Quantum statistics in complex networks

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In this work we discuss the symmetric construction of bosonic and fermionic networks and we present a case of a network showing a mixed quantum statistics. This model takes into account the different nature of nodes, described by a random parameter that we call energy, and includes rewiring of the links. The system described by the mixed statistics is an inhomogeneous system formed by two class of nodes. In fact there is a threshold energy ϵ_s such that nodes with lower energy ($\epsilon<\epsilon_s$) increase their connectivity, while nodes with higher energy ($\epsilon>\epsilon_s$) decrease their connectivity in time.

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I. INTRODUCTION

Recently, pushed by the need to fit the available experimental data on a large variety of networks, statistical physics is addressing its attention to complex networks [1–3] and in particular to scale-free networks characterized by power-law connectivity distribution. The topological properties of these networks are related to their dynamic evolution and play a key role in collective phenomena of complex systems [4–6]. Consequently there is an urgent need of a general formalism able to make a distinction between networks. Different approaches have already been proposed for equilibrium graphs [7,8].

In this paper we will restrict our study to inhomogeneous growing networks with different quality of nodes, described by quantum statistics. In fact we have recently presented a growing scale-free network with different qualities of the nodes and a thermal noise that is described by Bose statistics [9]. On the other hand we have found that a growing Cayley-tree with different qualities of the nodes and a thermal noise is described by Fermi statistics [10,11]. In order to be synthetic in the following we will refer to these two networks as the bosonic and the fermionic networks, respectively. Given the fact that the solution of the dichotomy between Bose and Fermi statistics is an attractive topic discussed in many different contexts, from supersymmetry [12] to quantum algebras [13], in the first part of the paper we compare the growth dynamics of the two networks. We find that the bosonic and the fermionic networks are obtained by continuous subsequent addition of an elementary fanshaped unit attached in two opposite directions. While the appearance of a classical system described by quantum statistics is not completely new [14,15], this is an example of the occurrence of two symmetrically constructed models following Bose and Fermi statistics, respectively. Always having in mind the general problem of the Bose-Fermi dichotomy, in the second part of this work we provide a more realistic example of network in which the two growth processes coexist. This is obtained by rewiring a bosonic network. This complex inhomogeneous system has two classes of nodes with increasing and decreasing connectivity and is fully described by a mixed statistic depending on two chemical potentials (μ_B and μ_F).

II. SYMMETRIC CONSTRUCTION OF BOSONIC AND FERMIONIC NETWORKS

The bosonic network [9] is a scale-free network in which each node has an intrinsic quality ϵ from a time-independent distribution $p(\epsilon)$. At each time step a new node is added to the network attaching m links preferentially to more connected low-energy nodes. The probability Π_i that a new link is attached to a node of energy ϵ_i and connectivity k_i is given by

$$\Pi_i^B \propto e^{-\beta\epsilon_i} k_i . \tag{1}$$

The fermionic network [10] is a growing Cayley tree of coordination number $m+1$ in which nodes have an intrinsic quality ϵ from a time-independent distribution $p(\epsilon)$. Nodes are distinguished between nodes at the interface (with connectivity 1) and nodes in the bulk (with connectivity $m+1$). At each time step a node at the interface can grow giving rise to m new nodes. The probability that a node i grows is given by the probability ρ_i that the node is at the interface (its survivability), times $e^{\beta\epsilon_i}$,

$$\Pi_i^F \propto e^{\beta\epsilon_i} \rho_i . \tag{2}$$

The dynamics of the two networks is parameterized by β that is a characteristic of the network growth and plays the role of the inverse temperature, i.e., $\beta=1/T$. For $T=0$ the dynamics became extremal and Π_i^B, Π_i^F are different form zero only for the lowest and the highest energy nodes of the network, respectively. As the temperature increases, the dynamics involves the other nodes also and in the $T\to\infty$ limit, Π_i^B and Π_i^F do not depend anymore on the energy of the nodes.

A generic bosonic network following Eq. (1) and a generic fermionic network following Eq. (2) can be constructed by attaching a fixed elementary unit to a number of nodes growing linearly with the size of the network N.

The fixed elementary unit playing the role of the "unitary cell" in crystal lattices, is a fan-shaped element constituted by a vertex node connected to m other nodes. But the way in which this unit is attached is symmetric in the two networks. In the bosonic network the vertex of the fan is a new node attached by m links to m of the N existing nodes of the network. On the contrary, in the fermionic network the el-

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ementary unit is reversed and the vertex is one of the $(1-1/m)N$ nodes at the interface, while the m nodes attached to it are new nodes of the network. Consequently both networks are constructed by the addition of the same elementary unit attached in the two opposite directions.

The mean-field equation for the bosonic and fermionic network describes respectively, the evolution of the connectivity k_i and the survivability ρ_i of the nodes. In the bosonic network, since every new link is attached to node i with probability (1), and m new links are attached at each time step, the mean-field equation for the connectivity k_i is given by

$$\frac{\partial k_i}{\partial t}=m\frac{e^{-\beta\epsilon_i}k_i}{\sum_j e^{-\beta\epsilon_j}k_j}, \tag{3}$$

where $\Sigma_j e^{-\beta\epsilon_j}k_j$ is the normalization sum of the probability Π_i^B, Eq. (1). Symmetrically, in a fermionic network every node grows with probability Π_i^F given by Eq. (2). Consequently, the probability $\rho_i(t)$ that a node i is at the interface decreases in time following the mean-field equation

$$\frac{\partial \rho_i}{\partial t}=-\frac{e^{\beta\epsilon_i}\rho_i}{\sum_j e^{\beta\epsilon_j}\rho_j}, \tag{4}$$

where the denominator sum is needed in order to normalize the probability Π_i^F. In both networks the resulting structure optimizes the system by minimizing the "free energy" of each node of the network $\epsilon_i - T\log(k_i)$ (bosonic network) or $\epsilon_i - T|\log\rho_i|$ (fermionic network). In the two networks this optimization is achieved in different ways. In the bosonic network the low-energy nodes are more likely to be awarded a new link while in the fermionic network high-energy nodes are more likely to be removed from the interface. While geometrically the two networks are related by the reversal of the elementary unit, the mean-field equations (3) and (4), in the case $m=1$, are symmetric under time reversal $(t\to -t)$ and the change of sign of the energies $(\epsilon_i\to-\epsilon_i)$. Self-consistent calculations [9,10] show that the connectivity $k(t|\epsilon,t')$ [the survivability $\rho(t|\epsilon,t')$] of a node of energy ϵ added to the network at time t', follows a power law in time with an exponent dependent on its energy,

$$k(t|\epsilon,t')=m\left(\frac{t}{t'}\right)^{f_B(\epsilon)} \quad \text{with} \quad f_B(\epsilon)=e^{-\beta(\epsilon-\mu_B)},$$
$$\rho(t|\epsilon,t')=\left(\frac{t'}{t}\right)^{f_F(\epsilon)} \quad \text{with} \quad f_F(\epsilon)=e^{\beta(\epsilon-\mu_F)}. \tag{5}$$

The time reversal of the two mean-field solutions implies here that the connectivity of the nodes always increases in time in the bosonic networks while the survivability of the nodes always decreases in time in the fermionic network. The dynamics described by Eq. (5) depends on the two constants μ_B and μ_F given, respectively, by the solutions of the two equations

$$1=\int d\epsilon\, p(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu_B)}-1}=\int d\epsilon\, p(\epsilon)n_B(\epsilon),$$
$$1-\frac{1}{m}=\int d\epsilon\, p(\epsilon)\frac{1}{e^{\beta(\epsilon-\mu_F)}+1}=\int d\epsilon\, p(\epsilon)n_F(\epsilon), \tag{6}$$

where $n_B(\epsilon)$ and $n_F(\epsilon)$ indicate the bosonic and fermionic occupation numbers, respectively. Thus, the evolution of each node of the network is completely determined by a number, μ_B or μ_F, defined as the chemical potentials of a bosonic or fermionic system with specific volumes $v_B=1$ and $v_F=1+1/(m-1)$, respectively.

The quantum occupation numbers $n_B(\epsilon)$ and $n_F(\epsilon)$ appear spontaneously in the solution of the mean-field equations (3) and (4) and assume a clear meaning when we look at the static picture of the networks. In fact, in the bosonic network the total number of links attached to nodes with energy ϵ, $N_B(\epsilon)$, is given by

$$N_B(\epsilon)=mtp(\epsilon)[1+n_B(\epsilon)]. \tag{7}$$

In the left-hand side of Eq. (7) the first and second terms represent the number of outgoing and incoming links connected to nodes of energy ϵ. Similarly, in the fermionic network, the total number of nodes with energy ϵ found below the interface, $N_F(\epsilon)$ is given by the difference between all the nodes of the network and those that are at the interface, i.e.,

$$N_F(\epsilon)=mtp(\epsilon)[1-n_F(\epsilon)]. \tag{8}$$

Nevertheless, $n_B(\epsilon)$ and $n_F(\epsilon)$ acquire also a very specific role in the single time evolution of the network. In fact, at time t, the probability $\pi_B^{(t)}(\epsilon)$ of attaching a new link to a generic node of energy ϵ (bosonic network) and the probability $\pi_F^{(t)}(\epsilon)$ that a generic node with energy ϵ will grow in the fermionic network, are given by

$$\pi_B^{(t)}(\epsilon)=\int_1^t dt'\,\delta(\epsilon-\epsilon_{t'})\frac{\partial k(t|\epsilon_{t'},t')}{\partial t}\to p(\epsilon)n_B(\epsilon),$$
$$\pi_F^{(t)}(\epsilon)=\int_1^t dt'\,\delta(\epsilon-\epsilon_{t'})\frac{\partial \rho(t,|\epsilon_{t'},t')}{\partial t}\to p(\epsilon)[1-n_F(\epsilon)]. \tag{9}$$

These results explain the interconnection between the dynamics of the networks and their self-similar aspect. In fact, for the bosonic network we have that the probability for a new node to be linked to a node with energy ϵ converges in time to the same limit as the density of existing links pointing to nodes of energy ϵ. Similarly, for the fermionic network we have that the probability that a node with energy ϵ is chosen to grow converges to the same limit than the density of nodes in the bulk.

The occurrence of the two quantum statistics in the description of such networks is due to the fact that the networks are growing by the continuous addiction of the unitary cell but they try also to minimize the energy of the system (by the choice of the node to which attach a new link in the bosonic network or by the choice of the growing node in the fermionic network). The stochastic model behind the construction of the two networks always involves the choice of a node in between a growing number of nodes, but while in the Cayley tree a chosen node is removed from the interface and cannot be chosen anymore, in a scale-free network there is no limit to the number of links a node can acquire. Consequently, the Cayley tree is described by a Fermi distribution while the scale-free network is described by a Bose distribution.

The framework of quantum statistics clarifies the relation between the self-organized critical processes and scale-free models. In fact, the fermionic network evolution in the $T\rightarrow 0$ limit reduces to the invasion percolation dynamics on a Cayley tree [16–18], a well known self-organized process [19], while the bosonic network in the $T\rightarrow\infty$ limit reduces to the Barabási-Albert model [20] for growing scale-free networks.

III. MIXED STATISTICS IN SCALE-FREE NETWORK WITH REWIRING

Our purpose here is to expand on the previous results and to discuss systems which are governed by additional processes on top of the simple growth discussed before. For example, in real networks, in addition to the appearance of new nodes, one can observe new links as well, or rewiring of existing links. In fact rewiring of the link in a scale-free network has been used to model increasing disorder in more realistic networks [21,22]. We show that the presence of such additional processes can create a coexistence of Fermi and Bose statistics within the same system. This implies that most real systems, for which such additional processes are present, exist in a mixed state, whose statistics can be described only by simultaneously involving both Bose and Fermi statistics. It is not our purpose to model any particular system at this point. Thus next we discuss a simple system that displays this mixed behavior.

A simple example of mixed statistics is given by introducing rewiring into a bosonic network. This network is constructed iteratively in the following way: at each time step a new node and m links are added to the network. The new node has an energy ϵ chosen from a distribution $p(\epsilon)$ and the m links connect the new node preferentially to well connected, low-energy nodes of the system. As in the bosonic network without rewiring we assume that a new link is attached with probability

$$\Pi_i^+ \propto e^{-\beta\epsilon_i} k(t|\epsilon_i,t_i) \tag{10}$$

to node i arrived in the network at time t_i, with energy ϵ_i and connectivity $k(t|\epsilon_i,t_i)$ at time t. Furthermore we assume also that at each time step m' edges detach from existing nodes and are rewired to the new node. Consequently, every new node will have $m+m'$ links attached to it. We assume that the edges connected to high-energy nodes are more unstable, so that the probability that an edge connected to a node of energy ϵ_i detaches from it is proportional to $e^{\beta\epsilon_i}$. Consequently, the probability that a node i will loose a link because of the rewiring is given by

$$\Pi_i^- \propto e^{\beta\epsilon_i} k(t|\epsilon_i,t_i), \tag{11}$$

where t_i is the time node i added in the network , ϵ_i is its energy, and $k(t|\epsilon_i,t_i)$ is its connectivity at time t. The continuous equation describing the time evolution of the connectivities of the nodes is given by

$$\frac{\partial k(t|\epsilon_i,t_i)}{\partial t} = m\frac{e^{-\beta\epsilon_i}k(t|\epsilon_i,t_i)}{\sum_j e^{-\beta\epsilon_j}k(t|\epsilon_j,t_j)} - m'\frac{e^{\beta\epsilon_i}k(t|\epsilon_i,t_i)}{\sum_j e^{\beta\epsilon_j}k(t|\epsilon_j,t_j)} \tag{12}$$

with the initial condition

$$k_0 = k(t|\epsilon_i,t) = m + m'. \tag{13}$$

To solve Eq. (12) we assume that in the thermodynamic limit the normalization sums Z_B and Z_F, given by

$$\begin{aligned} Z_B &= \sum_j e^{-\beta\epsilon_j}k(t|\epsilon_j,t_j), \\ Z_F &= \sum_j e^{\beta\epsilon_j}k(t|\epsilon_j,t_j), \end{aligned} \tag{14}$$

self-average and converge to their mean value $\langle Z_B\rangle_\epsilon$ and grow linearly in time, with the asymptotic behavior given by the constants μ_B and μ_F, $\langle Z_F\rangle_\epsilon$,

$$\begin{aligned} Z_B &\rightarrow \langle Z_B\rangle_\epsilon \rightarrow m t e^{-\beta\mu_B}, \\ Z_F &\rightarrow \langle Z_F\rangle_\epsilon \rightarrow m' t e^{\beta\mu_F}. \end{aligned} \tag{15}$$

Using Eq. (15), the dynamic equation (12) reduces to

$$\frac{\partial k(t|\epsilon_i,t_i)}{\partial t} = (e^{-\beta(\epsilon_i-\mu_B)} - e^{\beta(\epsilon_i-\mu_F)})\frac{k(t|\epsilon_i,t_i)}{t}. \tag{16}$$

Consequently, we have found that the time evolution of the connectivity $k(t|\epsilon_i,t_i)$ follows a power law

$$k(t|\epsilon,t') = k_0\left(\frac{t}{t'}\right)^{f_{mix}(\epsilon)} \tag{17}$$

with

$$f_{mix}(\epsilon) = e^{-\beta(\epsilon-\mu_B)} - e^{\beta(\epsilon-\mu_B)}. \tag{18}$$

The characteristic difference of this network from the bosonic scale-free network is that the connectivity of the nodes, due to the rewiring process, can either increase or decrease in time. In fact, $f_{mix}(\epsilon)$ [defined in Eq. (18)] change sign at a threshold energy value

$$\epsilon_s = \frac{\mu_B + \mu_F}{2}. \tag{19}$$

Consequently, the nodes with energy $\epsilon < \epsilon_s$ increase their connectivity in time while nodes with energy higher then the threshold ϵ_s, i.e., $\epsilon > \epsilon_s$, have a decreasing connectivity.

After substituting $k(t|\epsilon_i, t_i)$ from Eq. (17) with $f_{mix}(\epsilon)$ given by Eq. (18) into Eq. (12) and the sum over j with an integral , we get the self-consistent equations for the chemical potential μ_B and μ_F,

$$\frac{m}{m+m'} = \int d\epsilon p(\epsilon) \frac{e^{-\beta(\epsilon-\mu_B)}}{1 - e^{-\beta(\epsilon-\mu_B)} + e^{\beta(\epsilon-\mu_F)}},$$

$$\frac{m'}{m+m'} = \int d\epsilon p(\epsilon) \frac{e^{\beta(\epsilon-\mu_F)}}{1 - e^{-\beta(\epsilon-\mu_B)} + e^{\beta(\epsilon-\mu_F)}}. \tag{20}$$

At the same time, the distribution of edges attached to nodes with energy ϵ converges to the mixed statistics

$$n_{mix}(\epsilon)p(\epsilon) = (m+m')p(\epsilon) \frac{1}{1 - e^{-\beta(\epsilon-\mu_B)} + e^{\beta(\epsilon-\mu_F)}}, \tag{21}$$

while the number $n_+(\epsilon)p(\epsilon)$ of the edges stochastically attached to the nodes of energy ϵ or the number $n_-(\epsilon)p(\epsilon)$ of the nodes detached from nodes with energy ϵ are given by

$$n_+(\epsilon)p(\epsilon) = mp(\epsilon) \frac{e^{-\beta(\epsilon-\mu_B)}}{1 - e^{-\beta(\epsilon-\mu_B)} + e^{\beta(\epsilon-\mu_F)}},$$

$$n_-(\epsilon)p(\epsilon) = m'p(\epsilon) \frac{e^{\beta(\epsilon-\mu_F)}}{1 - e^{-\beta(\epsilon-\mu_B)} + e^{\beta(\epsilon-\mu_F)}}, \tag{22}$$

respectively. The distribution $n_{mix}(\epsilon)$ appears as a natural candidate of a mixed statistics going from the $\mu_F \to \infty$ limit where $n_{mix}(\epsilon) \propto 1 + n_B(\epsilon)$ to the $\mu_B \to \infty$ limit where $n_{mix}(\epsilon) \propto n_F(\epsilon)$.

We have simulated a network with $m=2$ and $m'=1$ and uniform energy distribution $p(\epsilon)=1$ for $\epsilon \in (0,1)$, with chemical potentials $\mu_B = 0.03$, $\mu_F = 0.51$, and $\epsilon_s = 0.27$. In Fig. 1 we show the connectivity of the nodes of the network with energy values above and below the threshold $\epsilon_s = 0.27$. The figure shows that nodes with energy $\epsilon < \epsilon_s$ increase their connectivity in time while nodes with energy $\epsilon > \epsilon_s$ decrease their connectivity in time. In Fig. 2 we report the number of links attached to the nodes of energy ϵ, $n_{mix}(\epsilon)$, for a system size $N = 10^4$ with the data averaged over 100 runs. In the same figure we report also the number of nodes stochastically attached to (detached from) nodes of energy ϵ, $n_+(\epsilon)$ $[n_-(\epsilon)]$.

The connectivity distribution $P(k)$ is given by the sum of the probabilities $P(k|\epsilon)$ that a node with energy ϵ has connectivity k. Thus, if $k > m+m'$ we have to sum over all the nodes with energy lower then the threshold ϵ_s, while if $k < m+m'$ the summation will be over the nodes with energies higher than the threshold,

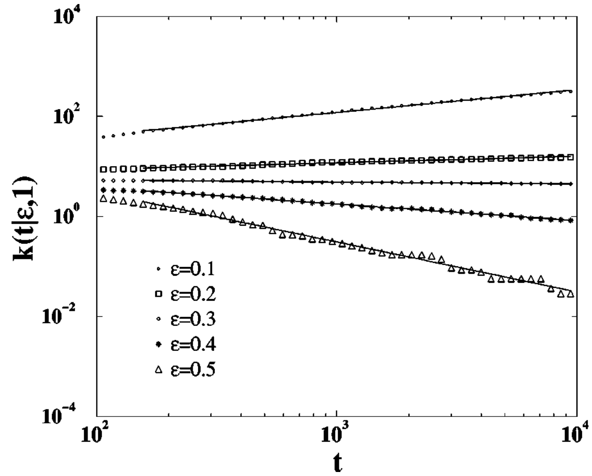

FIG. 1. Dynamical evolution of the connectivity of nodes with different energies. The connectivity of the nodes always follows a power law, increasing or decreasing in time depending on the energy ϵ and the threshold value ϵ_s.

$$P(k) = \theta(k-k_0)\frac{t}{k_0}\int_{\epsilon<\epsilon_s} d\epsilon p(\epsilon) \frac{1}{|f_{mix}(\epsilon)|}\left(\frac{k}{k_0}\right)^{-\gamma(\epsilon)} + \theta(k_0-k)\frac{t}{k_0}\int_{\epsilon>\epsilon_s} d\epsilon p(\epsilon) \frac{1}{|f_{mix}(\epsilon)|}\left(\frac{k}{k_0}\right)^{-\gamma(\epsilon)}, \tag{23}$$

with

$$\gamma(\epsilon) = 1 + 1/f_{mix}(\epsilon) \tag{24}$$

with

$$\gamma(\epsilon) > 1 \quad \text{for} \quad \epsilon < \epsilon_s, \tag{25}$$

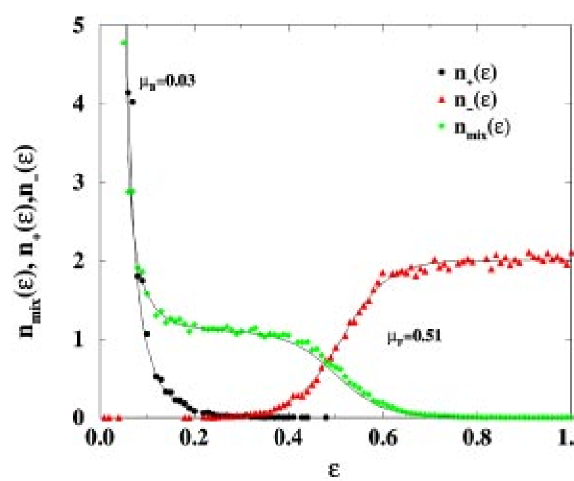

FIG. 2. The number n_{mix} of edges attached to the nodes with energy ϵ, the number $n_+(\epsilon)$ of the edges stochastically attached to the nodes with energy ϵ, and the number $n_-(\epsilon)$ of the nodes detached from nodes with energy ϵ are plotted as a function of energy. The simulations have been obtained with a uniform energy distribution in the interval [0,1]. The data for 10^5 time steps are averaged over 100 runs.

$$\gamma(\epsilon)<1 \quad \text{for} \quad \epsilon>\epsilon_s.$$

In the *limit* $\beta\to 0$ all the nodes of the network evolve in the same way with

$$f_{mix}(\epsilon)=\frac{m-m'}{2m}=\Delta. \tag{26}$$

Thus, if $\Delta>0$ every node increases its connectivity in time while if $\Delta<0$ all the nodes have decreasing connectivity. In the case $\Delta=0$ the mean-field equation describes a system in which the connectivities remain constant in time.

On the contrary in the *limit* $\beta\to\infty$ the difference between nodes with different energy is strongly enhanced.

We have to observe that as Δ goes from its highest value $\Delta=1/2$ to negative values, the energy distribution goes from a pure Bose distribution to a mixed distribution with an increasing Fermi character, i.e., with a decreasing Fermi potential μ_F. But it is impossible to reach the pure Fermi statistics in this way. In fact, if we consider the limit $m=0$, the number of links in the network is not increasing in time, and the new nodes only acquire edges from the rewiring process. In this case the connectivity of the nodes decreases exponentially as

$$k(t|\epsilon,t_i)=k_0\exp[-e^{-\beta(\epsilon-\mu_F)}(t-t_i)] \tag{27}$$

with the chemical potential defined by

$$N=\int d\epsilon p(\epsilon)m' e^{\beta(\epsilon-\mu_F)}. \tag{28}$$

We observe that in this case the network does not grow anymore and the number of edges attached to nodes with energy ϵ is simply given by the Boltzmann occupation factor. In this case, the self-consistent equation and the mass conservation relation are not anymore equivalent, the first one reducing in the thermodynamic limit to an identity. For this network, the probability $P(k)$ to find a node with connectivity k is given by

$$P(k)=\frac{1}{k}\int d\epsilon p(\epsilon)k_0 e^{\beta(\epsilon-\mu_F)}, \tag{29}$$

i.e., goes like $P(k)\sim k^{-1}$.

IV. CONCLUSIONS

In conclusion we have shown the symmetry between the fermionic and the bosonic networks emphasizing the role of quantum statistics.

These two particular evolving networks are related by the time reversal evident in the continuum equations describing their dynamics and in the reversed unitary unit by which the two networks are built. This time reversal implies that the connectivity increases in time while the survivability of each node decreases in time as an energy-dependent power law. The time reversal of the single process generates two different structures with properties and dynamics only described by the functionals μ_B and μ_F, at every temperature $T=1/\beta$. Having introduced these two limit simple cases and having illustrated their symmetry we have shown that it is possible to construct a new class of networks described by a mixed statistics that can be applied to real systems where the two different growth processes coexist.

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