

Multifractal stationary random measures and multifractal random walks with log infinitely divisible scaling laws

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We define a large class of continuous time multifractal random measures and processes with arbitrary log infinitely divisible exact or asymptotic scaling law. These processes generalize within a unified framework both the recently defined log-normal multifractal random walk [J.F. Muzy, J. Delour, and E. Bacry, Eur. J. Phys. B **17**, 537 (2000), E. Bacry, J. Delour, and J.F. Muzy, Phys. Rev. E **64**, 026103 (2001)] and the log-Poisson “product of cylindrical pulses” [J. Barral and B.B. Mandelbrot, Cowles Foundation Discussion Paper No. 1287, 2001 (unpublished)]. Our construction is based on some “continuous stochastic multiplication” [as introduced in F. Schmitt and D. Marsan, Eur. J. Phys. B. **20**, 3 (2001)] from coarse to fine scales that can be seen as a continuous interpolation of discrete multiplicative cascades. We prove the stochastic convergence of the defined processes and study their main statistical properties. The question of genericity (universality) of limit multifractal processes is addressed within this new framework. We finally provide a method for numerical simulations and discuss some specific examples.

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I. INTRODUCTION

Multifractal processes are now widely used models in many areas including nonlinear physics, geophysics, or econophysics. They are used to account for scale-invariance properties of some observed data. Our purpose in this article is to introduce a wide class of random measures and stochastic processes with stationary increments that possess exact multifractal scaling without *any preferred scale ratio*. This construction generalizes (and unifies) the recently introduced log-normal “multifractal random walk” (MRW) model [1,2] and the log-Poisson compounded “multifractal product of cylindrical pulses” [3]. Technical mathematical proofs are reported in the companion paper [4].

In the multifractal framework, scale-invariance properties of a one-dimensional (1D) stochastic process $X(t)$ are generally characterized by the exponents ζ_q which govern the power-law scaling of the absolute moments of the “fluctuation” $\delta_l X(t)$ of $X(t)$ at any scale l up to a scale T , i.e.,

$$m(q, l) = K_q l^{\zeta_q}, \quad \forall l \leq T, \quad (1)$$

where $m(q, l)$ is defined as the expectation (from now on, the symbol $E(\cdot)$ will always refer to the mathematical expectation):

$$m(q, l) = E[|\delta_l X(t)|^q]. \quad (2)$$

The so-called “fluctuation” $\delta_l X(t)$ can be defined in various ways, the most commonly used definition being

$$\delta_l X(t) = X(t+l) - X(t). \quad (3)$$

When the exponent ζ_q is linear in q , i.e., $\exists H, \zeta_q = qH$, the process is referred to as a *monofractal* process. Let us note that the so-called *self-similar* processes (e.g., fractional Brownian motions, Levy walks) are a particular case of monofractal processes. On the contrary, if ζ_q is a nonlinear function of q it is referred to as a *multifractal* process (or as a process displaying *multiscaling* or *intermittency*).

The scale-invariance property (1) can be qualified as a *continuous* scale-invariance in the sense that the relation is a strict equality for the continuum of scales $0 < l \leq T$. Alternative “weaker” forms of scale-invariance have been widely used. All of them assume at least the asymptotic scale-invariance relation

$$m(q, l) \sim K_q l^{\zeta_q}, \quad l \rightarrow 0^+. \quad (4)$$

The *discrete* scale-invariance property adds strict equality for discrete scale values l_n (with $l_n \rightarrow 0^+$, when $n \rightarrow +\infty$)

$$m(q, l_n) = K_q l_n^{\zeta_q}, \quad n \rightarrow +\infty. \quad (5)$$

The image implicitly associated with a multifractal process is a random multiplicative cascade from coarse to fine scales. Such cascading processes can be explicitly defined in very different ways ranging from the original construction proposed by Mandelbrot in its early work [5] to recent wavelet-based variants [6]. Though these different constructions do not lead to the same objects, they all aim at building a stochastic process $X(t)$ which fluctuation $\delta_{\lambda l} X(t)$ at a scale λl (where l is an arbitrary scale smaller than the large scale T and $\lambda < 1$) is obtained from its fluctuation $\delta_l X(t)$ at the larger scale l through the simple “cascading” rule

$$\delta_{\lambda l} X(t) = W_{\lambda}^{law} \delta_l X(t), \quad \forall l \leq T, \quad (6)$$

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where W_λ is a positive random variable independent of X and which law depends only on λ .

In the multiplicative construction by Mandelbrot [5] [which log-normal variant was originally introduced by Yaglom [7]], a positive multifractal measure $M(dt)$ is built, i.e., $X(t) = M([0, t])$ is an increasing process and the fluctuation $\delta_l X(t)$ at scale l is defined by Eq. (3). In the more recent wavelet-based constructions $X(t)$ is a process, not necessarily increasing, and the fluctuation $\delta_l X(t)$ is defined as the wavelet coefficient at scale l and time t [6]. Such cascade processes have been extensively used for modeling scale-invariance properties as expressed in Eq. (1). However, apart from self-similar processes which are monofractal processes with continuous dilation-invariance properties, all cascade processes involve some arbitrary discrete scale ratio and consequently only have discrete scale-invariance properties [Eq. (5)]. Indeed, they rely on a “coarse to fine” approach consisting first in fixing the fluctuations at the large scale T and then, using Eq. (6) iteratively, deriving fluctuations at smaller and smaller scales. However, the (t, l) time-scale half-plane is very constrained: one cannot choose freely $\delta_l X(t)$ at all scales and times. Thus, the cascade is generally not built using the whole time-scale half-plane but only using a sparse (e.g., dyadic) grid in this half-plane. Consequently an arbitrary scale ratio (e.g., $\lambda = 1/2$) is introduced in the construction. The continuous scale-invariance property (1) is thus broken and replaced by the weaker discrete scale-invariance property (5) with $l_n = \lambda^n T$. Moreover, as a consequence, the fluctuations $\delta_l X(t)$ in these approaches are no longer stationary.

Let us point out that despite the potential interest of stochastic processes with stationary fluctuations and continuous invariance-scaling properties, until recently [1–3], explicit constructions possessing such properties were lacking. Let us assume that one can build a continuous cascade process $X(t)$ satisfying Eq. (6) with a continuous dilation parameter λ . As first pointed out by Novikov [8], if such a construction is possible, a simple transitivity argument shows that $\ln W_\lambda$ must have an infinitely divisible law: its characteristic function is of the form $\hat{G}_\lambda(q) = \lambda^{F(q)}$. One can then easily prove that $m(q, \lambda l) = m(q, l) \lambda^{F(-iq)}$, and consequently $\zeta_q = F(-iq)$. Continuous cascade statistics and log infinite divisibility have been the subject of many works with applications in various domains ranging from turbulence to geophysics [8–16]. In the case $W_\lambda = \lambda^H$ is deterministic, one gets $\zeta_q = qH$ and therefore, $X(t)$ is a monofractal process. This is the case of the so-called self-similar processes. The simplest nonlinear (i.e., multifractal) case is the so-called log-normal cascade that corresponds to a log-normal law for W_λ and thus to a parabolic ζ_q spectrum. Other well known log infinitely divisible models often used in the context of fully developed turbulence are log-Poisson [17] and log-Levy [18] models.

From our knowledge, among all the attempts to build multifractal processes with continuous scale invariance properties and stationary fluctuations, only the recent works by Bacry *et al.* [1,2] and Barral and Mandelbrot [3] refer to a precise mathematical construction. Bacry *et al.* have built the

so-called multifractal random walk processes as limit processes based on discrete time random walks with stochastic log-normal variance. Independently, Barral and Mandelbrot [3] have proposed a new class of stationary multifractal measures. Their construction is based on distributing, in a half-plane, Poisson points associated with independent identically distributed (i.i.d.) random weights and then taking a product of these weights over conical domains. In the same time, Schmitt and Marsan considered an extension of discrete cascades to a continuous scale framework [19] and introduced infinitely divisible stochastic integrals over conelike structures similar to those of Barral and Mandelbrot. However, they did not consider any continuous time limit of their construction nor study its scaling properties as defined by Eq. (1).

In this paper, we propose a model that is based on the stochastic approach developed in Ref. [19] and that unifies all previous constructions within a single framework. Starting from original discrete multiplicative cascades, we will use a conelike construction as in Refs. [3,19] in order to get rid of discrete scale ratios and to consider any log infinitely divisible multifractal statistics. We will show that this allows us to build a very large class of multifractal measures and processes [including original MRW [1,2] and Barral-Mandelbrot multifractal measures [3]] for which both long-range correlations and multiscaling properties can be controlled very easily.

The paper is organized as follows. In Sec. II, we review the discrete multiplicative cascades in order to naturally introduce the notion of stochastic integral over a conelike structure in some “time-scale” half-plane. We then define a class of log infinitely divisible stationary multifractal random measures (MRM) which statistical properties are studied in Sec. III. In Sec. IV, we define the log infinitely divisible multifractal random walks. We show how their scaling properties can be inferred from the associated MRM. In Sec. V we address some questions related to numerical simulations and provide explicit examples. In Sec. VI, we discuss some links of the present work with previous connected approaches. Conclusions and prospects for future research are reported in Sec. VII.

II. FROM DISCRETE MULTIPLICATIVE MEASURES TO MULTIFRACTAL RANDOM MEASURES

In this section, we provide some heuristics about how one can build a positive stationary stochastic measure $M(dt)$ with continuous scale-invariance properties, i.e., such that the associated increasing process $X(t) = M([0, t])$ satisfies Eq. (6) with a continuous dilation parameter λ .

A. Discrete multiplicative cascades

For the sake of illustration, let us start with simple discrete multiplicative cascades. In the original construction, as proposed by Mandelbrot, one builds the measure $M(dt)$ as the limit of a sequence of stochastic measures M_{l_n} , indexed by a discrete scale parameter $l_n = T\lambda^n$ (we choose $\lambda = 1/2$). The measure M_{l_n} , at the step n of the construction, has a

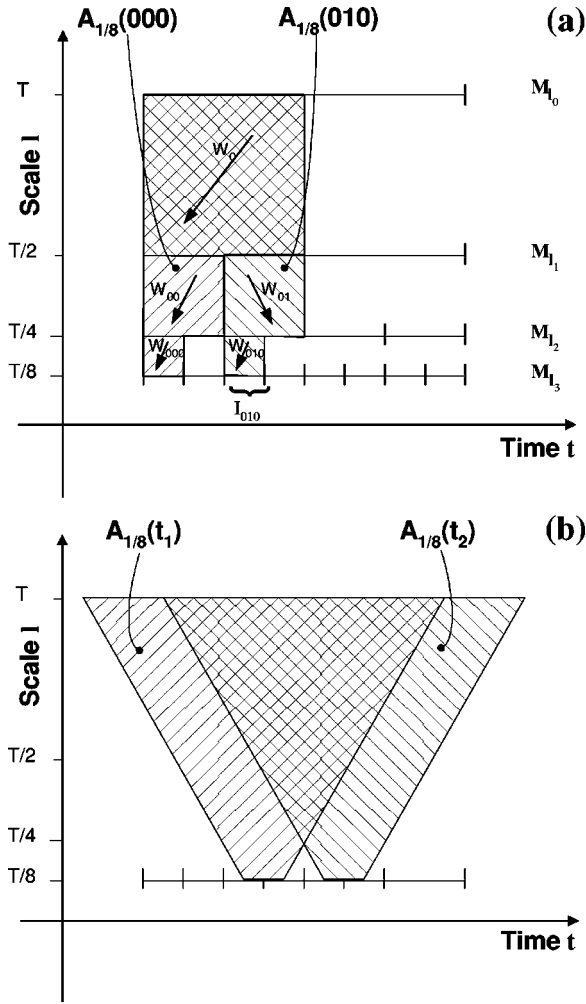


FIG. 1. Defining a continuous cascade interpolating Mandelbrot cascades. (a) Mandelbrot cascade: One starts from the coarse scale T and constructs the sequence M_{l_n} that is resolved at scale $T2^{-n}$ by iterating a multiplicative rule along the dyadic tree. At each construction step, the random multiplicative factor $W_{s_0 \dots s_n}$ can be represented as the exponential of the integral of an infinitely divisible noise over a square domain indicated by hatched sets. (b) Continuous cascade conical domains obtained as linear smoothing of the hatched domains in (a). Such cones are involved in the definition of the process $\omega_{2^{-3}}(t)$.

uniform density on successive dyadic intervals of size l_n . The idea is to build the sequence M_{l_n} so that it satisfies (when n is varying) a scale-invariance property. The n th measure is obtained from the $(n-1)$ th measure by multiplication with a positive random process W which law does not depend on n . One can naturally index the dyadic intervals along the dyadic tree using a kneading sequence $\{s_1 \dots s_n\}$ where $s_i=0$ ($s_i=1$) if, at “depth” i , the interval is on the left (on the right) boundary of its parent interval. Thus, for instance, one gets the following dyadic intervals: $I_0 = [0, 2^{-1}]$, $I_1 = [2^{-1}, 1]$, $I_{00} = [0, 2^{-2}]$, $I_{01} = [2^{-2}, 2^{-1}]$, and so on. With these notations, the multiplicative rule reads [see Fig. 1(a)]

$$M_{l_n}(I_{s_0 \dots s_n}) = W_{s_0 \dots s_n} M_{l_{n-1}}(I_{s_0 \dots s_{n-1}}), \quad (7)$$

where $I_{s_0 \dots s_n}$ is of size $T2^{-n}$ and is one of the two “sons” of the interval $I_{s_0 \dots s_{n-1}}$ and $W_{s_0 \dots s_n}$ are i.i.d. Since the construction is invariant with respect to a rescaling by a factor 2, the limit measure $M(dt)$ will then satisfy the same scale-invariance property and will be multifractal in the discrete sense of Eq. (5). There is a huge mathematical literature devoted to the study of such a construction and we refer the reader to Refs. [20–23] for rigorous results about the existence, regularity and statistical properties of Mandelbrot cascades. In physics or other applied sciences, as recalled in the Introduction, the previous construction (and many of its variants) is considered as the paradigm for multifractal objects and has been often used as a reference model in order to reproduce observed multiscaling. But because of its lack of continuous scale invariance and translation invariance, such models cannot be fully satisfactory in many contexts where the considered phenomena possess some degree of stationarity and do not display any preferred scale ratio.

B. Revisiting discrete cascades

In order to generalize Eq. (7) to a continuous framework one can try to perform the limit $\lambda \rightarrow 1$ in the discrete construction as in Ref. [19]. Another way to proceed is to represent a Mandelbrot cascade as a discretization of an underlying continuous construction.

For that purpose, we suppose that the random weights $W_{s_0 \dots s_n}$ in the cascade are log infinitely divisible. For a continuous cascade, this choice can be motivated as follows: Let us suppose that the large scale density, $M_T(dt)$ is equal (or proportional) to the Lebesgue measure dt . We would like to define iteratively the densities $M_l(t)$, for all $l \leq T$. To go from resolution l' to resolution $l \leq l'$, we thus write

$$M_l(t) = W_{l/l'}(t) M_{l'}(t), \quad \forall l \leq l', \quad (8)$$

where $W_{l/l'}(t)$ is a positive stationary discrete random process independent of $M_{l'}(t)$. Let us define

$$\omega_l(t) = \ln W_{l/T}(t). \quad (9)$$

Thus, one gets

$$M_l(t) dt = e^{\omega_l(t)} M_T(t) dt = e^{\omega_l(t)} dt. \quad (10)$$

By iterating Eq. (8), we see that $W_{l/l'}(t)$ can be obtained as a “continuous product” of positive i.i.d. random variables. Consequently $\omega_l(t)$ can be written as the sum of an arbitrary number of i.i.d. random variables. This is precisely the definition of an infinitely divisible random variable [24]. In order to go from discrete to continuous cascades, it is therefore natural to assume that, in the discrete situation, $\omega_l(s_0 \dots s_n) = \ln(\prod_{i=0}^n W_{s_0 \dots s_i})$ is infinitely divisible. A “simple” way of building such an infinitely divisible process ω_l is to represent it by a stochastic integral of an infinitely divisible stochastic 2D measure $P(dt, dl)$ over a domain $\mathcal{A}_{l_n}(s_0 \dots s_n)$:

$$\omega_{l_n}(s_0 \dots s_n) = P[\mathcal{A}_{l_n}(s_0 \dots s_n)] = \int_{\mathcal{A}_{l_n}(s_0 \dots s_n)} P(dt, dl). \tag{11}$$

The stochastic measure $P(dt, dl)$ is uniformly (with respect to a measure $\mu(dt, dl)$) distributed on the time-scale half-plane $\mathcal{S}^+ = \{(t, l), t \in \mathbb{R}, l \in \mathbb{R}^+\}$.

Let us recall that, by definition,¹ if $P(dt, dl)$ is a stochastic infinitely divisible measure uniformly (with respect to a measure $\mu(dt, dl)$) distributed on \mathcal{S}^+ , then, for any two μ -measurable sets \mathcal{A} and \mathcal{A}' such that $\mu(\mathcal{A}) = \mu(\mathcal{A}')$, then $P(\mathcal{A})$ and $P(\mathcal{A}')$ are identically distributed random variables which characteristic function is nothing but

$$E(e^{ipP(\mathcal{A})}) = e^{\varphi(p)\mu(\mathcal{A})}, \tag{12}$$

where $\varphi(p)$ depends only on a centering parameter m and the so-called canonical Lévy measure $\nu(dx)$ which is associated with P . The general shape of φ is described by the celebrated Lévy-Khintchine formula [24,26]:

$$\varphi(p) = imp + \int \frac{e^{ipx} - 1 - ip \sin x}{x^2} \nu(dx), \tag{13}$$

with $\int_{-\infty}^{-y} \nu(dx)/x^2 < \infty$ and $\int_y^{\infty} \nu(dx)/x^2 < \infty$ for all $y > 0$.

The sets $\mathcal{A}_{l_n}(s_0 \dots s_n)$ associated with each of the 2^n values $\omega_{l_n = T2^{-n}}(s_0 \dots s_n)$ in Mandelbrot construction can be chosen naturally as the union of all similar squares $I_{s_0 \dots s_k} \times [T2^{-k}, T2^{-(k-1)}]$, $n \leq k \leq 1$:

$$\mathcal{A}_{l_n}(s_0 \dots s_n) = \cup_{k=0}^n I_{s_0 \dots s_k} \times [T2^{-k}, T2^{-(k-1)}]. \tag{14}$$

The sets $\mathcal{A}_{2^{-3}}(000)$ and $\mathcal{A}_{2^{-3}}(010)$ are indicated as hatched domains in Fig. 1(a). Since the $W_{s_0 \dots s_n}$'s are i.i.d., we want to choose the measure $\mu(dt, dl)$ such that the measure of each square $I_{s_0 \dots s_n} \times [T2^{-n}, T2^{-(n-1)}]$ is a constant. The natural measure to choose is $\mu(dt, dl) = dt dl / l^2$. It is the natural measure associated with the time-scale plane \mathcal{S}^+ in the sense that it is (left) invariant by the translation-dilation group.

Fixing $\mu(dt, dl) = dt dl / l^2$ one then gets $\forall n$,

$$\int_{I_{s_0 \dots s_n} \times [T2^{-n}, T2^{-(n-1)}]} d\mu = \int_{2^{-n}}^{2^{-(n-1)}} l^{-2} dl \int_0^{2^{-n}} dt = 1/2$$

and thus, from Eq. (12), $W_{s_0 \dots s_n}$ are i.i.d.: we recover exactly the Mandelbrot construction.

¹We refer the reader to Ref. [25] for a rigorous definition of the so-called ‘‘infinitely divisible independently scattered random measure.’’

C. Towards multiplicative cascades with continuous scale invariance

We would like to apply the previous scheme in the case the construction is no longer indexed by a discrete scale parameter l_n but by a continuous scale parameter l .

In order to ‘‘interpolate’’ smoothly this construction both in time and scale, one can interpolate the previous union of similar squares $\mathcal{A}_{l_n}(s_0 \dots s_n)$ [Eq. (14)] using domains $\mathcal{A}_l(t)$ where (t, l) can take any value in \mathcal{S}^+ . In order the limit measure $M(dt) = \lim_{l \rightarrow 0} M_l(dt)$ to be stationary, it is clear that one has to choose the set $\mathcal{A}_l(t)$ to be ‘‘translation invariant’’ in the sense that

$$(t + \tau, l') \in \mathcal{A}_l(t + \tau) \Leftrightarrow (t, l') \in \mathcal{A}_l(t), \quad \forall \tau. \tag{15}$$

One thus just needs to specify the set $\mathcal{A}_l(0)$. A ‘‘natural’’ choice (though, as we will see in Sec. III F, it only leads to asymptotic, and not exact, scaling properties) seems to be the conical set [see Fig. 1(b)]

$$\mathcal{A}_l(0) = \{(t, l'), l' \geq l, -f(l')/2 < t \leq f(l')/2\}, \tag{16}$$

where $f(l)$ is the function

$$f(l) = \begin{cases} l & \text{for } l \leq T \\ 0 & \text{for } l \geq T, \end{cases} \tag{17}$$

in case, we get exactly the construction originally proposed, using a densification argument, by Schmitt and Marsan in Ref. [19]. Other choices for the function $f(l)$ will be discussed in Sec. III F.

Let us remark that the choice of the linear conical shape $f(l) = l$ is consistent with the interpretation of the parameter l as a scale parameter. Indeed, the value of the measure P around some position (t_0, l) in the half-plane \mathcal{S}^+ influences the values of the measure over the time interval $[t_0 - l/2, t_0 + l/2]$, i.e., exactly over a time scale l .

III. STATIONARY MULTIFRACTAL RANDOM MEASURES

A. Defining MRM

According to the arguments of the previous section, we thus propose the following definition for the class of log infinitely divisible multifractal random measures. Let us introduce an infinitely divisible stochastic 2D measure P uniformly distributed on the half-plane $\mathcal{S}^+ = \{(t, l), t \in \mathbb{R}, l \in \mathbb{R}^+\}$ with respect to the measure $\mu(dt, dl) = dt dl / l^2$ and associated with the Lévy measure $\nu(dx)$. Let us recall that for any set $\mathcal{A} \subset \mathcal{S}^+$, $P(\mathcal{A})$ has an infinitely divisible law whose moment generating function is

$$E(e^{pP(\mathcal{A})}) = e^{\varphi(-ip)\mu(\mathcal{A})}, \tag{18}$$

where $\varphi(p)$ is defined by Eq. (13). Henceforth, we define the real convex cumulant generating function, as, when it exists,

$$\psi(p) \equiv \varphi(-ip). \tag{19}$$

Let $\omega_l(t)$ the stationary stochastic process defined by

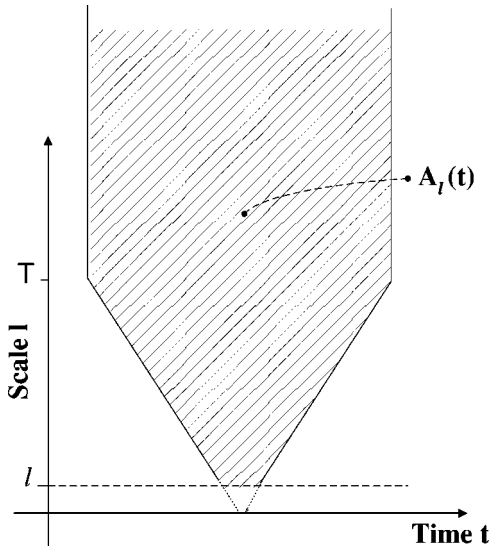


FIG. 2. Conical domain [Eq. (22)] in the (t, l) half-plane involved in the definition of ω_l [Eq. (20)].

$$\omega_l(t) = P[\mathcal{A}_l(t)], \tag{20}$$

where $\mathcal{A}_l(t)$ is the 2D subset of S^+ defined by

$$\mathcal{A}_l(t) = \{(t', l'), \quad l' \geq l, -f(l')/2 < t' - t \leq f(l')/2\}, \tag{21}$$

where $f(l)$ satisfies

$$f(l) = \begin{cases} l & \text{for } l \leq T \\ T & \text{for } l \geq T. \end{cases} \tag{22}$$

As it will be shown in the following sections, the choice of the large scale behavior $f(l) = T$ for $l \geq T$ is the unique one that ensures the convergence of the construction and the exact scaling of the limit measure. The conelike domain $\mathcal{A}_l(t)$ is indicated as a hatched domain in Fig. 2.

We finally define the stochastic positive measure $M_l(dt)$ as

$$M_l(dt) = e^{\omega_l(t)} dt, \tag{23}$$

meaning that for any Lebesgue measurable set I , one has²

$$M_l(I) = \int_I e^{\omega_l(t)} dt. \tag{24}$$

The MRM M is then obtained as the limit measure (the meaning and the existence of this limit will be addressed in the following section)

$$M(dt) = \lim_{l \rightarrow 0^+} M_l(dt). \tag{25}$$

²Since the paths of $\omega_l(t)$ are continuous to the right and limited to the left, the integral (24) is well defined [see Ref. [4]].

Since a simple change in the mean of the stochastic measure P would lead to the same measure up to a deterministic multiplicative factor, we will assume, without loss of generality, that ψ satisfies

$$\psi(1) = 0. \tag{26}$$

We can consider some generalizations of the previous construction. The first one consists in changing the function $f(l)$, i.e., to change the shape \mathcal{A}_l the measure P is integrated on. The second one consists in changing the measure $\mu(dt, dl)$, i.e., to change the way the measure P is distributed in the half-plane S^+ . Actually, from Eq. (12) one can easily show that the construction only depends on the function $\mu(\mathcal{A}_l)$, consequently, changing the shape of $f(l)$ basically amounts changing $\mu(dt, dl)$. A simple example that illustrates such a freedom is the choice $\mu(dt, dl) = dt dl$, i.e., μ is nothing but the 2D Lebesgue measure. In that case, $\mu(\mathcal{A}_l)$ remains unchanged if one chooses $f(l) = 1/l$ in the definition (22). The parameter l is no longer a scale but can be interpreted as a frequency and thus S^+ is the time-frequency half-plane. Therefore, in the following sections, without loss of generality, we choose to fix $\mu(dt, dl) = dt dl/l^2$ (i.e., to work within the time-scale half-plane S^+) and we will discuss, in Sec. III F, the consequences which rise from other choices than Eq. (22) for the function $f(l)$.

B. Existence of the limit MRM $M(dt)$

In Ref. [4], we prove, within the framework of positive continuous martingales that, almost surely, M_l converges to a well defined limit measure when $l \rightarrow 0^+$. Moreover, we prove that if $\psi'(1) < 1$ then there exists $\epsilon > 0$ such that $\psi(1 + \epsilon) < 1$ and the moment of order $1 + \epsilon$, $E(M_l([0, t])^{1 + \epsilon})$, is finite. Then, using the fact that $\psi(1) = 0$, it is straightforward to prove that

$$E(M([0, t])) = \lim_{l \rightarrow 0^+} E(M_l([0, t])) = 1. \tag{27}$$

Consequently the limit measure $M(dt)$ is nondegenerated (i.e., different from zero). The overall proof is very technical and, for this reason, has not been reproduced in this paper.

However, if $\psi(2) < 1$, one can prove (see Appendix A) that $\sup_l E(M_l([0, t])^2)$ is bounded and that the sequence $M_l(dt)$ converges in the mean square sense. Again, using the fact that $\psi(1) = 0$, it follows that $M(dt)$ is nondegenerated. (Let us note that, as explained in Sec. III D, one can prove [4] that assuming $\psi(2) < 1$ basically amounts assuming that $E[M([0, t])^2] < +\infty$).

C. Exact multifractal scaling of $M(dt)$

In order to study the scaling properties of the limit measure $M(dt)$, let us establish the scale invariance properties of the process $\omega_l(t)$.

1. Characteristic function of $\omega_l(t)$

Let $q \in \mathbb{N}^*$, $\mathbf{t}_q = t_1, t_2, \dots, t_q$ with $t_1 \leq t_2 \leq \dots \leq t_n$ and $\mathbf{p}_q = p_1, p_2, \dots, p_q$. The characteristic function of the vector $\{\omega_l(t_m)\}_{1 \leq m \leq q}$ is defined by

$$Q_l(\mathbf{t}_q, \mathbf{p}_q) = E \left\{ \exp \sum_{m=1}^q i p_m P[\mathcal{A}_l(t_m)] \right\}. \quad (28)$$

Relation (18) allows us to get an expression for quantities like $E[\exp \sum_{m=1}^q a_m P(\mathcal{B}_m)]$ where $\{\mathcal{B}_m\}_m$ would be disjoint subsets of \mathcal{S}^+ and a_m arbitrary numbers. However, the $\{\mathcal{A}_l(t_m)\}_m$ in Eq. (28) have no reason to be disjoint subsets. We need to find a decomposition of $\{\mathcal{A}_l(t_m)\}_m$ onto disjoint domains. This is naturally done by considering the different intersections between these domains.

Let us define the cone intersection domains as

$$\mathcal{A}_l(t, t') = \mathcal{A}_l(t) \cap \mathcal{A}_l(t') \quad (29)$$

and

$$\rho_l(t) = \mu[\mathcal{A}_l(0, t)]. \quad (30)$$

Using the definition of $\mathcal{A}_l(t)$ with the shape of $f(l)$ as given by Eq. (22), the expression for $\rho_l(t)$ reads

$$\rho_l(t) = \begin{cases} \ln\left(\frac{T}{l}\right) + 1 - \frac{t}{l} & \text{if } t \leq l \\ \ln\left(\frac{T}{t}\right) & \text{if } T \geq t \geq l \\ 0 & \text{if } t > T. \end{cases} \quad (31)$$

Notice that $\rho_l(t)$ satisfies the remarkable property (for $t \leq T$ and $\lambda \leq 1$):

$$\rho_{\lambda l}(\lambda t) = \rho_l(t) - \ln(\lambda). \quad (32)$$

In Ref. [4], $Q_l(\mathbf{t}_q, \mathbf{p}_q)$ is computed using a recurrence on q . We obtain the following result:

$$Q_l(\mathbf{t}_q, \mathbf{p}_q) = e \sum_{j=1}^q \sum_{k=1}^j \alpha^{(j,k)} \rho_l(t_k - t_j), \quad (33)$$

where

$$\alpha(j, k) = \varphi(r_{k,j}) + \varphi(r_{k+1,j-1}) - \varphi(r_{k,j-1}) - \varphi(r_{k+1,j}) \quad (34)$$

and

$$r_{k,j} = \begin{cases} \sum_{m=k}^j p_m & \text{for } k \leq j \\ 0 & \text{for } k > j. \end{cases} \quad (35)$$

Moreover, let us remark that

$$\sum_{j=1}^q \sum_{k=1}^j \alpha(j, k) = \varphi\left(\sum_{k=1}^q p_k\right). \quad (36)$$

2. Multifractal properties of $M(dt)$

The multifractal scaling properties of the limit MRM $M(dt)$ result from the scale-invariance property of the process $\omega_l(t)$ that is itself a direct consequence of previous exponential expression (33) for the characteristic function of the process ω_l and the particular shape of the conical domains leading to expression (31). Indeed, using these equations together with Eq. (36), it can be proven that $\forall n, \forall t_1, \dots, t_n \in [0, T]^n, \forall p_1 \dots p_n \in \mathbb{R}^n$, one has

$$Q_{\lambda l}(\lambda \mathbf{t}_q, \mathbf{p}_q) = \lambda^{-\varphi(\sum_{j=1}^n p_j)} Q_l(\mathbf{t}_q, \mathbf{p}_q).$$

It follows that, for $\lambda \leq 1$, the process $\omega_l(t)$ satisfies, for $l \in [0, T]$, the following invariance property:

$$\{\omega_{\lambda l}(t\lambda)\}_t \stackrel{law}{=} \{\Omega_\lambda + \omega_l(t)\}_t, \quad (37)$$

where Ω_λ is an infinitely divisible random variable (i.e., it does not depend on t) which is independent of $\omega_l(t)$ and which infinitely divisible law is defined by

$$E(e^{ip\Omega_\lambda}) = \lambda^{-\varphi(p)}. \quad (38)$$

We deduce the following scale-invariance relationship for the sequence of measures $M_l([0, t])$:

$$\begin{aligned} \{M_{\lambda l}([0, \lambda t])\}_t &= \left\{ \int_0^{\lambda t} e^{\omega_{\lambda l}(u)} du \right\}_t \lambda \left\{ \int_0^t e^{\omega_{\lambda l}(\lambda u)} du \right\}_t \\ &\stackrel{law}{=} \lambda e^{\Omega_\lambda} \left\{ \int_0^t e^{\omega_l(u)} du \right\}_t = \lambda e^{\Omega_\lambda} \{M_l([0, t])\}_t \\ &= W_\lambda \{M_l([0, t])\}_t, \end{aligned} \quad (39)$$

where $W_\lambda = \lambda e^{\Omega_\lambda}$ is independent of $\{M_{\lambda l}([0, \lambda t])\}_t$.

By taking the limit $l \rightarrow 0^+$, one gets the continuous cascade equation for MRM as defined in Eq. (6):

$$\{M([0, \lambda t])\}_t \stackrel{law}{=} W_\lambda \{M([0, t])\}_t, \quad \forall \lambda \in [0, 1], \quad (40)$$

where $\ln(W_\lambda)$ is an infinitely divisible random variable independent of $\{M([0, t])\}_t$.

The exact multifractal scaling follows immediately: $\forall q \in \mathbb{R}, \forall t < T$, we get

$$E[M([0, t])^q] = K_q t^{\zeta_q}, \quad (41)$$

were the multifractal spectrum and the prefactor read

$$\zeta_q = q - \psi(q), \quad (42)$$

$$K_q = T^{-\zeta_q} E[M([0, T])^q]. \quad (43)$$

Let us notice that the moment $E[M([0, t])^q]$ in Eq. (41) can be infinite. Conditions for $E[M([0, t])^q]$ to be finite will be discussed in Sec. III D.

3. Multiscaling of correlation functions

The exact (multi)scaling law (41) for the absolute moments can be easily extended, along the same line, to n -points correlation functions. Indeed, let us define the n -point correlation function, when it exists, as

$$C_n(t_1, \dots, t_n; \tau_1, \dots, \tau_n; p_1, \dots, p_n) = E[M([t_1, t_1 + \tau_1])^{p_1} \dots M([t_n, t_n + \tau_n])^{p_n}], \tag{44}$$

where $\tau_k > 0$, and $t_k + \tau_k \leq t_{k+1}$. It is easy to show that if $t_n + \tau_n - t_1 < T$, then C_n is an homogeneous function of degree $\zeta_{\Sigma p_k}$:

$$C_n(\{\lambda t_k\}; \{\lambda \tau_k\}; \{p_k\}) = \lambda^{\zeta_p} C_n(\{t_k\}; \{\tau_k\}; \{p_k\}) \tag{45}$$

with

$$p = \sum_{k=1}^n p_k. \tag{46}$$

This equation extends the scaling law (41) for the moments to multipoints correlation functions.

D. Algebraic tails of probability density functions

Let us note that, if $q > 1$,

$$E[M([0, t])^q] = E[(M([0, t/2]) + M([t/2, t]))^q] \geq 2E[M([0, t/2])^q].$$

Using the multifractal scaling (41), one gets $l^{\zeta_q} \geq 2(1/2)^{\zeta_q}$ leading to $1 \geq 2^{1-\zeta_q}$. It follows that

$$K_q < \infty \Rightarrow \zeta_q \geq 1. \tag{47}$$

Thus, if $q > 1$, the q order moment is infinite if $\zeta_q < 1$. In Ref. [4], we show that the ‘‘reverse’’ implication is also true, i.e.,

$$\zeta_q > 1 \Rightarrow K_q < \infty. \tag{48}$$

Let us notice that this infinite moment condition $\zeta_q < 1, q > 1$, is exactly the same as for discrete multiplicative cascades established in Refs. [20–22]. Divergence of moments for multifractals have also been discussed in e.g., [5,14].

Therefore, if there exists some value $1 < q_* < \infty$ such that

$$\zeta_{q_*} = 1 \tag{49}$$

then,

$$\text{Prob}\{M([0, t]) \geq x\} \sim x^{-q_*} \quad \text{when } x \rightarrow +\infty. \tag{50}$$

The pdf of the measure $M([0, t])$ is thus heavy tailed with a tail exponent that can be, unlike classical α -stable laws, arbitrary large.

E. Examples

In order to illustrate previous considerations let us consider some specific examples.

(i) *Deterministic case.* The simplest situation is when the Lévy $\nu(dx)$ measure is identically zero: This case corresponds to the self-similar, monofractal situation where $\psi(p) = pm$. The constraint $\psi(1) = 0$ implies $m = 0$, and we thus get the Lebesgue measure.

(ii) *Log-normal MRM.* The log-normal MRM is obtained when the canonical measures attributes a finite mass at the origin: $\nu(dx) = \lambda^2 \delta(x) dx$ and $\lambda^2 > 0$. From Eqs. (13) and (19), the cumulant generating function is that of a normal distribution: $\psi(p) = pm + \lambda^2 p^2/2$. The condition $\psi(1) = 0$ implies the relationship $m = -\lambda^2/2$. The log-normal ζ_q spectrum is a parabola:

$$\zeta_q^{\text{ln}} = q \left(1 + \frac{\lambda^2}{2} \right) - \frac{\lambda^2}{2} q^2. \tag{51}$$

Let us note that the so-obtained increasing process $M([0, t])$ is the same as the increasing MRW process mentioned in the conclusion of Ref. [2]. This similarity will be further discussed in Sec. VIA 1.

(iii) *Log-Poisson MRM.* When there is a finite mass at some finite value $x_0 = \ln(\delta)$, of intensity $\lambda^2 = \gamma(\ln \delta)^2$: $\nu(dx) = \lambda^2 \delta(x - x_0)$. The corresponding distribution is Poisson of scale parameter γ and intensity $\ln(\delta)$: $\psi(p) = p[m - \sin(\ln(\delta))] - \gamma(1 - \delta^p)$. The log-Poisson ζ_q spectrum is therefore exactly the same as the one proposed by She and L ev eque in their cascade model for turbulence [17]:

$$\zeta_q^{\text{lp}} = qm' + \gamma(1 - \delta^q), \tag{52}$$

where m' is such that $\zeta_1 = 1$. Notice that in the limit $\delta \rightarrow 1^-$, $\gamma(\ln \delta)^2 \rightarrow \lambda^2$, one recovers the log-normal situation. In the original She-L ev eque model, $\gamma = 2$ and $\delta = 2/3$ [17].

(iv) *Log-Poisson compound MRM.* When the canonical measure $\nu(dx)$ satisfies $\int \nu(dx) x^{-2} = C < \infty$ [e.g., $\nu(dx)$ is concentrated away from the origin], one can see that $F(dx) = \nu(dx) x^{-2}/C$ is a probability measure. In that case,

$$\varphi(p) = im' p + C \int (e^{ipx} - 1) F(dx)$$

is exactly the cumulant generating function associated with a Poisson process with scale C and compound with the distribution F [24]. Let us now consider a random variable W such that $\ln W$ is distributed according to $F(dx)$. It is easy to see that $\int e^{px} F(dx) = E(W^p)$. It turns out that the log-Poisson compound MRM has the following multifractal spectrum:

$$\zeta_q^{\text{lpc}} = qm - C[E(W^q) - 1]. \tag{53}$$

This is exactly the spectrum obtained by Barral and Mandelbrot in their construction of ‘‘product of cylindrical pulses’’ [3]. The similarity between our construction and Barral-Mandelbrot construction will be further discussed in Sec. VIA 2.

(v) *Log- α -stable MRM.* When $\nu(dx) \sim x^{1-\alpha} dx$ for $0 < \alpha < 2$, one has an log- α -stable MRM:

$$\zeta_q^{\text{ls}} = qm - \sigma^\alpha |q|^\alpha. \tag{54}$$

Such laws have been used in the context of turbulence and geophysics [14,18]. They have been often referred to as “universal multifractals” because α -stable laws are fixed points of infinitely divisible laws under a suitable renormalization procedure.

Many other families of ζ_q spectra can be obtained (e.g., log- Γ , log-Hyperbolic, . . .) for other choices of the Lévy measure. Let us remark that in the case of a normal random variable ω_l , the function $\rho_l(\tau)$ is nothing but the covariance of the process as introduced in Refs. [1,2]. This function, that measures the areas of domains \mathcal{A}_l intersections, is therefore the analog of the covariance for general infinitely divisible distributions. The Eq. (33) shows that our construction can be seen as a natural extension of Gaussian processes (or multivariate Gaussian laws) within the class of infinitely divisible processes (multivariate infinitely divisible laws) in the sense that it is completely characterized by a cumulant generating function $\varphi(p)$ (specifying the mean and the variance parameters in the Gaussian case) and a two-points “covariance” function $\rho(t)$ (or covariance matrix).

F. Asymptotic scaling and universality

In the last section, we have seen that the choice (22) for $f(l)$ leads to exact scaling of the moments of the associated MRM. In this section we study the scaling behavior of the moments for other choices of $f(l)$.³ Let us remark that $f(l)$ is defined up to a multiplicative constant. This just amounts to a choice in the scale of $\psi(p)$.

In the following, $f^{(e)}(l)$ will refer to the “exact scaling” choice (22) that was made in the previous sections. The so-obtained sets in the \mathcal{S}^+ half-plane will be referred to as $\mathcal{A}_l^{(e)}(t)$, the ω process as $\omega_l^{(e)}(t)$ and the associated MRM as $M^{(e)}(dt)$.

1. Large scale perturbation of $f(l)$

Rigorous mathematical proofs can be found in Ref. [4].

Let us first study the case when one builds an MRM using a function $f(l)$ which differs from $f^{(e)}(l)$ only for scales larger than a large (fixed) scale L , i.e.,

$$f(l) = \begin{cases} f^{(e)}(l) & \text{for } l \leq L \\ g(l) & \text{for } l \geq L. \end{cases} \quad (55)$$

For the measure $M_l(dt)$ to remain finite, $\rho_l(t)$ must be finite and thus we must have $\int_l^{+\infty} f(u)u^{-2}du < \infty$. Therefore, the large scale behavior $g(l)$ must be such that, for some $\epsilon > 0$, $g(l) = O(l^{1-\epsilon})$ as $l \rightarrow +\infty$. An example of such large scale modification is the function defined by Eq. (17), where $g = g^{(s)}$ where $g^{(s)}(l) = 0$ and $L = T$. This function is the one which was used by Barral and Mandelbrot in Ref. [3]. Let us first choose the particular case $g = g^{(s)}$ (L being any strictly

positive number). The so-obtained sets in the \mathcal{S}^+ half-plane will be referred to as $\mathcal{A}_l^{(s)}(t)$, the ω process as $\omega_l^{(s)}(t)$ and the associated MRM as $M^{(s)}(dt)$. Since, $\forall t, t', [\mathcal{A}_l^{(e)}(t) \setminus \mathcal{A}_l^{(s)}(t)] \cap \mathcal{A}_l^{(s)}(t') = \emptyset$, one has

$$\omega_l^{(e)}(t) = \omega_l^{(s)}(t) + \delta_L(t),$$

where $\delta_L(t) = P[\mathcal{A}_l^{(e)}(t) \setminus \mathcal{A}_l^{(s)}(t)]$ is a process which is independent of the process $\omega_l^{(s)}(t)$ and which does not depend on the value of l (as long as $l < L$). It follows that

$$E(M^{(e)}([0,t])^q) \leq E(\sup_{[0,t]} e^{q\delta_L(t)})E(M^{(s)}([0,t])^q)$$

$$E(M^{(e)}([0,t])^q) \geq E(\inf_{[0,t]} e^{q\delta_L(t)})E(M^{(s)}([0,t])^q).$$

Because the process $\delta_L(t)$ is (right) continuous, $\lim_{t \rightarrow 0^+} \sup_{[0,t]} e^{q\delta_L(t)} = \lim_{t \rightarrow 0^+} \inf_{[0,t]} e^{q\delta_L(t)} = e^{q\delta_L(0)}$, we get

$$E[M^{(s)}([0,t])^q] \sim_{t \rightarrow 0} C_q E[M^{(e)}([0,t])^q]. \quad (56)$$

Thanks to the exact scaling of $M^{(e)}([0,t])$ [Eq. (41)], we see that,

$$E[M^{(s)}([0,t])^q] \sim_{t \rightarrow 0} C_q t^{\zeta_q}, \quad (57)$$

where ζ_q is defined in Eq. (42). Thus, we find that $M^{(s)}(dt)$, corresponding to the specific choice $g(l) = g^{(s)}(l) = 0$, satisfies the asymptotic scale invariance property (4). If one chooses a different function $g(l)$, using exactly the same arguments as above [in which $M^{(e)}(dt)$ is replaced by $M(dt)$, i.e., the measure obtained when using the new function $g(l)$], one can prove [4] that

$$E(M([0,t])^q) \sim_{t \rightarrow 0} D_q E(M^{(s)}([0,t])^q), \quad (58)$$

and consequently

$$E(M([0,t])^q) \sim_{t \rightarrow 0} D_q C_q t^{\zeta_q}. \quad (59)$$

Thus, using any function $g(l)$ in Eq. (55) [satisfying $g(l) = O(l^{1-\epsilon})$ ($l \rightarrow +\infty$)] leads to an MRM measure which satisfies the asymptotic scale invariance property (4).

2. Small scale perturbation of $f(l)$

Let us now study the consequences of a small scale perturbation of $f^{(e)}(l)$. Let us suppose that $f(l) \sim l^\alpha$ for $l \rightarrow 0$. In that case,

$$\rho_l(0) \sim_{l \rightarrow 0} \begin{cases} l^{\alpha-1} & \text{if } \alpha < 1 \\ Cst & \text{if } \alpha > 1 \\ -\ln(l) & \text{if } \alpha = 1, \end{cases} \quad (60)$$

In the first case, $\alpha < 1$, one can show any moment of order $1 + \epsilon$ [$\epsilon > 0$] cannot be bounded. Since $E[M_l([0,t])] = 1$,

³We recall that we do not consider other choices for the uniform measure $\mu(dt, dl)$ in the half space \mathcal{S}^+ because $f(l)$ and $\mu(dt, dl)$ are involved in the properties of the limit measure $M(dt)$ only through the function $\rho_l(t) = \mu[\mathcal{A}_l(0,t)]$. Hence, up to a change of variable $l' = h(l)$, one can always set $\mu(dt, dl) = dt dl / l^{-2}$.

general martingale arguments can be used to prove that $M_l([0,t])$ converges towards the trivial zero measure.

If $\alpha > 1$, the limit measure is proportional to the Lebesgue measure and thus one obtains the trivial asymptotic scaling:

$$E[M([0,t])^q] \underset{t \rightarrow 0}{\sim} C_q t^q.$$

Let us now consider the marginal case, $f(l) = l + o(l)$ at small scales. From the results of previous section, with no loss of generality, as far as the asymptotic scaling is concerned, we can suppose that the large scale component is identical to the exact scaling situation: $f(l) = T$ for $l \geq T$. We can then show that, once again, the MRM satisfies a multifractal scaling in the asymptotic sense [Eq. (4)]:

$$E[M([0,t])^q] \underset{t \rightarrow 0}{\sim} K'_q t^{\zeta_q}. \tag{61}$$

Indeed, let us index all the quantities by the cutoff scale l and the integral scale T , i.e., we add an explicit reference to the integral scale T : $\rho_{l,T}(t)$ is the area of domain intersections and $M_{l,T}([0,t])$ is the associated MRM. After some little algebra, using the definition of $\rho_{l,T}(t)$ and the fact that $f(l) = l + o(l)$, one can show that

$$\rho_{\lambda l, \lambda T}(\lambda t) \underset{\lambda \rightarrow 0}{\rightarrow} \rho_{l,T}^{(e)}(t). \tag{62}$$

Thus, thanks to Eq. (33), we have

$$\{\omega_{\lambda l, \lambda T}(\lambda t)\}_t \underset{\lambda \rightarrow 0}{\xrightarrow{law}} \{\omega_{l,T}^{(e)}(t)\}_t.$$

Hence, because $M_{l,T}([0,t]) = \int_0^t e^{\omega_{l,T}(u)} du$, by taking the limit $l \rightarrow 0$,

$$\lambda^{-1} M_{0, \lambda T}([0, \lambda t]) \underset{\lambda \rightarrow 0}{\xrightarrow{law}} M_{0,T}^{(e)}([0,t])$$

and therefore, from Eqs. (41), (42), and (43),

$$\lambda^{-q} E[M_{0, \lambda T}([0, \lambda t])^q] \underset{\lambda \rightarrow 0}{\rightarrow} E[M_{0,T}^{(e)}([0,T])^q] T^{-\zeta_q t^{\zeta_q}}.$$

By choosing $T = T' \lambda^{-1}$ and using the identity $M_{0, \lambda^{-1} T'}^{(e)}([0, \lambda^{-1} T']) =^{law} \lambda^{-1} M_{0,T'}^{(e)}([0,T'])$, we conclude that

$$E[M_{0,T'}([0, \lambda t])^q] \underset{\lambda \rightarrow 0}{\sim} \lambda^{\zeta_q}.$$

This achieves the proof.

We can therefore see, that, as far as asymptotic multifractality is concerned, the pertinent parameter is the small scale behavior of the function $f(l)$ or equivalently the small time behavior of $\rho_{l=0}(t)$. As pointed out previously, in the case of a Gaussian field $\omega(t)$ (i.e., the infinitely divisible law has only a Gaussian component), $\rho(t)$ is nothing but the covariance of the process. The previous discussion leads thus to the conclusion that *nontrivial limit multifractal measures arise*

only in the marginal situation when the correlation function of the logarithm of the fluctuations decreases as a logarithmic function.

G. An alternative discrete time construction for MRM

In the case $\int x^{-2} \nu(dx) < \infty$ (e.g., the Lévy measure has no mass in an interval around $x=0$) a realization of the measure $P(dt, dl)$ is made of dirac functions distributed in the S^+ half-plane. Thus the process $M_l([0,t]) = \int_0^t e^{\omega_l(t)} dt$ is a jump process that can be simulated with no approximation.

However, if $\int x^{-2} \nu(dx)$ has a nonfinite integral (e.g., it has a Gaussian component), this is no longer the case. Thus, one has to build another sequence of stochastic measures $\tilde{M}_l(dt)$ that converges in law towards $M(dt)$ and that can be seen as a discretized version of $M(dt)$. We will see, in following sections, that such a discrete time approach is also interesting for multifractal stochastic processes construction.

We choose $\tilde{M}_l(dt)$ to be uniform on each interval $[kl, (k+1)l]$, $\forall k \in \mathbb{N}$ and with density $e^{\omega_l(kl)}$. Thus, for any $t > 0$ such that $t = pl$ with $p \in \mathbb{N}^*$, one gets

$$\tilde{M}_l([0,t]) = \sum_{k=0}^{p-1} e^{\omega_l(kl)} l. \tag{63}$$

In the same way as for the measure $M_l(dt)$, one can prove [4], within the framework of positive martingales, that, almost surely, $\tilde{M}_{l=2^{-n}}(dt)$ converges towards a well defined limit measure when $n \rightarrow +\infty$ (i.e., $l \rightarrow 0^+$). Moreover, in the same way as in Sec. III B, if we suppose $\psi(2) < 1$, then one can show (the proof is very similar as the one in Appendix A) that, in the mean square sense,

$$\lim_{n \rightarrow +\infty} \tilde{M}_{l=2^{-n}}(dt) = M(dt). \tag{64}$$

As long as $\psi(2) < 1$, this construction gives therefore a way of generating a measure which is arbitrarily close (by choosing l small enough) to the limit measure $M(dt)$.

IV. LOG INFINITELY DIVISIBLE MULTIFRACTAL RANDOM WALKS

In this section we build and study a class of multifractal stochastic processes that are no longer, as before, strictly increasing processes (measures). They can be basically built in two different ways: (i) By subordinating a fractional Brownian motion (fBm) with the previously defined MRM $M(t)$ or (ii) by a stochastic integration of a MRM against a fractional Gaussian noise (fGn). As we will see, most of the statistical properties of the so-obtained random processes are directly inherited from those of the associated MRM.

A. MRW with uncorrelated increments

In the same spirit as the log-normal MRW constructions in Refs. [1,2] [see also Ref. [27]], we use a stochastic integration of $e^{\omega_l(t)}$ against the (independent) Wiener measure $dW(t)$.

1. Definition

Since $E(e^{\omega_l(t)}) = 1$, one can consider the process

$$X_l(t) = \int_0^t e^{(1/2)\omega_l(u)} dW(u), \tag{65}$$

where $dW(t)$ is a Gaussian white noise independent of ω_l . The MRW is then defined as the limit of $X_l(t)$ when $l \rightarrow 0^+$:

$$X(t) = \lim_{l \rightarrow 0^+} X_l(t). \tag{66}$$

One can easily prove that for fixed t and l , one has

$$X_l(t) \stackrel{law}{=} \sigma_l(t) \epsilon, \tag{67}$$

where ϵ is a standardized normal random variable independent of $\sigma_l(t)$ which is itself nothing but the associated MRM as defined previously:

$$\sigma_l^2(t) = \int_0^t e^{\omega_l(u)} du = M_l(t). \tag{68}$$

Let us note that the (nondecreasing) increments of $\sigma_l^2(t)$ are referred to (in the field of mathematical finance [28,29]) as the *stochastic volatility*.

Using the same kind of arguments on finite dimensional laws, one can also prove that the finite dimensional laws of the process $X_l(t)$ converge to those of the subordinated process $B[M(t)]$. Actually, one can show [4] that, as long as $\psi'(1) < 1$ (i.e., $\exists \epsilon > 0, \psi(1 + \epsilon) < 1$ which is, as mentioned in Sec. II B, the condition for the limit measure $M(dt)$ to be nondegenerated), one has

$$X(t) = \lim_{l \rightarrow 0^+} X_l(t) \stackrel{law}{=} \lim_{l \rightarrow 0^+} B(\sigma_l(t)) \stackrel{law}{=} B(M(t)). \tag{69}$$

The so-obtained MRW can be thus understood as a Brownian motion in a ‘‘multifractal time’’ $M(t)$. The subordination of a Brownian process with a nondecreasing process has been introduced by Mandelbrot and Taylor [30] and is the subject of an extensive literature in mathematical finance. Multifractal subordinators have been considered by Mandelbrot [31] and widely used to build multifractal processes from multifractal measures (see below). In a forthcoming section we will see that multifractal subordination and stochastic integration do not lead to the same processes when one considers long-range correlated Gaussian noises (fGn).

2. Expression of the moments and multifractal properties

Thanks to Eq. (69) [assuming $\psi'(1) < 1$], one gets the expression of the absolute moments of $X(t)$ [or $X(t_0 + t) - X(t_0)$]:

$$\forall q, E(|X(t)|^q) = E(|\epsilon|^q) E(M([0,t])^{q/2})$$

$$= \sigma^q \frac{2^{q/2} \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} E(M([0,t])^{q/2}),$$

where the first factor comes the order q moment of a centered Gaussian variable of variance σ^2 .

If $M(dt)$ is an exact multifractal stationary random measure, then, $X(t)$ obeys the exact multifractal scaling equation

$$E(|X(t)|^q) = \sigma^q \frac{2^{q/2} \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} K_{q/2} t^{\zeta_q}, \tag{70}$$

where K_q is defined in Eq. (43) and

$$\zeta_q = q/2 - \psi(q/2). \tag{71}$$

Using Eqs. (47), (48), and (49), one deduces that, for $q > 2$,

$$E(|X(t)|^q) < +\infty \Rightarrow \zeta_q \geq 1, \tag{72}$$

and conversely

$$\zeta_q < 1 \Rightarrow E(|X(t)|^q) < \infty. \tag{73}$$

Moreover, let us note that if $M(dt)$ verifies only an asymptotic scaling, so does the MRW process $X(t)$.

3. An alternative discrete time construction for MRW processes

As in Refs. [1,2] or in Sec. III G, one can also try to build an MRW process using a discrete approach. A discrete construction can be useful for numerical simulations. Let us, for instance, choose $l_n = 2^{-n}$. And let $\epsilon_l[k] = \int_{kl}^{(k+1)l} dW(u)$ be a discrete Gaussian white noise. We define the piece-wise constant process $\tilde{X}_{l_n}(t)$ as ($t = pl_n$):

$$\tilde{X}_{l_n}(t) = \sum_{k=0}^{p-1} e^{1/2\omega_{l_n}(kl_n)} \epsilon_{l_n}[k]. \tag{74}$$

The MRW $\tilde{X}_{l_n}(t)$ can be rewritten as

$$\tilde{X}_{l_n}(t) = \sum_{k=0}^{p-1} \sqrt{\tilde{M}_{l_n}([kl_n, (k+1)l_n])} \epsilon_{l_n}[k] / l_n, \tag{75}$$

where \tilde{M}_{l_n} is defined by Eq. (63). One then deduces easily the convergence of \tilde{X}_{l_n} from the convergence of \tilde{M}_{l_n} . Thus, for instance, one can prove [4] that, as long as $\psi(2) < 1$, one has

$$\lim_{n \rightarrow +\infty} \tilde{X}_{l_n}(t) \stackrel{law}{=} B[M(t)]. \tag{76}$$

B. MRW processes with long-range correlations

1. Definitions

In order to construct long-range correlated MRW, it is natural to replace the Wiener noise (Brownian motion) in previous construction by a fractional Gaussian noise (fractional Brownian motion). A fBm, $B_H(t)$ is a continuous, self-similar, zero-mean Gaussian process which covariance reads [see, e.g., Ref. [32] for a precise definition and properties]

$$E(B_H(t)B_H(s)) = \frac{\sigma^2}{2}(s^{2H} + t^{2H} - |t-s|^{2H}), \quad (77)$$

where $0 < H < 1$ is often called the Hurst parameter. Standard Brownian motion corresponds to $H = 1/2$.

The simplest approach to construct a long-range correlated MRW follows the idea of Mandelbrot that simply consists in subordinating a fractional Brownian motion of index H with the MRM $M(t)$, i.e.,

$$X_H^s(t) = B_H[M(t)]. \quad (78)$$

An alternative would consist in building a stochastic integral against a fGn $dW_H(t)$

$$X_{l,H}^i(t) = \int_0^t e^{\omega_l(t)} dW_H(t) \quad (79)$$

and considering some appropriate limit $l \rightarrow 0$. However, stochastic integrals against fGn cannot be defined as easily as for the white Gaussian noise and the proposed constructions require the complex machinery of Malliavin calculus or Wick products [33,34]. One simple way to define the previous integral could be to see it as the limit of a Riemann sum

$$\int_0^t e^{\omega_l(t)} dW_H(t) \equiv \lim_{\Delta t \rightarrow 0} \sum_{k=0}^{t/\Delta t} e^{\omega_l(k\Delta t)} \epsilon_{H,\Delta t}[k], \quad (80)$$

where $\epsilon_{H,\Delta t}(k) = B_H(k\Delta t) - B_H((k-1)\Delta t)$. We have not proved yet that this is a mathematically sound definition. However, if one assumes that (79) makes sense, one can address the question of the existence of the limit process $\lim_{l \rightarrow 0} X_{l,H}^i(t)$. In the Appendix B, we provide heuristic arguments for mean square convergence. We obtain a condition

$$H > 1/2 + \psi(2)/2, \quad (81)$$

where $\psi(p)$ is the cumulant generating function associated with ω .

2. Multifractal properties

In the case of the subordinated version $X_H^s(t)$ of the MRW, the scaling properties can be directly deduced by the self-similarity of $B_H(t)$ [31]. Since $B_H(t+\tau) - B_H(t) = {}^{law} \tau^H [B_H(t+1) - B_H(t)]$, and $M(t)$ is independent of B_H , one has $X_H[M(t)] = {}^{law} M(t)^H B_H(1)$. The scaling of the absolute moments of the increments of X_H is therefore

$$E[|X_H(t)|^q] = E[M(t)^{qH}] E[|B(1)|^q] = K_{qH} G_{q,t} \zeta_q^s$$

with

$$\zeta_q^s = qH - \psi(qH). \quad (82)$$

For the second version, X_H^i , the scaling of the moments is determined using the scale invariance of the process $\omega_l(t)$ and the self-similarity of the fGn $dW_H(t)$. Using the same method as for the measure $M(t)$, one obtains

$$E[|X_H^i(t)|^q] = M_{q,t} \zeta_q^i \quad (83)$$

with

$$\zeta_q^i = qH - \psi(q). \quad (84)$$

We can see that the multifractal spectra of X_H^s and X_H^i are different: they do not correspond to identical processes as it was the case for the uncorrelated construction. Notice that the existence criterion (81), can be simply rewritten as $\zeta_2^i > 1$. According to the considerations developed in Sec. III D, this condition ensures the existence of the second order moment of X_H^i . Whether the class of processes $X_H^i(t)$ can be extended, in some weak probabilistic sense, to values of $H < 1/2 + \psi(2)/2$ is still an open problem (such processes would have an infinite variance). The condition of finite variance for $X_H^s(t)$ is less restrictive since it comes to the condition $K_{2H} < \infty$, where K_{2H} is defined in Eq. (43). For $H < 1/2$ such a moment is always finite.

C. A remark on subordination

Let us remark that, in some sense, the subordination by a MRM $M(dt)$ can be iterated. Indeed, if $M_1([0,t])$ and $M_2([0,t])$ are two independent MRM, the subordinated measure

$$M([t_1, t_2]) = M_1[M_2([0, t_1]), M_2([0, t_2])] \quad (85)$$

is well defined. Using the cascade equation (40), we deduce that

$$\begin{aligned} M([0, \lambda t]) &= {}^{law} M_1[0, W_\lambda^{(2)} M_2([0, t])] \quad (86) \\ &= {}^{law} W_{W_\lambda^{(2)}}^{(1)} M([0, t]), \quad (87) \end{aligned}$$

where $W_\lambda^{(1,2)}$ are the (independent) log infinitely divisible weights associated with the MRM $M_{1,2}$. The second equality is valid only if $W_\lambda^{(2)} < 1$, i.e., when the Lévy measure associated with M_2 is concentrated on $[0, +\infty]$. By computing the moment of order q of both sides of the equality, we see that the multifractal spectrum of the subordinated measure reads

$$\zeta_q = \zeta_{\zeta_q^{(1)}}^{(2)} = q - \psi^{(2)}[q - \psi^{(1)}(q)], \quad (88)$$

where $\zeta_q^{(i)}$ (respectively $\psi^{(i)}(q)$), $i = 1, 2$, is the spectrum associated with M_i . The Eq. (87) corresponds to a ‘‘random-

ization” of the rescaling factor λ that parametrizes the log infinitely divisible law of $\ln(W_\lambda^{(1)})$. It is easy to prove [24] that the law of $\ln(W_\lambda^{(2)})$ remains infinitely divisible. The class of log infinitely divisible MRM is therefore closed under subordination. The family of subordinated spectra (88) is thus included in the family of log infinitely divisible spectra and the operation (85) does not allow us to build new MRM with exact scale invariance properties.

V. NUMERICAL SIMULATIONS

A. Principles

In order to generate realizations of $\tilde{M}_{l_n}(dt)$ [defined by Eq. (63)], one needs to be able to generate realizations of $\omega_{l_n}(kl_n)$. However, following the definition of ω_l one would need realizations of the 2D random measure $P(dt, dl)$ for $l \geq l_n$. In the case of a compound Poisson process, the process $M_l([0, t]) = \int_0^t e^{\omega_l(t)} dt$ is a jump process that can be synthesized easily.

In the general case, we need to find a set of disjoint “elementary” domains of the half-plane \mathcal{S}^+ such that, for any k , there exists a subset of this set such that $\mathcal{A}_{l_n}(kl_n)$ can be expressed as a union over the elementary domains of this subset. Since, at fixed l_n , the boundaries of the domains $\mathcal{A}_{l_n}(kl_n)$ ($k \in \mathbb{Z}$) define a tiling of \mathcal{S}^+ , it is natural to consider the elementary cells of this tiling. Each cell is the intersection between left and right strips limited by left and right boundaries of conical domains: Let us define the cell $\mathcal{B}_l(t, t')$ (with $t < t'$) as

$$\mathcal{B}_l(t, t') = [\mathcal{A}_l(t) \setminus \mathcal{A}_l(t-l)] \cap [\mathcal{A}_l(t') \setminus \mathcal{A}_l(t'+l)]. \tag{89}$$

Then, by definition the cells $\{\mathcal{B}_{l_n}(kl_n, k'l_n)\}_{k < k'}$ are disjoint domains and form a partition of the subspace (of \mathcal{S}^+) $\{(t, l) \in \mathcal{S}^+, l \geq l_n\}$. Moreover

$$\mathcal{A}_{l_n}(kl_n) = \bigcup_{-\infty \leq i \leq k} \bigcup_{k \leq j \leq +\infty} \mathcal{B}_{l_n}(il_n, jl_n). \tag{90}$$

On the other hand, for a fixed $s \geq l_n$, one has

$$(u, s) \in \mathcal{A}_l(t) \setminus \mathcal{A}_l(t-l) \Leftrightarrow t-l+f(s)/2 \leq u \leq t+f(s)/2,$$

and

$$(u, s) \in \mathcal{A}_l(t+l) \setminus \mathcal{A}_l(t) \Leftrightarrow t-f(s)/2 \leq u \leq t+l-f(s)/2.$$

Thus setting $Y_{i,j} = P[\mathcal{B}_{l_n}(il_n, jl_n)]$, straightforward computations lead to the following representation of discrete process $\omega_{l_n}(kl_n)$:

$$\omega_{l_n}(kl_n) = \sum_{i=-\infty}^k \sum_{j=k}^{+\infty} Y_{i,j}, \tag{91}$$

where $\{Y_{i,j}\}_{i,j}$ are independent infinitely divisible random variables which satisfy

$$E(e^{pY_{i,j}}) = e^{\psi(p)\rho_{i,j}}, \tag{92}$$

with

$$\rho_{i,j} = \int_{s \geq l_n} ds/s^2 \int dt H_{i,j,s}(t), \tag{93}$$

where $H_{i,j,s}(t)$ is the indicator function

$$H_{i,j,s}(t) = \mathbb{1}_{[a_{i,j}(s), b_{i,j}(s)]}(t), \tag{94}$$

with

$$a_{i,j}(s) = \max[(i-1)l_n + f(s)/2, jl_n - f(s)/2], \tag{95}$$

and

$$b_{i,j}(s) = \min[il_n + f(s)/2, (j+1)l_n - f(s)/2]. \tag{96}$$

Let us note that if the function f is bounded [which is the case if we are under the hypothesis (22), i.e., in the case of “exact” scaling], the number of terms in Eq. (91) is finite.

In the Gaussian case, the situation is simpler because ω_l is completely characterized by its covariance function $\rho_l(\tau)$. In that case, the 2D synthesis problem can be easily transposed as a 1D filtering procedure by finding a filter $\phi_l(t)$ such that

$$\phi_l * \phi_l = \rho_l, \tag{97}$$

where $*$ stands for the convolution product. The process

$$\omega'_l(t) = \int \phi_l(t-t') W(dt') \tag{98}$$

(where W is here a 1D Wiener noise) will be thus identical to the normal process $\omega_l(t)$ as defined in Eq. (20). In the original study of Refs. [1,2] the MRW has been defined along this method.

B. Numerical examples

In Fig. 3 are shown two samples of MRW which are respectively log-normal and log-Poisson [see Eqs. (51) and (52)]. In both cases we have chosen $T=512$ sample units, $H=1/2$ (ϵ is a Gaussian white noise). For the log-normal process, $\lambda^2=0.05$ while $\gamma=4$ and $\delta=e^{-\sqrt{0.057\gamma}}$ for the log-Poisson process. The log-Poisson process has been synthesized using Eq. (91) while a simple filtering method was used for the log-normal process. Equation (97) was solved numerically in the Fourier domain. In Fig. 4 are plotted the ζ_q functions estimated for both processes. These functions have been obtained from the scaling of the moments estimated using 256 MRW trials of 64 integral scales long. The ζ_q values for negative q have been obtained using the so-called WTMM method that is a wavelet-based method introduced to study multifractal functions [35–37]. The superimposed analytical formulas obtained with Eqs. (71), (51), and (52), fit very well with statistical estimates except a large negative q values for the log-normal case. This can be explained as a finite statistics effect.

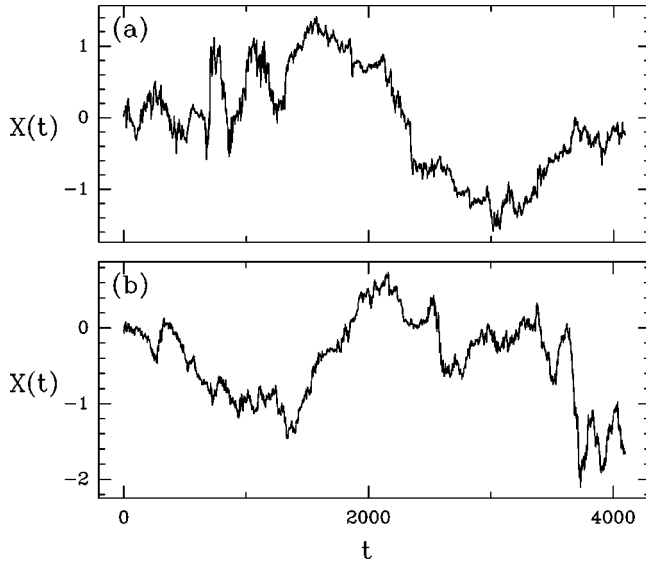


FIG. 3. MRW signals sampled at rate 1. (a) Log-normal MRW with $T=512$ and $\lambda^2=0.05$. (b) Log-Poisson sample with $T=512$, $\gamma=4$ and $\gamma \ln(\delta)^2=0.05$. In both signals, the cutoff l has been fixed to $1/8$ for the numerical synthesis.

VI. CONNECTED APPROACHES

In this section we review some specific results concerning respectively log-normal and log-Poisson compound MRW.

A. Log-normal MRW

The log-normal MRW has been originally defined in Refs. [1,2]. It corresponds to the simplest situation when the Lévy measure has only a Gaussian component. In that case, the

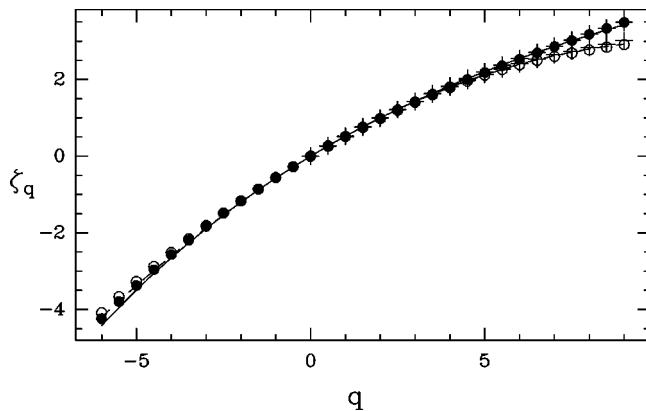


FIG. 4. ζ_q spectrum estimation for log-normal and log-Poisson MRWs. The exponents have been directly estimated from linear regression of increment order q absolute moments in doubly logarithmic representations. These moments have been estimated using a statistical sample of 256 signals of 64 integral scales. The symbols (●) correspond to log-Poisson estimates while (○) correspond to log-normal estimates. The solid line represents the log-Poisson analytical spectrum (52) and the dashed line to log-normal analytical spectrum (51) rescaled using Eq. (71). The parameters of both processes are those of Fig. 2 and have been chosen so that the so-called intermittency parameter $-\zeta''(0)$, is the same (0.05).

Gaussian process $\omega_l(t)$ can be directly constructed from a 1D white noise, without any reference to 2D conical domains. This model is interesting because its multifractal properties are described by only two parameters, the integral scale T and the so-called intermittency parameter λ^2 . Some simple estimators of these quantities have been proposed in Ref. [1]. Moreover, many exact analytical expressions can be obtained and notably the value of the prefactor K_q in Eq. (42). In Ref. [38], it is shown that this prefactor can be written as a Selberg integral [39]. Its analytical expression reads

$$K_q = T^q \prod_{k=0}^{q-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2(k+1))}{\Gamma(2-2\lambda^2(q+k-1)) \Gamma(1-2\lambda^2)}. \quad (99)$$

It is easy to check that K_q is defined only if $q < q_* = 2/\lambda^2$. We recover the finite moment condition (49), $q < q_*$ with $\zeta_{q_*} = 1$. Notice that, as emphasized by Frisch [40], the existence of infinite moments can be a drawback of a log-normal multifractal as a model for experimental situations like turbulence. However, for a typical value $\lambda^2 = 5 \cdot 10^{-2}$, $q_* = 40$, so a log-normal approximation can be very good in a range of q values far beyond the limit associated with the finite size of experimental samples. Let us finally mention that the log-normal MRW can be naturally generalized to a “multivariate multifractal model” which is a multifractal vector of processes characterized by an intermittency matrix Λ_{ij} [1]. This notion of “joint multifractality” can be very interesting in many applications.

B. Multifractal products of cylindrical pulses

In Ref. [3], Mandelbrot and Barral introduced a positive multifractal random measure using products of positive random variables associated with Poisson points within 2D conical domains. Their construction is a particular case of MRM. It actually reduces to the case where $\nu(dx)$ is a Lévy measure satisfying $\int \nu(dx) x^{-2} < \infty$ (see Sec. III E) and where the set $\mathcal{A}_l(t)$ is delimited by the function $f(l)$ as defined by Eq. (17). Let us note that, since they have not considered the full domain $\mathcal{A}_l(t)$ associated with the function $f(l)$ as defined by Eq. (22), as explained in Sec. III F, this construction performs asymptotic scaling (4) but not exact scaling (1). However, these authors did not study the scaling properties of the random measures. They rather focused on the pathwise regularity properties. More precisely, they proved the validity of the so-called “multifractal formalism” [see, e.g., Refs. [35–37,41,42]] that relates the function ζ_q to the singularity spectrum $D(h)$ associated with (almost) all realizations of the process. For a given path of the increasing process associated with an MRM, $D(h)$ is defined as the Hausdorff dimension of the set of “isoregularity” points, i.e., the points where the (Hölder) regularity is h . Barral and Mandelbrot proved that $D(h)$ and ζ_q are related by a Legendre transformation. Since we proved that ζ_q is the scaling exponent of MRM moments, it follows that one can estimate the singularity spectrum of the MRM paths in the case of log-Poisson compound statistics. It should be interesting to extend the Mandelbrot-Barral theorem to the general log-infinitely divisible MRM and MRW paths.

VII. CONCLUSION AND PROSPECTS

A. Summary and open questions

In this paper, we have constructed a class of stationary continuous time stochastic measures and random processes that have exact or asymptotic multifractal scaling properties in the sense of Eqs. (1) and (4). We have shown how stochastic integration of an infinitely divisible noise over cone-like domains, as originally proposed in Ref. [19], naturally arises when one wants to “interpolate” discrete multiplicative cascades over a continuous range of scales within a construction that is invariant by time translations. The exponential of these stochastic integrals ($e^{\omega_l(t)}$) can be interpreted as a “continuous product” from coarse to fine scales and thus as the continuous extension of the multiplicative rule involved in the definition of discrete cascades. We have shown that the probability density functions associated with MRM and MRW processes can have, like discrete cascades, fat tails with arbitrary large exponents. However, unlike their discrete analog, our “continuous cascades” have stationary fluctuations, do not involve any particular scale ratio and can be defined in a causal way. This “sequential” formulation, as opposed to the classical “top to bottom” definition of multifractals, can be very interesting for modeling dynamical processes (see the next section). Let us note that we focused in this study on 1D processes but our construction can easily be extended to higher dimensions.

It is well known [see Refs. [1,2]] that the multiscaling (1) or (4) with a nonlinear convex ζ_q function cannot extend over an unbounded range of scales and there necessarily exists an “integral scale” T above which the scaling of the moments changes. The existence of such an integral scale can be found in the general shape of function $f(l)$ as discussed in Sec. III F: One must have $f(l) \sim l$ when $l \rightarrow 0$ and $f(l) = o(l)$ when $l \rightarrow +\infty$, so the scale T is a scale that separates these two asymptotic regimes. It is remarkable that there exists a particular expression for the function $f(l)$ [Eq. (22)] for which the moments in the multifractal regime $l \leq T$ satisfy an *exact* scaling. The existence of processes with such properties was not *a priori* obvious. In the same section, we have shown that $f(l) \sim l$ is a necessary and sufficient condition for the existence of a limit multifractal object. From a fundamental point of view, one important question concerns the unicity of our construction: Can any process satisfying (1) be represented within the framework we have introduced?

It remains many open mathematical problems related to the processes which we introduced in this paper. Some of them have been already mentioned, notably the questions related to the construction of stochastic integrals in Sec. IV B. As discussed in Sec. VI B, it should be interesting to generalize the results of Ref. [3] in order to link scaling properties and pathwise regularity within a multifractal formalism. Another interesting problem concerns the study of limit probability distributions associated with MRM for which very few features are known. Like infinitely divisible laws, they appear to be related to some semi-group structure. Finally, one can wonder if log infinitely processes we have defined are not the natural candidates to be described within

the framework of “Markovian continuous cascades” as introduced in Refs. [43,44].

B. Possible applications

One of the main issues of the present work was to construct a wide family of multifractal processes (or measures) that are likely to be pertinent models in many fields where multiscaling laws are observed. Naturally, the first application of which one can think, is fully developed turbulence. Turbulence and multifractals share a long history and we refer the reader to Ref. [40] for a review on the “intermittent” nature of turbulent fields. Recently, new aspects of turbulence were studied by considering fluid dynamics from a Lagrangian point of view. This was possible because two groups developed new experimental devices based on a fast imaging system [45] or ultrasound techniques [46,47] allowing for a direct measurement of the velocity of a single tracer in a turbulent flow. In a recent works, Pinton and his collaborators [46,47] studied the intermittency of Lagrangian trajectories and related it to the slow (logarithmic) decay of the particle acceleration correlations [47,48,49], very much like for a MRW model. In other words, these authors found that the turbulent Lagrangian dynamics is very well described by an equation of Langevin type with a driving force amplitude similar to $e^{\omega_l(t)}$ involved in the MRW definition. The understanding of the physical origin of such dynamical correlations and the link between Lagrangian and Eulerian statistics is a very promising path towards the explanation of the intermittency phenomenon in fully developed turbulence.

Besides turbulence, “econophysics” [50–52] is an emerging field where fractal and multifractal concepts have proven to be fruitful. Indeed, many recent studies brought empirical evidences for the multifractal nature of the fluctuations of financial markets [see Ref. [1] and reference therein]. Some physicists raised an interesting analogy between turbulence and finance [51,53–55]. Logarithmic decaying correlations and “1/f” power spectrum have been directly observed for various time series [54,56], so it is reasonable to think that MRW models are well suited for modeling financial time series [1]. The versatility of infinitely divisible MRW is very interesting to account for various stylized facts of financial times series such that the multiscaling, the power-law tail behavior of return pdf and therefore, such models can be very helpful for financial engineering and risk management.

Among all the remaining disciplines where multifractality has been observed, one can mention the study of network traffic [57–59], geophysics, and climatology [60–62], biomedical engineering [63], the modeling of natural images [64], . . . On a more theoretical ground, “continuous branching trees” and log-correlated random processes have also been considered in the physics of disordered systems [65,66]. The study of free energy density in the thermodynamic limit in presence of log-correlated disorder (or equivalently on topologies involving random disordered trees) raise questions very similar to the study of limit MRM addressed in this paper. One can hope that pushing forward this analogy is a promising prospect to get significant results in both fields.

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APPENDIX A: MEAN SQUARE CONVERGENCE OF A MRM

Let $M_l(dt)$ defined as in Eq. (23). In this section we prove that, assuming $\psi(2) < 1$, one has

$$M_l([0,t]) \xrightarrow[l \rightarrow 0]{m.s.} M([0,t]). \tag{A1}$$

Let us define

$$R_{l,l'}(\tau) = E(e^{\omega_l(u) + \omega_{l'}(u+\tau)}). \tag{A2}$$

In order to prove Eq. (A1), let us first show that, if $l' \leq l$:

$$R_{l,l'}(\tau) = R_{l,l}(\tau) = e^{\rho_l(\tau)\psi(2)} \tag{A3}$$

with $\rho_l(\tau)$ as defined as in Eqs. (30) and (31).

The first equality in Eq. (A3) comes directly from the assumption $\psi(1) = 0$ while the second equality is a particular case of the identity (33), where $q = 2$, $p_1 = p_2 = -i$ and $t_1 - t_2 = \tau$.

Let us show, that, $\forall \epsilon, \exists l_0, \forall l, l' < l_0, E[(M_l(t) - M_{l'}(t))^2] < \epsilon$. Let us suppose that $l' \leq l$. Then,

$$\begin{aligned} C(l, l', t) &= E[(M_l([0,t]) - M_{l'}([0,t]))^2] = E[M_l^2([0,t])] \\ &\quad + E[M_{l'}^2([0,t])] - 2E[M_l([0,t])M_{l'}([0,t])] \\ &= \int_0^t \int_0^t [E(e^{\omega_l(u) + \omega_l(v)} + E(e^{\omega_{l'}(u) + \omega_{l'}(v)})] dudv \\ &\quad - 2 \int_0^t \int_0^t E(e^{\omega_l(u) + \omega_{l'}(v)}) dudv. \end{aligned}$$

Thus, thanks to Eq. (A3), we get, after a little algebra

$$\begin{aligned} C(l, l', t) &\leq Dt \int_0^t (R_{l,l'}(u) - R_{l,l}(u)) du, \\ &\leq Dt \int_0^l R_{l',l'}(u) du, \\ &\leq Dt \int_0^{l'} R_{l',l'}(u) du + Dt \int_{l'}^l R_{l',l'}(u) du \\ &\leq Et l'^{1-\psi(2)} + Et (l^{1-\psi(2)} - l'^{1-\psi(2)}), \end{aligned}$$

where D and E are positive constants. Since $\psi(2) < 1$, we see that $M_l([0,t])$ is a Cauchy sequence and thus converges in mean square sense.

The previous computations also prove the convergence of all finite dimensional vector $\{M_l[t_1, t_1 + \tau_1], \dots, M_l[t_n, t_n + \tau_n]\}$. In order to prove the existence of the limit $M([0,t])$

as a stochastic process, one further needs a ‘‘tightness’’ condition. Such a condition can be obtained along the same line as for previous computation. Indeed, provided $\psi(2) < 1$, one can bound the order 2 moment of $M_l([0,t])$, $\forall t, l$:

$$E(M_l([0,t]^2)) \leq Ct^{2-\psi(2)},$$

where C does not depend on l . This achieves the proof.

APPENDIX B: MEAN SQUARE CONVERGENCE OF A MRW WITH A FRACTIONAL GAUSSIAN NOISE ϵ_H

In order to simplify the proof and to avoid technical complications, let us show the mean square convergence of the process

$$X_l(t) = \int_0^t dW_H(t) e^{\omega_l(t)}, \tag{B1}$$

where dW_H is a continuous fGn which covariance is ($H \neq 1/2$)

$$\gamma_H(\tau) = \sigma^2 H(2H-1) \tau^{2H-2}. \tag{B2}$$

Rigorously speaking the previous integral is not well defined but the proof of the convergence of the discrete version (80) is very similar. Let us now

$$X_l(t) \xrightarrow[l \rightarrow 0]{m.s.} X(t) \tag{B3}$$

provided H satisfies

$$H > 1/2 + \psi(2)/2. \tag{B4}$$

We proceed along the same line as in the Appendix A. In order to prove Eq. (B3), we have to prove, that, $\forall l, l' \leq l, E[(X_l(t) - X_{l'}(t))^2] \rightarrow_{l \rightarrow 0} 0$. Let

$$E = E[(X_l(t) - X_{l'}(t))^2]. \tag{B5}$$

Thanks to Eq. (A3), we have

$$\begin{aligned} E &= E[X_l^2(t)] + E[X_{l'}^2(t)] - 2E[X_l(t)X_{l'}(t)] \\ &= \int_0^t \int_0^t [E(e^{\omega_l(u) + \omega_l(v)} + E(e^{\omega_{l'}(u) + \omega_{l'}(v)})] \gamma_H(|u - v|) dudv - 2 \int_0^t \int_0^t E(e^{\omega_l(u) + \omega_{l'}(v)}) \gamma_H(|u - v|) dudv \\ &= \int_0^t \int_0^t [R_{l,l'}(|u - v|) - R_{l,l}(|u - v|)] \gamma_H(|u - v|) dudv. \end{aligned}$$

The last integral behavior in the limit $l \rightarrow 0$ can be easily evaluated. After some simple algebra we get

$$E[(X_l(t) - X_{l'}(t))^2] = O(l^{2H-\psi(2)-1}).$$

Thus if condition (B4) is satisfied, i.e., $2H - \psi(2) - 1 > 0$, $X_l(t)$ is a Cauchy sequence and thus converges in mean square sense.

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