expressed in terms of the generator function according to Eq. (2.41) so that Eq. (4.1) becomes

$$n \frac{d}{dn} g^{-1}\left(\frac{n^\kappa - n^{-\kappa}}{2}\right) + g^{-1}\left(\frac{n^\kappa - n^{-\kappa}}{2}\right) - \lambda g^{-1}\left(\frac{(n/\alpha)^\kappa - (n/\alpha)^{-\kappa}}{2}\right) = 0. \tag{4.2}$$

In the above equation the function to be determined is now the generator function g. To do so we make the following changes of variables:

$$t = \kappa \ln n, \tag{4.3}$$

$$z(t)=g^{-1}(\sinh t), \tag{4.4}$$

$$c=-\kappa\ln\alpha, \tag{4.5}$$

so that Eq. (4.2) assumes the following simple form:

$$\kappa z'(t)+z(t)-\lambda z(t+c)=0. \tag{4.6}$$

The property (v) of the generator $g(x)$ imposes that $z(t)$ obeys the two conditions $z(0)=0$ and $z'(0)=1$. These conditions, if combined with Eq. (4.6), can be equivalently written in the form

$$\lambda z(c)=\kappa, \tag{4.7}$$

$$\lambda z'(c)=1. \tag{4.8}$$

It is more convenient to take into account these conditions and write Eq. (4.6) under the form

$$z(t+c)=z(t)z'(c)+z'(t)z(c). \tag{4.9}$$

After recalling the property (ii) of $g(x)$, which imposes that $z(-t)=-z(t)$, Eq. (4.9) can be written as

$$2z(c)z'(t)=z(c+t)+z(c-t). \tag{4.10}$$

Let us introduce the new function $w(t)=z'(t)$. We can see that Eq. (4.10), after deriving with respect to c, transforms into the following functional equation:

$$2w(c)w(t)=w(c+t)+w(c-t). \tag{4.11}$$

Finally the nonlinear transformation defined through $w(t)=\cosh\xi(t)$ permits us to write the last equation under the form

$$\cosh[\xi(c)+\xi(t)]+\cosh[\xi(c)-\xi(t)]=\cosh[\xi(c+t)]+\cosh[\xi(c-t)]. \tag{4.12}$$

It is trivial to verify that the most general solution of Eq. (4.12) is given by

$$\xi(t)=rt, \tag{4.13}$$

with r an arbitrary real parameter.

Shannon solution. We note that in the case $r=0$ we obtain $g(x)=\sinh x$ and then $\ln_{\{\kappa\}}(n)=\ln n$. This is the well known standard logarithm which, inserted in Eq. (3.12), produces the Shannon entropy.

The new solution. We consider now the case $r\neq 0$. It is easy to realize that in this case the generator $g(x)$ assumes the form

$$g(x)=\sinh\left[\frac{1}{r}\operatorname{arcsinh}(rx)\right]. \tag{4.14}$$

For simplicity of the exposition we first discuss the case $r=1$ for which $g(x)=x$. After some simple calculations we obtain that only when

$$-1<\kappa<1, \tag{4.15}$$

it is possible to determine the real constants λ and α obtaining

$$\lambda=\sqrt{1-\kappa^2}, \tag{4.16}$$

$$\alpha=\left(\frac{1-\kappa}{1+\kappa}\right)^{1/2\kappa}. \tag{4.17}$$

In this case the generator $g(x)=x$ imposes the following expressions for the deformed logarithm and exponential

$$\ln_{\{\kappa\}}(x)=\frac{x^{\kappa}-x^{-\kappa}}{2\kappa}, \tag{4.18}$$

$$\exp_{\{\kappa\}}(x)=(\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}. \tag{4.19}$$

For the general case where $r\neq 1$ we obtain the same solution given by Eqs. (4.15)–(4.19) with the only difference that in place of κ now the scaled parameter $r\kappa$ appears. Then we can set $r=1$ without loosing the generality of the theory. Moreover for $\kappa=0$ we obtain the Shannon solution as a particular and limiting case of the new solution.

Finally, after inserting the expression of the κ-logarithm given by Eq. (4.18) into Eq. (3.12), we can write the new additive and extensive entropy in the following simple form:

$$S_\kappa[n]=-k_B\int d^3p\,\frac{n^{1+\kappa}-n^{1-\kappa}}{2\kappa}. \tag{4.20}$$

We can write the entropy S_κ also in terms of the distribution f obtaining

$$S_\kappa[f]=-k_B\int d^3p(c_\kappa f^{1+\kappa}+c_{-\kappa}f^{1-\kappa}), \tag{4.21}$$

where the coefficient $c_\kappa=\alpha^{1+\kappa}/2\kappa$ depends exclusively on the deformation parameter κ and is given by

$$c_\kappa=\frac{1}{2\kappa}\left(\frac{1-\kappa}{1+\kappa}\right)^{(1+\kappa)/2\kappa}. \tag{4.22}$$

The above entropy is contained, as a particular case, in the class of entropies introduced previously in Ref. [1] [In Eq. (65) of this reference it appears the nonspecified parameter α, while the Boltzmann constant is absent because setted to have $k_B(1+\kappa)\alpha=1$].

We recall that the entropy given by Eq. (4.20) is different from the nonextensive entropy introduced in Refs. [14,15]. Of course the statistical distribution defined through Eqs. (3.1) and (4.19), introduced previously in Ref. [1], is also different from the distribution of the nonextensive statistics [15] and of the plasmas physics [16].

V. THE κ EXPONENTIAL AND κ LOGARITHM

Let us report here the main mathematical properties, some of these reported in Refs. [1–3], of the functions $\exp_{\{\kappa\}}(x)$ and $\ln_{\{\kappa\}}(x)$ defined through Eqs. (4.18) and (4.19), respectively.

We start by observing that the generator of the deformation is the function $g(x)=x$ and then from (2.1) and (2.2) we obtain

$$x_{\{\kappa\}}=\frac{1}{\kappa}\,\mathrm{arcsinh}\,(\kappa x), \tag{5.1}$$

$$x^{\{\kappa\}}=\frac{1}{\kappa}\sinh\,(\kappa x). \tag{5.2}$$

We have also (Ref. [17], p. 58)

$$x_{\{\kappa\}}=x\,F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};-\kappa^2x^2\right),\qquad \kappa^2x^2\leqslant 1. \tag{5.3}$$

The following function:

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}}, \tag{5.4}$$

which has a central role in quantum group theory [18,19], results in being proportional to $x^{\{\kappa\}}$; namely, we have the relation

$$[x]=\frac{1}{\ln_{\{\kappa\}}(e)}\,x^{\{\kappa\}}, \tag{5.5}$$

which can be written also in the form

$$[x]=\frac{\ln_{\{\kappa\}}(e^x)}{\ln_{\{\kappa\}}(e)}, \tag{5.6}$$

with $q=e^{\kappa}$. We note that the well known symmetry of quantum group theory $q\leftrightarrow q^{-1}$ is related the symmetry $\kappa\leftrightarrow-\kappa$ of the present theory. We also observe that, exploiting Eqs. (2.6) and (2.7), we can obtain the two following properties of the function $[x]$:

$$[x+y]=[x]\overset{\kappa'}{\oplus}[y], \tag{5.7}$$

$$[xy]=[x]\overset{\kappa'}{\otimes}[y], \tag{5.8}$$

with $\kappa'=(q-q^{-1})/2$.

The definitions of κ-sum and κ-product given through Eqs. (2.3) and (2.4), respectively transform as follows:

$$x\overset{\kappa}{\oplus}y=\frac{1}{\kappa}\sinh[\mathrm{arcsinh}(\kappa x)+\mathrm{arcsinh}(\kappa y)], \tag{5.9}$$

$$x\overset{\kappa}{\otimes}y=\frac{1}{\kappa}\sinh\left(\frac{1}{\kappa}\mathrm{arcsinh}(\kappa x)\,\mathrm{arcsinh}(\kappa y)\right). \tag{5.10}$$

In particular, the κ-sum assumes a very simple form

$$x\overset{\kappa}{\oplus}y=x\sqrt{1+\kappa^2y^2}+y\sqrt{1+\kappa^2x^2}. \tag{5.11}$$

Starting directly from the κ sum given by Eq. (5.11), one obtains the following expressions for the κ differential and κ derivative:

$$dx_{\{\kappa\}}=\frac{dx}{\sqrt{1+\kappa^2x^2}}, \tag{5.12}$$

$$\frac{d\,f(x)}{dx_{\{\kappa\}}}=\sqrt{1+\kappa^2x^2}\,\frac{d\,f(x)}{d\,x}. \tag{5.13}$$

We consider now the functions $\exp_{\{\kappa\}}(x)$ and $\ln_{\{\kappa\}}(x)$ which can be written also as

$$\exp_{\{\kappa\}}x=\exp\left(\frac{1}{\kappa}\mathrm{arcsinh}\,\kappa x\right), \tag{5.14}$$

$$\ln_{\{\kappa\}}(x)=\frac{1}{\kappa}\sinh(\kappa\ln x). \tag{5.15}$$

We point out the following concavity properties:

$$\frac{d^2}{dx^2}\exp_{\{\kappa\}}(x)>0,\qquad x\in\mathbf{R}, \tag{5.16}$$

$$\frac{d^2}{dx^2}\ln_{\{\kappa\}}(x)<0,\qquad x>0. \tag{5.17}$$

A very interesting property of these functions is their power law asymptotic behavior

$$\exp_{\{\kappa\}}(x)\underset{x\to\pm\infty}{\sim}|2\kappa x|^{\pm 1/|\kappa|}, \tag{5.18}$$

$$\ln_{\{\kappa\}}(x)\underset{x\to 0^+}{\sim}-\frac{1}{2|\kappa|}x^{-|\kappa|}, \tag{5.19}$$

$$\ln_{\{\kappa\}}(x)\underset{x\to+\infty}{\sim}\frac{1}{2|\kappa|}x^{|\kappa|}. \tag{5.20}$$

The Taylor expansion of the κ exponential is given by

$$\exp_{\{\kappa\}}(x)=\sum_{n=0}^{\infty}a_n(\kappa)\frac{x^n}{n!},\qquad \kappa^2x^2<1 \tag{5.21}$$

(Ref. [17], p. 26), with the coefficients a_n defined as

$$a_0(\kappa)=1,\qquad a_1(\kappa)=1,$$
$$a_{2m}(\kappa)=\prod_{j=0}^{m-1}[1-(2j)^2\kappa^2], \tag{5.22}$$
$$a_{2m+1}(\kappa)=\prod_{j=1}^{m}[1-(2j-1)^2\kappa^2].$$

It results that $a_n(0)=1$ and $a_n(-\kappa)=a_n(\kappa)$. We note that the first three terms in the above Taylor expansion are the same as the ordinary exponential, namely,

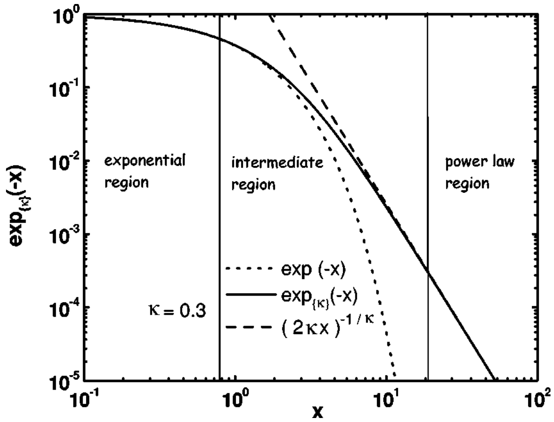

FIG. 1. Plot of the function $\exp_{\{\kappa\}}(-x)$ versus x for $\kappa=0.3$. This function is compared with the ordinary exponential and with a pure power law.

$$\exp_{\{\kappa\}}(x)=1+x+\frac{x^2}{2}+(1-\kappa^2)\frac{x^3}{3!}+\cdots. \quad (5.23)$$

In Fig. 1, the function $\exp_{\{\kappa\}}(-x)$ for a fixed value of κ is plotted. We note that the bulk of this function is very close to the standard exponential. Indeed the Taylor expansion of $\exp_{\{\kappa\}}(-x)$ is the same, up to second order, of the one of $\exp(-x)$. The tail of $\exp_{\{\kappa\}}(-x)$ behaves as a power law. Between the bulk and the tail an intermediate region whose extension depends on the value of κ exists.

In Fig. 2, the function $\exp_{\{\kappa\}}(-x)$ for some different values of κ is plotted. We note that when $\kappa\to 0$ the κ-exponential approaches the ordinary one.

The Taylor expansion of $\ln_{\{\kappa\}}(1+x)$ converges if $-1<x\leqslant 1$ and assumes the form

$$\ln_{\{\kappa\}}(1+x)=\sum_{n=1}^{\infty} b_n(\kappa)(-1)^{n-1}\frac{x^n}{n}, \quad (5.24)$$

(Ref. [17], p. 25), with $b_1(\kappa)=1$, while for $n>1$, $b_n(\kappa)$ is given by

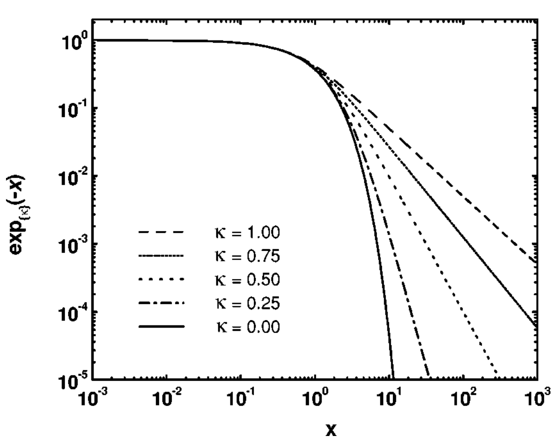

FIG. 2. Plot of the function $\exp_{\{\kappa\}}(-x)$ versus x for some different values of κ. The case $\kappa=0$ corresponds to the ordinary exponential.

$$b_n(\kappa)=\frac{1}{2}(1-\kappa)\left(1-\frac{\kappa}{2}\right)\cdots\left(1-\frac{\kappa}{n-1}\right)+\frac{1}{2}(1+\kappa)\left(1+\frac{\kappa}{2}\right)\cdots\left(1+\frac{\kappa}{n-1}\right). \quad (5.25)$$

It results in $b_n(0)=1$ and $b_n(-\kappa)=b_n(\kappa)$. The first terms of the expansion are

$$\ln_{\{\kappa\}}(1+x)=x-\frac{x^2}{2}+\left(1+\frac{\kappa^2}{2}\right)\frac{x^3}{3}-\cdots. \quad (5.26)$$

Another expansion involving the κ exponential ($x^2\leqslant 1$) is the following:

$$\exp_{\{\kappa\}}(x)=\exp\left(\sum_{n=0}^{\infty} d_n\kappa^{2n}x^{2n+1}\right) \quad (5.27)$$

(Ref. [17], p. 58), being

$$d_n=\frac{(-1)^n(2n)!}{(2n+1)2^{2n}(n!)^2}. \quad (5.28)$$

Exploiting this expansion, we can write the κ exponential as an infinite product of standard exponentials

$$\exp_{\{\kappa\}}(x)=\prod_{n=0}^{\infty}\exp(d_n\kappa^{2n}x^{2n+1}). \quad (5.29)$$

On the other hand the κ exponential can be viewed as a continuous linear combination of an infinity of standard exponentials. Namely, for Re $s\geqslant 0$ it results (Ref. [17], p. 1108)

$$\exp_{\{\kappa\}}(-s)=\int_0^{\infty}\frac{1}{\kappa x}J_{1/\kappa}\left(\frac{x}{\kappa}\right)\exp(-sx)dx. \quad (5.30)$$

The following two integrals can be useful:

$$\int_0^{\infty}x^{r-1}\exp_{\{\kappa\}}(-x)dx=\frac{[1+(r-2)|\kappa|]|2\kappa|^{-r}}{[1-(r-1)|\kappa|]^2-\kappa^2}\frac{\Gamma\left(\frac{1}{|2\kappa|}-\frac{r}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|}+\frac{r}{2}\right)}\Gamma(r), \quad (5.31)$$

$$\int_0^1\left(\ln_{\{\kappa\}}\frac{1}{x}\right)^{r-1}dx=\frac{|2\kappa|^{1-r}}{1+(r-1)|\kappa|}\frac{\Gamma\left(\frac{1}{|2\kappa|}-\frac{r-1}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|}+\frac{r-1}{2}\right)}\Gamma(r). \quad (5.32)$$

We conclude the present section focusing our attention to another interesting property of κ logarithm. We consider the following eigenvector equation:

$$\mathcal{D}_\kappa(x)L(x)=l_\kappa L(x), \quad (5.33)$$

and examine the case of the eigenvector $L(x)=n^x$. It is trivial to verify that when $\mathcal{D}_0(x)=d/dx$ we obtain that the eigenvalue l_0 of this operator is the standard logarithm $l_0=\ln n$. We pose now the question if it is possible to determine the operator $\mathcal{D}_\kappa(x)$ associated to the same eigenvector $L(x)=n^x$ and having as eigenvalue the κ logarithm, namely,

$$l_\kappa=\ln_{\{\kappa\}}n. \tag{5.34}$$

We obtain that this operator is the finite difference operation

$$\mathcal{D}_\kappa(x)L(x)=\frac{L(x+\kappa)-L(x-\kappa)}{2\kappa}, \tag{5.35}$$

which reduces to the standard derivative as the increment 2κ of the independent variable approaches to zero.

One can find many other elegant and useful mathematical properties for the κ functions which obviously we cannot report here.

VI. RELATIVISTIC KINETICS

In this section we treat the statistical system, considered previously in stationary conditions, within a relativistic and kinetic framework. The new relativistic kinetics here presented, in the limit $c\to\infty$, reduces to the classical kinetics considered in Ref. [1].

By using the standard notation of the relativistic theory we denote with $x=x^\nu=(ct,\boldsymbol{x})$ the four-vector position and with $p=p^\nu=(p^0,\boldsymbol{p})$ the four-vector momentum, being $p^0=\sqrt{\boldsymbol{p}^2+m^2c^2}$ and employ the metric $g^{\mu\nu}=diag(1,-1,-1,-1)$ [20].

Let us consider the following relativistic kinetic equation:

$$p^\nu\partial_\nu f-mF^\nu\frac{\partial f}{\partial p^\nu}=\int\frac{d^3p'}{p'^0}\frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p_1'^0}G\times[a(f'\underset{\kappa}{\otimes}f_1')-a(f\underset{\kappa}{\otimes}f_1)], \tag{6.1}$$

where the distribution f is a function of the four vectors x and p, namely $f=f(x,p)$. We note that the left hand side of Eq. (6.1) is the same of the standard relativistic Boltzmann equation but the collision integral in the right hand side results to be more complicated, containing the deformed product $\underset{\kappa}{\otimes}$ and the arbitrary function $a(f)$ which we suppose to be positive and increasing. The factor G is the transition rate which depends only on the nature of the two body particle interaction.

The above equation in the case $\kappa=0$ and $a(f)=f$ reduces to the already known relativistic Boltzmann equation describing the standard relativistic kinetics [20]. Clearly in the case $\kappa\neq 0$ and $a(f)\neq f$, the above equation describes a new relativistic kinetics, radically different from the standard one.

We anticipate that this new relativistic kinetics, which we will consider here, defines a statistics resulting to be independent on the particular form of the function $a(f)$.

Steady states. We consider now the steady states of Eq. (6.1) for which the collision integral becomes equal to zero. Then we have

$$f\underset{\kappa}{\otimes}f_1=f'\underset{\kappa}{\otimes}f_1', \tag{6.2}$$

and after taking into account the property (2.61) of the κ logarithm, we obtain

$$\ln_{\{\kappa\}}f+\ln_{\{\kappa\}}f_1=\ln_{\{\kappa\}}f'+\ln_{\{\kappa\}}f_1'. \tag{6.3}$$

This last equation represents a conservation law and then we can conclude that $\ln_{\{\kappa\}}f$ is a summational invariant; in the most general case it is a linear combination of the microscopic relativistic invariants, namely a constant and the four-vector momentum. In Ref. [20] it is shown that in presence of external electromagnetic fields the more general microscopic relativistic invariant has the form $(p^\nu+qA^\nu/c)U_\nu+$ constant, being U_ν the hydrodynamic four-vector velocity with $U^\nu U_\nu=c^2$. Then we can pose

$$\ln_{\{\kappa\}}f=-\frac{(p^\nu+qA^\nu/c)U_\nu-mc^2-\mu}{\lambda k_BT}. \tag{6.4}$$

Consequently we obtain the following stationary distribution:

$$f=\exp_{\{\kappa\}}\left(-\frac{(p^\nu+qA^\nu/c)U_\nu-mc^2-\mu}{\lambda k_BT}\right). \tag{6.5}$$

In the case $\kappa=0$ this distribution reduces to the already known relativistic distribution [20].

The above equilibrium distribution, in the global rest frame where $U^\nu=(c,0,0,0)$ and in absence of external forces $(A^\nu=0)$, simplifies as

$$f=\exp_{\{\kappa\}}\left(-\frac{E-\mu}{\lambda k_BT}\right), \tag{6.6}$$

and assumes the same form of the distribution (3.1).

We remark that for $E-\mu\gg\lambda k_BT$ this distribution presents a power law behavior, namely,

$$f\approx\left(\frac{E_*}{E}\right)^{1/\kappa}, \tag{6.7}$$

being $E_*=k_BT\sqrt{1-\kappa^2}/2\kappa$.

In order to introduce explicitly the dependence on the velocity variable in the distribution (6.6), we consider the expression of the relativistic kinetic energy $E=E(v)$,

$$E=\sqrt{m^2c^4+\mathbf{p}^2c^2}-mc^2, \tag{6.8}$$

with $\mathbf{p}=m\gamma(v)\mathbf{v}$ and the Lorentz factor given by

$$\gamma(v)=\frac{1}{\sqrt{1-v^2/c^2}}. \tag{6.9}$$

After defining $\eta=\mu/mc^2$ we write Eq. (6.6) as follows:

$$f=\exp_{\{\kappa\}}\left(-\frac{mc^2}{\lambda k_B T}[\gamma(v)-1-\eta]\right). \tag{6.10}$$

Note that in the region $v\ll c$ this distribution assumes the form

$$f\approx\exp_{\{\kappa\}}\left(-\frac{\frac{1}{2}mv^2-\mu}{\lambda k_B T}\right), \tag{6.11}$$

which, after setting $\lambda k_B T=1/\beta$, coincides with the nonrelativistic statistical distribution, proposed in Ref. [1].

H theorem. In the standard relativistic kinetics it is well known from the *H* theorem that the production of entropy is never negative and in equilibrium conditions there is no entropy production. In the following we will demonstrate the *H* theorem for the system governed by the kinetic equation (6.1). For simplicity of the notation, hereafter we omit the letter κ in the symbol of κ entropy. We define the four-vector entropy $S=S^\nu=(S^0,\boldsymbol{S})$, in terms of the distribution $n=\alpha f$, as follows:

$$S^\nu=-k_B\int\frac{d^3p}{p^0}p^\nu n\ln_{\{\kappa\}}n, \tag{6.12}$$

and note that S^0 coincides with the κ entropy defined previously through Eq. (4.20) while $\boldsymbol{S}$ is the entropy flow. If we take into account the relation (3.10), the above four-vector entropy can be written in terms of the distribution f as

$$S^\nu=-k_B\lambda\alpha\int\frac{d^3p}{p^0}p^\nu\int df\ln_{\{\kappa\}}f. \tag{6.13}$$

It is trivial to verify that the entropy production $\partial_\nu S^\nu$ can be calculated starting from the definition of S^ν and the evolution equation (6.1), obtaining

$$\begin{aligned}\partial_\nu S^\nu&=-k_B\lambda\alpha\int\frac{d^3p}{p^0}(\ln_{\{\kappa\}}f)p^\nu\partial_\nu f\\&=-k_B\lambda\alpha\int\frac{d^3p'}{p'^0}\frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p'^0_1}\frac{d^3p}{p^0}G[a(f'\underset{\kappa}{\otimes}f_1')\\&\quad-a(f\underset{\kappa}{\otimes}f_1)]\ln_{\{\kappa\}}f-k_B\lambda\alpha m\int\frac{d^3p}{p^0}(\ln_{\{\kappa\}}f)F^\nu\frac{\partial f}{\partial p^\nu}.\end{aligned} \tag{6.14}$$

Since the Lorentz force F^ν has the properties $p^\nu F_\nu=0$ and $\partial F^\nu/\partial p^\nu=0$ the last term in the above equation involving F^ν is equal to zero [20], namely,

$$\partial_\nu S^\nu=-k_B\lambda\alpha\int\frac{d^3p'}{p'^0}\frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p'^0_1}\frac{d^3p}{p^0}G[a(f'\underset{\kappa}{\otimes}f_1')-a(f\underset{\kappa}{\otimes}f_1)]\ln_{\{\kappa\}}f. \tag{6.15}$$

Given the particular symmetry of the integral in Eq. (6.15) we can write the entropy production as follows:

$$\partial_\nu S^\nu=-\frac{1}{4}k_B\lambda\alpha\int\frac{d^3p'}{p'^0}\frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p'^0_1}\frac{d^3p}{p^0}G[a(f'\underset{\kappa}{\otimes}f_1')-a(f\underset{\kappa}{\otimes}f_1)][\ln_{\{\kappa\}}f+\ln_{\{\kappa\}}f_1-\ln_{\{\kappa\}}f'-\ln_{\{\kappa\}}f_1']. \tag{6.16}$$

Finally, we set this equation in the form

$$\partial_\nu S^\nu=\frac{1}{4}k_B\lambda\alpha\int\frac{d^3p'}{p'^0}\frac{d^3p_1}{p_1^0}\frac{d^3p_1'}{p'^0_1}\frac{d^3p}{p^0}G[a(f'\underset{\kappa}{\otimes}f_1')-a(f\underset{\kappa}{\otimes}f_1)][\ln_{\{\kappa\}}(f'\underset{\kappa}{\otimes}f_1')-\ln_{\{\kappa\}}(f\underset{\kappa}{\otimes}f_1)]. \tag{6.17}$$

After imposing that $a(h)$ is an increasing function, it results $[a(h_1)-a(h_2)][\ln_{\{\kappa\}}(h_1)-\ln_{\{\kappa\}}(h_2)]\geqslant0\ \forall h_1,h_2$ and then we can conclude that

$$\partial_\nu S^\nu\geqslant0. \tag{6.18}$$

This last relation is the local formulation of the relativistic *H* theorem which represents the second law of the thermodynamics for the system governed by the evolution equation (6.1).

Concerning the arbitrary positive and increasing function $a(f)$ appearing in the collision integral of the evolution equation, we note that, if we suppose that obeys to the following condition:

$$a(f\underset{\kappa}{\otimes}f_1)=a(f)a(f_1) \tag{6.19}$$

we recover the expression

$$a(f)=\exp(\ln_{\{\kappa\}}f), \tag{6.20}$$

proposed in Ref. [1].

VII. PHYSICAL MEANING OF THE κ DEFORMATION

In this section we will show that the deformation introduced by the parameter κ emerges naturally within Einstein's special relativity, so that one can see the κ deformation as a purely relativistic effect.

Let us consider in the one-dimensional frame $\mathcal{S}$ two identical particles of rest mass m. We suppose that the first particle moves toward right with velocity v_1 while the second particle moves toward the left with velocity v_2. The relativistic momenta of the particles are given by $p_1=p(v_1)$ and $p_2=p(v_2)$, respectively, being

$$p(v)=\frac{mv}{\sqrt{1-v^2/c^2}}. \tag{7.1}$$

We consider now the same particles in a new frame $\mathcal{S}'$ which moves at constant speed v_2 toward left with respect to the frame $\mathcal{S}$. In this new frame the particles have velocities given by $v_1' = v_1 \overset{c}{\oplus} v_2$ and $v_2' = 0$, respectively, being

$$v_1 \overset{c}{\oplus} v_2 = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}, \tag{7.2}$$

the well known relativistic additivity law for the velocities. In the same frame $\mathcal{S}'$ the particle relativistic momenta are given by $p_1' = p(v_1')$ and $p_2' = 0$, respectively. Up to now, we have simply recalled some well known concepts of the special relativity [21].

Let us pose the following question: if it is possible and how to obtain the value of the relativistic momentum p_1' starting directly from the values of the momenta p_1 and p_2 in the frame $\mathcal{S}$. The answer to this apparently innocent question is affirmative. One, after straightforward calculations (see the theorem in this section), arrives at the following surprising result:

$$p_1' = p(v_1) \overset{\kappa}{\oplus} p(v_2); \quad \kappa = \frac{1}{mc}. \tag{7.3}$$

In words, the relativistic momentum p_1' of the first particle in the rest frame of the second particle is the κ-deformed sum, with $\kappa = 1/mc$, of the momenta p_1 and p_2 of the particles in the frame $\mathcal{S}$.

Unexpectedly we discover that the κ sum is the additivity law for the relativistic momenta. Eq. (7.3) which we write in the form

$$p(v_1) \overset{\kappa}{\oplus} p(v_2) = p(v_1 \overset{c}{\oplus} v_2), \quad \kappa = \frac{1}{mc}, \tag{7.4}$$

says that the κ-deformed sum and the relativistic sum of the velocities are intimately related and reduce both, to the standard sum as the velocity c approaches to infinity. The deformations in both the cases are relativistic effects and are originated from the fact that c has a finite value. Eq. (7.4) follows as a particular case from the following theorem.

Theorem. Let

$$p_i(v_i) = \frac{m_i v_i}{\sqrt{1 - v_i^2/c^2}} \tag{7.5}$$

be the relativistic momenta of two particles ($i = 1,2$) of rest mass m_1 and m_2 which move in the one-dimensional frame $\mathcal{S}$ with speed v_1 and and v_2, respectively. If we indicate with $\overset{\kappa}{\oplus}$ the κ-sum defined through Eq. (5.11) and with $\oplus^c$ the velocity relativistic additivity law defined through Eq. (7.2), it results in

$$\frac{p_1(v_1)}{m_1} \overset{\kappa}{\oplus} \frac{p_2(v_2)}{m_2} = \frac{p_i(v_1 \overset{c}{\oplus} v_2)}{m_i}, \quad \kappa = \frac{1}{c}. \tag{7.6}$$

Proof. We start by using the definition of the κ sum, subsequently we use the explicit form of the relativistic momentum and finally we use the definition of the velocity relativistic additivity law

$$\begin{aligned}
\frac{p_1(v_1)}{m_1} \overset{\kappa}{\oplus} \frac{p_2(v_2)}{m_2}
&= \frac{p_1(v_1)}{m_1} \sqrt{1 + \left[\frac{p_2(v_2)}{m_2 c}\right]^2} + \frac{p_2(v_2)}{m_2} \sqrt{1 + \left[\frac{p_1(v_1)}{m_1 c}\right]^2} \\
&= \frac{v_1}{\sqrt{1 - (v_1/c)^2}} \sqrt{1 + \frac{(v_2/c)^2}{1 - (v_2/c)^2}} \\
&\quad + \frac{v_2}{\sqrt{1 - (v_2/c)^2}} \sqrt{1 + \frac{(v_1/c)^2}{1 - (v_1/c)^2}} \\
&= \frac{v_1 + v_2}{\sqrt{[1 - (v_1/c)^2][1 - (v_2/c)^2]}} \\
&= (v_1 \overset{c}{\oplus} v_2) \frac{1 + v_1 v_2/c^2}{\sqrt{[1 - (v_1/c)^2][1 - (v_2/c)^2]}} \\
&= \frac{v_1 \overset{c}{\oplus} v_2}{\sqrt{1 - \frac{1}{c^2}\left(\frac{v_1 + v_2}{1 + v_1 v_2/c^2}\right)^2}} \\
&= \frac{v_1 \overset{c}{\oplus} v_2}{\sqrt{1 - \frac{(v_1 \overset{c}{\oplus} v_2)^2}{c^2}}} = \frac{p_i(v_1 \overset{c}{\oplus} v_2)}{m_i}.
\end{aligned} \tag{7.7}$$

Trivially from Eq. (7.6) one obtains Eq. (7.4) as the particular case, when $m_1 = m_2 = m$. Note that the parameter κ has different values in these two equations because the summed quantities in the two cases are different.

We can easily explain the meaning of the deformed derivative. We indicate with p the relativistic momentum in the frame $\mathcal{S}$, and with dG/dp the derivative with respect to p of the Lorentz invariant scalar G. The same quantities in the frame $\mathcal{S}'$ are indicated with p' and dG/dp', respectively. It is trivial to verify that

$$\frac{dG}{dp_{\{\kappa\}}} = \frac{dG}{dp'}, \tag{7.8}$$

and then we can conclude that the κ-deformed derivative can be viewed as a standard derivative in an appropriate frame.

In the following section we will consider, in the framework of the special relativity, the κ statistics of N-identical particles, where κ is a dimensionless parameter and we will determine its value. To do so, it is more convenient to write Eq. (7.4) in the form

$$\frac{p(v_1)}{\kappa mc} \overset{\kappa}{\oplus} \frac{p(v_2)}{\kappa mc} = \frac{p(v_1 \overset{c}{\oplus} v_2)}{\kappa mc}. \tag{7.9}$$

which holds for any κ.

VIII. DETERMINATION OF THE PARAMETER κ

In this section we calculate the value of the parameter κ which, due to the symmetry $\kappa \leftrightarrow -\kappa$ of the theory, we consider positive, namely $0 \leqslant \kappa < 1$. Clearly if we start from the κ-statistical distributions f_1 and f_2 describing two independent statistical systems, we can construct the distribution $f_1 \underset{\kappa}{\otimes} f_2$ which describes a new composite system. This system in the case $\kappa=0$ reduces to the one described through the distribution $f_1 f_2$. In the following we will assume that *the distribution* $f_1 f_2$ *describes a state also in the case* $\kappa \neq 0$. Obviously this state is different from the one described by $f_1 \underset{\kappa}{\otimes} f_2$. As we will see, this simple but meaningful hypothesis is sufficient to determine the value of κ in the case of relativistic statistical systems.

Taking into account the form of the distributions f_1 and f_2 given by Eq. (6.6) (for simplicity we pose $\mu=0$), and the property (2.28) of the κ exponential one can write immediately

$$\exp_{\{\kappa\}}\left(-\frac{E_1}{\lambda k_B T}\right)\exp_{\{\kappa\}}\left(-\frac{E_2}{\lambda k_B T}\right) = \exp_{\{\kappa\}}\left(-\frac{E_1}{\lambda k_B T} \overset{\kappa}{\oplus} \frac{E_2}{\lambda k_B T}\right), \tag{8.1}$$

with $E_i = E(v_i)$. After some simple algebra we rewrite Eq. (8.1) as follows:

$$\exp_{\{\kappa\}}\left(-\frac{E_1}{\lambda k_B T}\right)\exp_{\{\kappa\}}\left(-\frac{E_2}{\lambda k_B T}\right) = \exp_{\{\kappa\}}\left(-\frac{E_3}{\lambda k_B T}\right), \tag{8.2}$$

with $E_3 = E_1\sqrt{1+E_2^2/E_0^2} + E_2\sqrt{1+E_1^2/E_0^2}$ and

$$E_0 = \frac{\lambda k_B T}{\kappa}. \tag{8.3}$$

We assume now that the κ exponential in the right-hand side of the Eq. (8.2) has the same structure of the one given by Eq. (6.6). Clearly we must impose that E_0 be exclusively expressed in terms of nonstatistical parameters. In the following we will show that $E_0 = mc^2$. To do this we exploit Eq. (7.9) obtained within the special relativity. Starting from this equation and taking into account the property (2.28) of the κ exponential we obtain

$$\exp_{\{\kappa\}}\left(-\frac{p(v_1)}{\kappa mc}\right)\exp_{\{\kappa\}}\left(-\frac{p(v_2)}{\kappa mc}\right) = \exp_{\{\kappa\}}\left(-\frac{p(v_1 \oplus_c v_2)}{\kappa mc}\right). \tag{8.4}$$

Recall that we wish to calculate the parameter κ which has a value that does not depend on the particle energy. Then, without losing generality, we can consider Eq. (8.4) in the ultrarelativistic region $(v \to c)$ where it results in $p(v) \approx E(v)/c$:

$$\exp_{\{\kappa\}}\left(-\frac{E(v_1)}{\kappa mc^2}\right)\exp_{\{\kappa\}}\left(-\frac{E(v_2)}{\kappa mc^2}\right) = \exp_{\{\kappa\}}\left(-\frac{E(v_1 \oplus_c v_2)}{\kappa mc^2}\right). \tag{8.5}$$

After comparing Eqs. (8.2) and (8.5), one obtains the relation

$$\kappa mc^2 = \lambda k_B T, \tag{8.6}$$

which is the same as the one given by Eq. (8.3), only if we impose that $E_0 = mc^2$. Equation (8.6) can be also written as

$$k_B T = mc^2 \frac{\kappa}{\sqrt{1-\kappa^2}}, \tag{8.7}$$

and results in being formally similar $(k_B T/c \leftrightarrow p,\ \kappa \leftrightarrow v/c)$ to the relation defining the relativistic momentum given by Eq. (7.1). We can extract finally the value of κ obtaining

$$\frac{1}{\kappa^2} = 1 + \left(\frac{mc^2}{k_B T}\right)^2. \tag{8.8}$$

It is important to emphasize that this expression of the parameter κ holds only under the above mentioned hypothesis and imposes that $|\kappa| < 1$. This condition on the value of κ coincides with the one expressed by Eq. (4.15) and obtained in a completely different way. We have $\kappa=0$ only if $T=0$ or if $c=\infty$. The limiting case $\kappa=1$ is obtained if $T=\infty$ or if $mc^2=0$.

At this point one can write the distribution (6.6) in the form

$$f = \exp_{\{\kappa\}}\left(-\frac{1}{\kappa}\,\frac{E-\mu}{mc^2}\right). \tag{8.9}$$

Note that the statistical information of the system, namely the temperature is hidden exclusively in the parameter κ. When $E \to \infty$ the distribution (8.9) shows a power law asymptotic behavior

$$f \approx \left(\frac{mc^2}{2E}\right)^{1/\kappa}. \tag{8.10}$$

The distribution (8.9) viewed as a function of the velocity becomes

$$f = \exp_{\{\kappa\}}\left(-\frac{\gamma(v)-1-\eta}{\kappa}\right), \quad v < c. \tag{8.11}$$

Concerning its derivative one obtains

$$\frac{df}{dv} = -\frac{v}{\kappa c^2}\,\frac{\gamma^3}{\sqrt{1+(\gamma-1-\eta)^2}}\,f, \tag{8.12}$$

and then for $v \to c$ it results in

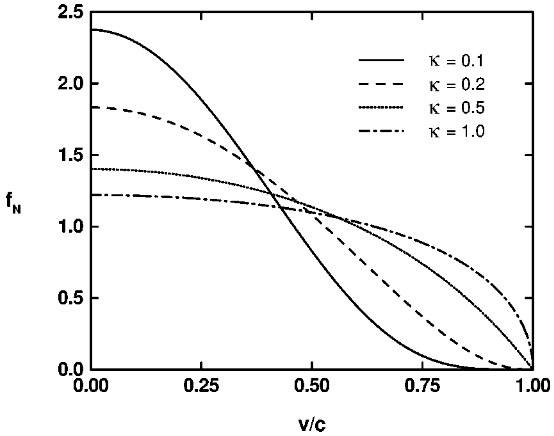

FIG. 3. Plot of the distribution function (after normalization) given by Eq. (8.16) versus v/c for some different values of κ.

$$\frac{df}{dv} \approx -\frac{1}{c\kappa 2^{1+1/2\kappa}}\left(1-\frac{v}{c}\right)^{-1+1/2\kappa}. \tag{8.13}$$

Then for $\kappa<1/2$ one has both $f=0$ and $df/dv=0$ in $v=c$. For $\kappa>1/2$ results $f=0$ and $df/dv=-\infty$ in $v=c$. Finally for $\kappa=1/2$ results $f=0$ and $df/dv=-1/2c$ in $v=c$.

In the nonrelativistic region for which $v\ll c$ we have

$$f \approx \exp_{\{\kappa\}}\left(-\frac{\frac{1}{2}mv^2-\mu}{\kappa mc^2}\right), \tag{8.14}$$

while in the limit $c\to\infty$ one recovers the standard Maxwellian distribution

$$f_M = \exp\left(-\frac{\frac{1}{2}mv^2-\mu}{k_BT}\right). \tag{8.15}$$

The explicit form of the distribution (8.11) when $\mu=-mc^2$ simplifies as follows:

$$f=\left(\sqrt{\frac{1-v^2/c^2}{1+\sqrt{2-v^2/c^2}}}\right)^{1/\kappa}, \quad v<c, \tag{8.16}$$

and in the limit $c\to\infty$ becomes

$$f_M = \exp\left(-\frac{mv^2}{2k_BT}\right). \tag{8.17}$$

In Fig. 3, the distribution function given by Eq. (8.16) (after normalization) versus v/c for different values of κ and then for different values of mc^2/k_BT, according to Eq. (8.8), is plotted.

IX. EXPERIMENTAL EVIDENCES

For a long time it has been known that the cosmic ray spectrum, which extends over 13 decades in energy, from a few hundred of MeV (10^8 eV) to a few hundred of EeV (10^{20} eV) and spans 33 decades in particle flux, from 10^4 to 10^{-29} $(\mathrm{m^2 sr\ s\ GeV})^{-1}$, is not exponential and then it violates the Boltzmann equilibrium statistical distribution $\propto\exp(-E/k_BT)$ [22–24]. Approximately this spectrum follows a power law E^{-a} and the spectral index a is near 2.7 below 5×10^{15} eV, near 3.1 above 5×10^{15} eV and again near 2.7 above 3×10^{18} eV. On the other hand it is known that the particles composing the cosmic rays are essentially the normal nuclei as in the standard cosmic abundances of matter. Then the cosmic rays can be viewed as an equivalent statistical system of identical relativistic particles with masses near the mass of the proton (938 MeV).

These above characteristics (relativistic particles with a very large extension both for their flux and energy) yield the cosmic rays spectrum an ideal physical system for a preliminary test of the correctness and predictability of the theory here proposed.

We consider the statistical distribution $f(E)$ given by Eq. (6.6) or (8.9). The particle flux $\Phi(E)\propto p^2 f(E)$ can be calculated trivially if we take into account the relativistic expression linking E and p obtaining

$$\Phi(E)=A\left[\left(\frac{E}{mc^2}+1\right)^2-1\right]\exp_{\{\kappa\}}[-\beta(E-\mu)]. \tag{9.1}$$

Note that this particle flux, in agreement with the observational data, decays following the power law

$$\Phi(E)\propto E^{-a}, \tag{9.2}$$

with

$$a=\sqrt{1+\left(\frac{mc^2}{k_BT}\right)^2}-2. \tag{9.3}$$

Analogously the particle flux obtained starting from the Boltzmann-Gibbs statistical distribution is given by

$$\Phi_0(E)=A_0\left[\left(\frac{E}{mc^2}+1\right)^2-1\right]\exp\left(-\frac{E}{k_BT}\right). \tag{9.4}$$

We use these two theoretical distributions of particle flux to fit the cosmic rays data reported in Ref. [24]. In Fig. 4, we show the observed data together with the theoretical curves $\Phi(E)$ (solid line) and $\Phi_0(E)$ (dotted line). The curve $\Phi(E)$ corresponding to $A=10^5$ $(\mathrm{m^2 sr\ s\ GeV})^{-1}$, $mc^2=938$ MeV, $\mu=-375$ MeV, and $\kappa=0.2165$ provides a high quality agreement with the observed data. This agreement over so many decades is quite remarkable. From the value of κ and mc^2 and adopting Eq. (8.7) we obtain that $k_BT=208$ MeV.

In the same figure the curve $\Phi_0(E)$ [with $A_0=1.3\times10^4$ $(\mathrm{m^2 sr\ s\ GeV})^{-1}$, $mc^2=938$ MeV, and $k_BT=208$ MeV] which decays exponentially and cannot fit the observed data violating the Boltzmann-Gibbs statistics, is reported .

A short remark must be made at this point. Clearly, the power law asymptotic behavior of the spectrum $\Phi(E)$ is imposed by the κ exponential whose origin is the κ sum. But the κ sum emerges naturally within the special relativity as the composition law of the relativistic momenta. Then we

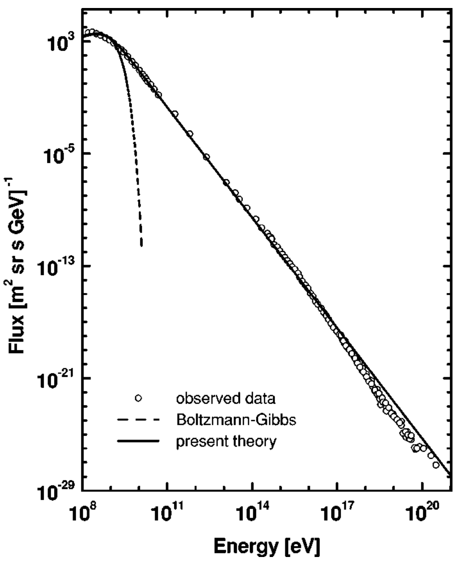

FIG. 4. Plot of the cosmic rays flux versus energy. The solid line is the curve obtained within the present theory and is given by Eq. (9.1) with $A=10^5$ $(\mathrm{m}^2\mathrm{sr\,s\,GeV})^{-1}$, $mc^2=938$ MeV, $\mu=-375$ MeV, $\kappa=0.2165$, and $k_BT=208$ MeV. The dotted line represents the theoretical curve obtained within the standard Boltzman-Gibbs statistics given by Eq. (9.4) with $A_0=1.3\times10^4$ $(\mathrm{m}^2\mathrm{sr\,s\,GeV})^{-1}$, $mc^2=938$ MeV, and $k_BT=208$ MeV. The observational data are collected by Swordy [24].

can conclude that the power law asymptotic behavior of the cosmic rays flux is simply the signature of the particle relativistic nature.

It is widely known that the Boltzmann-Gibbs distribution $\exp(-E/k_BT)$ originally proposed to describe a classical particle gas in thermodynamic equilibrium can be adopted to describe an enormous amount of phenomena in nature. On the other hand the power law tails have been observed experimentally in several fields of science. Some times in particular fields, this power law has a name (e.g., Pareto law in econophysics, Gutenberg-Richter law in seismology etc). Furthermore the power law tail is preceded by an exponential region and between the two regions exists a third intermediate region. It is worth remarking that the κ exponential defines a distribution which can describe simultaneously the three above regions (see Fig. 1) and then is particularly suitable to describe the above mentioned phenomena.

As a working example we analyze the experimental data reported in Ref. [8] related to the rain events in meteorology. In Fig. 5 is plotted the number density $N[\text{events/(year mm)}]$ of rain events versus the event size $M[\text{mm}]$ on a double logarithmic scale. We note that the data have a large extension (the abscissa spans 7 decades and the ordinate 5) and remark that its behavior is typical of a class of experimental data which we find in several areas of science. In order to fit the experimental data we adopt the distribution

$$N=A\exp_{\{\kappa\}}(-\beta M), \tag{9.5}$$

and, as one can see in Fig. 5, a remarkable agreement is obtained with $A=8\times10^4$ [events/(year mm)], $\beta=75$ (mm^{-1}) and $\kappa=0.7$.

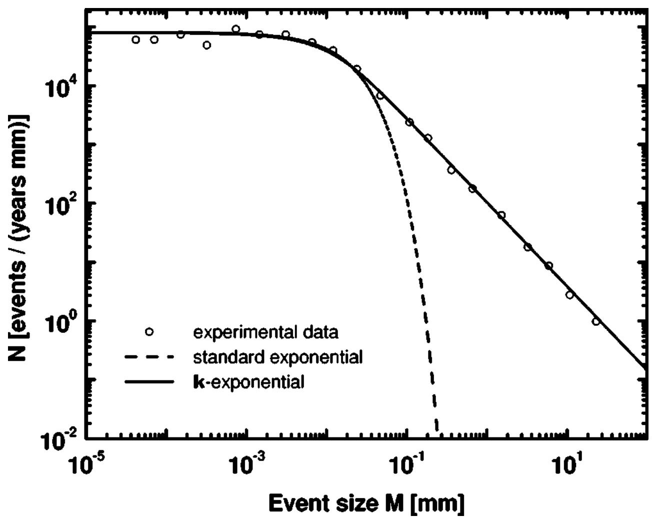

FIG. 5. Plot of the number density $N[\text{events/(year mm)}]$ of rain events versus the event size M (mm). The solid line is the curve obtained within the present theory and is given by Eq. (9.5) with $A=8\times10^4$ [events/(year mm)], $\beta=75$ (mm^{-1}) and $\kappa=0.7$. The dotted line corresponds to the ordinary exponential function ($\kappa=0$). The experimental data are from Ref. [8].

Clearly, one can hunt other mechanisms, different from the relativistic one, leading to κ statistics. Beside the important problem of the agreement between the theoretical curve and the observational data one does not neglect the epistemological problem concerning the structure of the theory which must be able both to explain the origin and to determine the value of any parameter appearing within the theory.

X. CONCLUSIONS

We summarize briefly the results obtained in the present effort. We have shown that beside the Boltzmann-Shannon entropy, the quantity

$$S_\kappa=-k_B\sum_i n_i\ln_{\{\kappa\}}n_i\,,$$

with $\ln_{\{\kappa\}}x=(x^\kappa-x^{-\kappa})/2\kappa$ and $-1<\kappa<1$, is the only existing entropy, simultaneously concave, additive and extensive. Starting from this entropy it is possible to construct a generalized statistical mechanics (and thermodynamics) having the same mathematical and epistemological structure of the Boltzmann-Gibbs one, which is recovered when the deformation parameter κ approaches to zero. Within this generalized statistics the distribution function assumes the form

$$n_i=\alpha\exp_{\{\kappa\}}[-\beta(E_i-\mu)],$$

with $\exp_{\{\kappa\}}(x)=(\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}$ while the constants α and β are given by $\alpha=[(1-\kappa)/(1+\kappa)]^{1/2\kappa}$, $1/\beta=\sqrt{1-\kappa^2}k_BT$. The chemical potential μ can be fixed by the normalization condition. This distribution has a bulk very close to the exponential one while its tail decays following a power law $n_i\propto E_i^{-1/\kappa}$.

The origin of this deformed statistics has its roots in the Einstein special relativity and the relativistic statistical mechanics kinetics obeying the H theorem.

We have shown that, within the special relativity, it is possible to determine the value of κ, obtaining

$$\frac{1}{\kappa^2}=1+\left(\frac{mc^2}{k_BT}\right)^2,$$

so that the relativistic statistcal mechanics does not contain free parameters.

The theory can describe observational data in many fields. In particular we find a high quality agreement in analyzing the spectrum of the cosmic rays which violates manifestly the Boltzmann-Gibbs statistics. This is an important test for the theory because the cosmic rays are relativistic particles and their spectrum has a very large extension (13 decades in energy and 33 decades in flux).

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