Multiple scattering by two impenetrable cylinders: Semiclassical theory

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Scattering of waves and particles by two identical, impenetrable, and parallel cylinders is studied here. The characteristic determinant of the scattering matrices is expanded in terms of simple traces that are semiclassically evaluated in order to extract the periodic orbits. Generalized formulas are derived for all the contributions that are purely geometrical or composite (including a creeping section). All the scattering resonances, interpreted as periodic orbits, are in excellent agreement with the exact results. The scalar problem of scattering by two impenetrable cylinders can be considered as a canonical problem.

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I. INTRODUCTION

Scattering problems by open systems have been extensively investigated in many fields of physics (for instance in quantum mechanics, electromagnetism, optics, and acoustics). Many semiclassical methods have been carried out in the past to study such problems. A very powerful one is the geometrical theory of diffraction (GTD) developed by Keller [1] in order to describe the evolution of waves in terms of rays. Another useful method is the semiclassical trace formula introduced by Gutzwiller [2,3] and extended by other authors [4–8], using cycle expansions of zeta functions or quantum Fredholm determinants. Afterwards, the GTD has been incorporated by Vattay, Wirzba, Rosenqvist, and Whelan [9–13] in the Gutzwiller trace formula in order to take account of the diffraction effects due to creeping waves. This periodic orbit theory of diffraction improves previous results, but errors still exist [11]. Furthermore, a nonscalar example in elastodynamics has been investigated by the authors of Ref. [14].

In this paper, we propose a semiclassical approach to extract and interpret all the scattering resonances of the two impenetrable cylinders scattering problem. The characteristic determinant of the scattering matrices involved in the problem is expanded in terms of simple traces which are evaluated using the Watson transformation [15]. Generalized formulas are obtained for all the contributions that are purely geometrical or composite, i.e., with a geometrical part (one or more reflections) and a diffractive part (creeping sections). It should be noted that Wirzba gives a semiclassical approximation and some generalized formulas interpreted in terms of periodic orbits for any geometry of a finite number of nonoverlapping disks [8], meanwhile we provide here a more detailed analysis for the particular two-dimensional scattering problem by two impenetrable cylinders.

We consider two infinite, identical, impenetrable, and parallel cylinders of radius a with a center-to-center distance d. In previous papers [16–18], an exact formalism has been developed by emphasizing the role of the symmetries of the scatterer. The two-cylinder system has a $\mathcal{C}_{2v}$ symmetry [19] with four one-dimensional irreducible representations labeled A_1, A_2, B_1, and B_2. The scattering resonances are the complex zeros of the characteristic determinants of the following matrices:

$$\mathbf{M}^{(\alpha)}=\mathbf{I}+\mathbf{A}^{(\alpha)}, \tag{1}$$

with $\alpha=A_1,\ A_2,\ B_1$, or B_2 and where

$$\mathbf{A}_{qp}^{(A_1)}=-\frac{\gamma_p}{4}(-1)^q[\mathcal{S}_q(ka)-1]\times[H_{q-p}^{(1)}(kd)+(-1)^p H_{q+p}^{(1)}(kd)], \tag{2}$$

$$\mathbf{A}_{qp}^{(A_2)}=-\frac{1}{2}(-1)^q[\mathcal{S}_q(ka)-1]\times[H_{q-p}^{(1)}(kd)-(-1)^p H_{q+p}^{(1)}(kd)], \tag{3}$$

$$\mathbf{A}_{qp}^{(B_1)}=+\frac{1}{2}(-1)^q[\mathcal{S}_q(ka)-1]\times[H_{q-p}^{(1)}(kd)-(-1)^p H_{q+p}^{(1)}(kd)], \tag{4}$$

$$\mathbf{A}_{qp}^{(B_2)}=+\frac{\gamma_p}{4}(-1)^q[\mathcal{S}_q(ka)-1]\times[H_{q-p}^{(1)}(kd)+(-1)^p H_{q+p}^{(1)}(kd)]. \tag{5}$$

Here γ_p denotes the Neumann factor given by $\gamma_0=1$ and $\gamma_p=2$ $(p>0)$. The vector $\mathcal{S}_q(ka)$ reads as follows for the particular boundary conditions (BC),

(i) Dirichlet BC in quantum mechanics, in acoustics, and in electromagnetism (particle scattering by hard disks [6–13,18], ultrasonic wave scattering by soft disks, and microwave scattering by metallic conductors [20]),

$$\mathcal{S}_q(ka)=-\frac{H_q^{(2)}(ka)}{H_q^{(1)}(ka)}. \tag{6}$$

(ii) Neumann BC in acoustics (ultrasonic wave scattering by hard disks),

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$$\mathcal{S}_q(ka) = -\frac{H_q^{(2)'}(ka)}{H_q^{(1)'}(ka)}. \tag{7}$$

(iii) Impedance BC in electromagnetism [transverse magnetic (TM) and transverse electric (TE) scattering by conductors with a given constant impedance ζ [21]],

$$\mathcal{S}_q(ka) = -\frac{\zeta H_q^{(2)'}(ka) + iH_q^{(2)}(ka)}{\zeta H_q^{(1)'}(ka) + iH_q^{(1)}(ka)}. \tag{8}$$

In connection with the above relation, it should be noted that the impedance BC are taken into account with a unit normal vector pointing in the direction of the exterior medium.

The scattering resonances of the two-cylinder system are the zeros of the characteristic determinants $\det \mathbf{M}^{(\alpha)}$ with $\alpha = \mathrm{A}_1$, A_2, B_1, or B_2 (see Refs. [6,18]) in the complex ka plane and they are classified according to the four irreducible representations of $\mathcal{C}_{2v}$. We propose here a semiclassical approach based on the cumulant expansion of the matrices (1) involved in the problem, the Watson transformation, the method of steepest descent, and high-frequency approximations [6,7]. Each term of the cumulant expansion is interpreted in terms of periodic orbits by applying once or several times the usual Watson transformation and by solving multiple integrals over complex variables. Our method provides all the periodic orbits for the considered scattering problem. Therefore, we can postulate that, in the scalar case, the scattering of a point particle—or in analogy the scattering of an (electromagnetic or acoustic) wave—from two identical, impenetrable and parallel cylinders is a canonical problem.

In Sec. II, we extract all the periodic orbits of the two-cylinder system for the first three orders of the cumulant expansion in case of the A_1 representation. A generalization for any truncation order of the cumulant expansion and for the four irreducible representations of the $\mathcal{C}_{2v}$ symmetry group is given in Sec. III. Furthermore, all the scattering resonances are expressed by generalized formulas. Section IV is devoted to the physical interpretation of the periodic orbits. The exact quantum-mechanical resonance data are compared to the predictions of our semiclassical approach.

II. SEMICLASSICAL THEORY

The aim of this section is to extract all the periodic orbits of the two-cylinder scatterer in a natural way using the Watson transformation [15], the method of steepest descent [22,23], the residue theorem [24], and high-frequency approximations. We present here our method for the A_1 representation. It will be shown in Sec. III that the results are easily generalized to the three other representations A_2, B_1, B_2 of $\mathcal{C}_{2v}$. The A_1 scattering resonances are the complex solutions of the characteristic determinant (see Ref. [6]),

$$\det \mathbf{M}^{(\mathrm{A}_1)} = 0. \tag{9}$$

From now on, to simplify the notation, the A_1 dependence is suppressed. We use the cumulant expansion [8]

$$\det \mathbf{M} = \det(\mathbf{I}+\mathbf{A}) = \sum_{q=0}^{+\infty} Q_q(\mathbf{A}), \tag{10}$$

with

$$Q_0(\mathbf{A}) = 1, \tag{11}$$

$$Q_q(\mathbf{A}) = \frac{1}{q}\sum_{m=1}^{q} (-1)^{m+1} Q_{q-m}(\mathbf{A}) \mathrm{Tr}(\mathbf{A}^m) \quad \text{for } q \geq 1. \tag{12}$$

Introducing the notations

$$\mathbf{f}_q = \mathrm{Tr}(\mathbf{A}^q) \quad \text{for } q \geq 1, \tag{13}$$

the first three cumulants read

$$Q_1(\mathbf{A}) = \mathbf{f}_1, \tag{14}$$

$$Q_2(\mathbf{A}) = -\frac{1}{2}[\mathbf{f}_2 - (\mathbf{f}_1)^2], \tag{15}$$

$$Q_3(\mathbf{A}) = \frac{1}{3}\left[\mathbf{f}_3 - \frac{3}{2}\mathbf{f}_1\mathbf{f}_2 + \frac{1}{2}(\mathbf{f}_1)^3\right]. \tag{16}$$

In what follows, we extract all the periodic orbits from the first three terms of the cumulant expansion.

A. The first term of the cumulant expansion

Using Eqs. (2) and (13), the first-order cumulant (14) reads

$$\mathbf{f}_1 = \sum_{p=0}^{\infty} \mathbf{A}_{pp} = -\frac{1}{4}\sum_{p=0}^{\infty} \gamma_p (-1)^p [\mathcal{S}_p(ka) - 1] \times [H_0^{(1)}(kd) + (-1)^p H_{2p}^{(1)}(kd)]. \tag{17}$$

We apply the usual Watson transformation [15] to convert the previous partial wave series into a contour integral,

$$\sum_{p=0}^{+\infty} (-1)^p F_p(ka) = \frac{i}{2}\int_C \frac{F(\nu,ka)}{\sin(\pi\nu)} d\nu, \tag{18}$$

therefore

$$\mathbf{f}_1 = -\frac{i}{4}\int_C \frac{\mathcal{S}_\nu(ka) - 1}{\sin(\pi\nu)} [H_0^{(1)}(kd) + e^{i\pi\nu} H_{2\nu}^{(1)}(kd)] d\nu, \tag{19}$$

where $\mathcal{S}_\nu(ka)$ and $H_{2\nu}^{(1)}(kd)$ are the analytic functions in the complex ν plane, interpolating $\mathcal{S}_p(ka)$ and $H_{2p}^{(1)}(kd)$. In Eq. (19), the contour C encircles the real positive axis in the clockwise sense (see Fig. 24 of Appendix A). It should be noted that the integration takes into account the Cauchy principle value at the origin. In what follows, the ka dependence of the $\mathcal{S}_\nu$ function will be suppressed so as to keep the notation simple, therefore $\mathcal{S}_\nu \equiv \mathcal{S}_\nu(ka)$. According to Appendix A, the deformation of the contour C permits one to extract from Eq. (19) a purely geometrical contribution $\mathbf{f}_{g,1}$ and a purely diffractive contribution $\mathbf{f}_{\mathrm{dif},1}$,

$$\mathbf{f}_1=\mathbf{f}_{g,1}+\mathbf{f}_{\mathrm{dif},1}, \tag{20}$$

with

$$\mathbf{f}_{g,1}=-\frac{1}{4}\int_\Gamma \mathcal{S}_\nu e^{-i\pi\nu}[H_0^{(1)}(kd)+e^{i\pi\nu}H_{2\nu}^{(1)}(kd)]d\nu, \tag{21}$$

$$\mathbf{f}_{\mathrm{dif},1}=-i\pi\sum_{\nu_n} r_{\nu_n}\frac{e^{i\pi\nu_n}}{1-e^{2i\pi\nu_n}}\times[H_0^{(1)}(kd)+e^{i\pi\nu_n}H_{2\nu_n}^{(1)}(kd)]. \tag{22}$$

Here ν_n denotes the poles of the $\mathcal{S}_\nu$ function in the complex ν plane and r_{ν_n} is the residue of $\mathcal{S}_\nu$ at the poles $\nu=\nu_n$. It should be noted that the notation $\mathbf{f}_{\mathrm{dif},q}$ refers to contributions including a creeping section. For $q=1$, $\mathbf{f}_{\mathrm{dif},1}$ corresponds to a purely diffractive contribution, whereas for $q>1$, $\mathbf{f}_{\mathrm{dif},q}$ stands for composite contributions (a geometrical part and a diffractive one).

1. Purely diffractive contribution

The residue-series contribution (22) reads

$$\mathbf{f}_{\mathrm{dif},1}=\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}, \tag{23}$$

with

$$\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}=-i\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}H_0^{(1)}(kd), \tag{24}$$

$$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}=-i\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(2i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}H_{2\nu_n}^{(1)}(kd). \tag{25}$$

Replacing $\mathrm{H}_0^{(1)}(kd)$ and $\mathrm{H}_{2\nu_n}^{(1)}(kd)$ by their Debye asymptotic expansions (32) and using the approximation

$$\nu_n\simeq ka, \tag{26}$$

which is valid for large values of ka, we obtain

$$\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}=-\sqrt{\frac{2i\pi}{kd}}\exp(ikd)\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}, \tag{27}$$

$$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}=-\sqrt{\frac{2i\pi}{k\sqrt{d^2-4a^2}}}\exp[ik\sqrt{d^2-4a^2}]\times\sum_{\nu_n} r_{\nu_n}\frac{\exp\left[i\nu_n\left(2\pi-2\arccos\frac{2a}{d}\right)\right]}{1-\exp(2i\pi\nu_n)}. \tag{28}$$

2. Purely geometrical contribution

The purely geometrical contribution (21) can be written as

$$\mathbf{f}_{g,1}=\mathbf{f}^{\mathrm{I}}_{g,1}+\mathbf{f}^{\mathrm{II}}_{g,1}, \tag{29}$$

with

$$\mathbf{f}^{\mathrm{I}}_{g,1}=-\frac{1}{4}\int_\Gamma \mathcal{S}_\nu e^{-i\pi\nu}H_0^{(1)}(kd)d\nu, \tag{30}$$

$$\mathbf{f}^{\mathrm{II}}_{g,1}=-\frac{1}{4}\int_\Gamma \mathcal{S}_\nu H_{2\nu}^{(1)}(kd)d\nu. \tag{31}$$

Each integral is approximated, in the high-frequency limit $ka\gg1$ and $kd\gg1$, using the method of steepest descent [22]. We insert the Debye asymptotic expansions for the Hankel functions [25]

$$H_\nu^{(1,2)}(z)\sim\sqrt{\frac{2}{\pm i\pi\sqrt{z^2-\nu^2}}}\times\exp\left[\pm i\left(\sqrt{z^2-\nu^2}-\nu\arccos\frac{\nu}{z}\right)\right]\quad\text{for } |z|>\nu, \tag{32}$$

and the $\mathcal{S}_\nu(x)$ function reads

$$\mathcal{S}_\nu(x)\sim-iR(\nu,x)\exp\left[-2i\left(\sqrt{x^2-\nu^2}-\nu\arccos\frac{\nu}{x}\right)\right]. \tag{33}$$

We have introduced in Eq. (33) the reflection coefficient $R(\nu,x)$ which is defined according to the boundary condition (BC)

$$R(\nu,x)\rightarrow+1\quad\text{(Dirichlet BC)}, \tag{34}$$

$$R(\nu,x)\rightarrow-1\quad\text{(Neumann BC)}, \tag{35}$$

$$R(\nu,x)\rightarrow-\frac{\zeta\sqrt{x^2-\nu^2}-x}{\zeta\sqrt{x^2-\nu^2}+x}\text{(impedance BC)}, \tag{36}$$

with $x=ka$. By using the steepest descent method, the integrals (30) and (31) asymptotically reduce to

$$\mathbf{f}^{\mathrm{I}}_{g,1}=\tfrac{1}{2}R(0,ka)\sqrt{\frac{a}{2d}}\exp[ik(d-2a)], \tag{37}$$

$$\mathbf{f}^{\mathrm{II}}_{g,1}=\tfrac{1}{2}R(0,ka)\sqrt{\frac{a}{2(d-2a)}}\exp[ik(d-2a)]. \tag{38}$$

Finally, the first cumulant is asymptotically approximated by

$$Q_1(\mathbf{A})=\mathbf{f}_{g,1}+\mathbf{f}_{\mathrm{dif},1}, \tag{39}$$

with

$$\mathbf{f}_{g,1}=\mathbf{f}^{\mathrm{I}}_{g,1}+\mathbf{f}^{\mathrm{II}}_{g,1},$$

$$\mathbf{f}_{\mathrm{dif},1}=\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}.$$

The previous contributions are given by Eqs. (27), (28), (37), and (38). In this part, four contributions (two purely geometrical and two purely diffractive) have been extracted from $Q_1(\mathbf{A})$. They will be interpreted in terms of periodic paths in Sec. IV.

B. The second term of the cumulant expansion

We focus in this part on the second cumulant $Q_2(\mathbf{A})$ defined by Eq. (15). The term $(\mathbf{f}_1)^2$ is directly deduced from the first-order cumulant (20),

$$(\mathbf{f}_1)^2=(\mathbf{f}_{g,1}+\mathbf{f}_{\mathrm{dif},1})^2, \tag{40}$$

whereas we have to evaluate the term $\mathbf{f}_2$ given by Eq. (13),

$$\mathbf{f}_2=\mathrm{Tr}(\mathbf{A}^2)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathbf{A}_{pq}\mathbf{A}_{qp}. \tag{41}$$

The expressions of the matrix elements (2) are inserted in the previous relation, thus $\mathbf{f}_2$ reads

$$\mathbf{f}_2=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\gamma_q}{4}(-1)^p(\mathcal{S}_p-1)\frac{\gamma_p}{4}(-1)^q(\mathcal{S}_q-1)X(p,q), \tag{42}$$

where

$$X(p,q)=[H^{(1)}_{p-q}(kd)+(-1)^qH^{(1)}_{p+q}(kd)]\times[H^{(1)}_{q-p}(kd)+(-1)^pH^{(1)}_{q+p}(kd)]. \tag{43}$$

Using the Watson transformation (18), we replace in Eq. (42) the two sums over the integers p,q by two contour integrals over the complex numbers ν_1,ν_2

$$\sum_{p=0}^{+\infty}(-1)^pF_p(ka)=\frac{i}{2}\int_{C_1}\frac{F(\nu_1,ka)}{\sin(\pi\nu_1)}d\nu_1, \tag{44}$$

$$\sum_{q=0}^{+\infty}(-1)^qF_q(ka)=\frac{i}{2}\int_{C_2}\frac{F(\nu_2,ka)}{\sin(\pi\nu_2)}d\nu_2. \tag{45}$$

The contours C_1 and C_2 encircle the real positive axis in the clockwise sense in the corresponding complex ν_1 plane and ν_2 plane. We then obtain

$$\mathbf{f}_2=-\frac{1}{16}\int_{C_1}\frac{\mathcal{S}_{\nu_1}-1}{\sin(\pi\nu_1)}\left[\int_{C_2}\frac{\mathcal{S}_{\nu_2}-1}{\sin(\pi\nu_2)}X(\nu_1,\nu_2)d\nu_2\right]d\nu_1, \tag{46}$$

where $X(\nu_1,\nu_2)$, $\mathcal{S}_{\nu_1}$, and $\mathcal{S}_{\nu_2}$ are the analytic functions interpolating $X(p,q)$, $\mathcal{S}_p$ and $\mathcal{S}_q$. The function $X(\nu_1,\nu_2)$ has useful symmetry properties required to evaluate the double integral (46),

$$X(\pm\nu_1,\pm\nu_2)=X(\nu_1,\nu_2), \tag{47}$$

moreover $X(\nu_1,\nu_2)$ reduces to

$$X(\nu_1,\nu_2)\equiv 2e^{i\pi\nu_1}H^{(1)}_{\nu_1+\nu_2}(kd)H^{(1)}_{\nu_1-\nu_2}(kd)+2e^{i\pi(\nu_1+\nu_2)}[H^{(1)}_{\nu_1+\nu_2}(kd)]^2. \tag{48}$$

Indeed, four terms appear in the expansion of Eq. (43), but the calculation of the double integral (46) over the complex numbers ν_1,ν_2 only provides two different contributions.

In order to evaluate Eq. (46), we independently proceed to the modifications of the C_2 contour in the complex ν_2 plane and of the C_1 contour in the complex ν_1 plane, following the method described in Appendix A. We define

$$F(\nu_1)=\int_{C_2}\frac{\mathcal{S}_{\nu_2}-1}{\sin(\pi\nu_2)}X(\nu_1,\nu_2)d\nu_2, \tag{49}$$

$$\mathbf{f}_2=-\frac{1}{16}\int_{C_1}\frac{\mathcal{S}_{\nu_1}-1}{\sin(\pi\nu_1)}F(\nu_1)d\nu_1. \tag{50}$$

Using Eq. (47) and according to Appendix A, the integral (49) reads

$$F(\nu_1)=F_g(\nu_1)+F_d(\nu_1), \tag{51}$$

with

$$F_g(\nu_1)=-i\int_{\Gamma_2}\mathcal{S}_{\nu_2}\exp(-i\pi\nu_2)X(\nu_1,\nu_2)d\nu_2, \tag{52}$$

$$F_d(\nu_1)=4\pi\sum_{\nu_n}r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}X(\nu_1,\nu_n). \tag{53}$$

Here Γ_2 corresponds in the complex ν_2 plane to the contour Γ of Fig. 24. Using Eq. (47), $F(\nu_1)$ presents the necessary property

$$F(-\nu_1)=F(\nu_1) \tag{54}$$

in order to apply the method described in Appendix A to $\mathbf{f}_2$. Therefore, Eq. (50) reads

$$\mathbf{f}_2=\mathbf{f}_g[F(\nu_1)]+\mathbf{f}_d[F(\nu_1)], \tag{55}$$

with

$$\mathbf{f}_g[F(\nu_1)]=\frac{i}{16}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1)F(\nu_1)d\nu_1, \tag{56}$$

$$\mathbf{f}_d[F(\nu_1)]=-\frac{\pi}{4}\sum_{\nu_1=\nu_n}r_{\nu_1}\frac{\exp(i\pi\nu_1)}{1-\exp(2i\pi\nu_1)}F(\nu_1). \tag{57}$$

Γ_1 corresponds to the contour Γ of Fig. 24 in the complex ν_1 plane. Consequently, inserting Eqs. (51)–(53) in Eqs. (56) and (57), three different contributions are obtained for $\mathbf{f}_2$,

$$\mathbf{f}_2=\mathbf{f}_d[F_d(\nu_1)]+2\mathbf{f}_g[F_d(\nu_1)]+\mathbf{f}_g[F_g(\nu_1)]. \tag{58}$$

For the sake of simplicity, Eq. (58) can be written again such as

$$\mathbf{f}_2 \equiv \mathbf{f}_{dd,2} + 2\mathbf{f}_{\text{dif},2} + \mathbf{f}_{g,2}, \tag{59}$$

with

$$\mathbf{f}_{dd,2} = -\frac{\pi}{4}\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)} \times\left[4\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}X(\nu_n,\nu_n)\right], \tag{60}$$

$$\mathbf{f}_{\text{dif},2} = \frac{i\pi}{4}\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)} \times\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1)X(\nu_1,\nu_n)d\nu_1, \tag{61}$$

$$\mathbf{f}_{g,2} = \frac{1}{16}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1) \times\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}\exp(-i\pi\nu_2)X(\nu_1,\nu_2)d\nu_2\right]d\nu_1. \tag{62}$$

Now we have to evaluate the previous contributions.

1. Purely diffractive contribution

$\mathbf{f}_{dd,2}$ is a purely diffractive contribution. Using Eq. (43), Eq. (60) reads

$$\mathbf{f}_{dd,2} = \left(-i\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)} \times[H_0^{(1)}(kd)+\exp(i\pi\nu_n)H_{2\nu_n}^{(1)}(kd)]\right)^2, \tag{63}$$

and from Eq. (22), we directly obtain

$$\mathbf{f}_{dd,2} = (\mathbf{f}_{\text{dif},1})^2. \tag{64}$$

2. Composite contribution

$\mathbf{f}_{\text{dif},2}$ is a composite contribution, i.e., it contains a geometrical part and a diffractive one. After inserting relation (48) in Eq. (61), two different terms have to be evaluated,

$$\mathbf{f}_{\text{dif},2} = \mathbf{f}^{\text{I}}_{\text{dif},2} + \mathbf{f}^{\text{II}}_{\text{dif},2}, \tag{65}$$

with

$$\mathbf{f}^{\text{I}}_{\text{dif},2} = \frac{i\pi}{2}\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)} \times\int_{\Gamma_1}\mathcal{S}_{\nu_1}H^{(1)}_{\nu_n+\nu_1}(kd)H^{(1)}_{\nu_1-\nu_n}(kd)d\nu_1, \tag{66}$$

$$\mathbf{f}^{\text{II}}_{\text{dif},2} = \frac{i\pi}{2}\sum_{\nu_n} r_{\nu_n}\frac{\exp(2i\pi\nu_n)}{1-\exp(2i\pi\nu_n)} \times\int_{\Gamma_1}\mathcal{S}_{\nu_1}[H^{(1)}_{\nu_1+\nu_n}(kd)]^2 d\nu_1. \tag{67}$$

$\mathbf{f}^{\text{I}}_{\text{dif},2}$ and $\mathbf{f}^{\text{II}}_{\text{dif},2}$ contain a diffractive part (the residue series) and a geometrical part (the integration in the complex ν_1 plane) evaluated by applying the method of steepest descent [22]. We obtain after calculations

$$\mathbf{f}^{\text{I}}_{\text{dif},2} = -\sum_{\nu_n} r_{\nu_n}\frac{\exp\left\{2i\nu_n\left[\pi-\arccos\left(\frac{a}{d}\right)\right]\right\}}{1-\exp(2i\pi\nu_n)} \times R(0,ka)\sqrt{\frac{i\pi a}{k(\sqrt{d^2-a^2}-a)\sqrt{d^2-a^2}}} \times\exp[2ik(\sqrt{d^2-a^2}-a)], \tag{68}$$

$$\mathbf{f}^{\text{II}}_{\text{dif},2} = -\sum_{\nu_n} r_{\nu_n}\frac{\exp\left\{2i\nu_n\left[\pi-\arccos\left(\frac{a}{d-a}\right)\right]\right\}}{1-\exp(2i\pi\nu_n)} \times R\left(\nu_n\frac{a}{d-a},ka\right)\sqrt{\frac{i\pi a}{kd\sqrt{d(d-2a)}}} \times\exp[2ik\sqrt{d(d-2a)}]. \tag{69}$$

3. Purely geometrical contribution

$\mathbf{f}_{g,2}$ given by Eq. (62) is a purely geometrical contribution obtained applying twice the method of steepest descent on the variables ν_1 and ν_2 (see details in Appendix B). We obtain

$$\mathbf{f}_{g,2} = \mathbf{f}^{\text{I}}_{g,2} + \mathbf{f}^{\text{II}}_{g,2}, \tag{70}$$

with

$$\mathbf{f}^{\text{I}}_{g,2} = \tfrac{1}{2}R(0,ka)^2\frac{a}{2(d-a)}\exp[2ik(d-2a)], \tag{71}$$

$$\mathbf{f}^{\text{II}}_{g,2} = \tfrac{1}{2}R(0,ka)^2\frac{a}{2\sqrt{d(d-2a)}}\exp[2ik(d-2a)]. \tag{72}$$

Finally using Eqs. (40), (59), and (64), the second cumulant $Q_2(\mathbf{A})$ reads

$$Q_2(\mathbf{A}) = -\frac{1}{2}[\mathbf{f}_{g,2}-(\mathbf{f}_{g,1})^2] + \mathbf{f}_{\text{dif},1}\mathbf{f}_{g,1} - \mathbf{f}_{\text{dif},2}, \tag{73}$$

with

$$\mathbf{f}_{g,1} = \mathbf{f}^{\text{I}}_{g,1} + \mathbf{f}^{\text{II}}_{g,1},$$

$$\mathbf{f}_{\text{dif},1} = \mathbf{f}^{\text{I}}_{\text{dif},1} + \mathbf{f}^{\text{II}}_{\text{dif},1},$$

$$\mathbf{f}_{g,2}=\mathbf{f}^{\mathrm{I}}_{g,2}+\mathbf{f}^{\mathrm{II}}_{g,2},$$

$$\mathbf{f}_{\mathrm{dif},2}=\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2}+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2}.$$

The previous terms are given by the relations (37), (38), (27), (28), and (68)–(72) and will be physically interpreted in Sec. IV. It should be noted that only purely geometrical contributions and diffractive (or composite) contributions appear in $Q_2(\mathbf{A})$. We refer to the terms $\mathbf{f}_{\mathrm{dif},q}$ by diffractive (or composite) contributions composed by one diffractive part (residue series) and $(q-1)$ geometrical parts (evaluated by the method of steepest descent).

C. The third term of the cumulant expansion

In this part we study the third cumulant $Q_3(\mathbf{A})$ given by Eq. (16),

$$Q_3(\mathbf{A})=\tfrac{1}{3}[\mathbf{f}_3-\tfrac{3}{2}\mathbf{f}_1\mathbf{f}_2+\tfrac{1}{2}(\mathbf{f}_1)^3].$$

The evaluation of $\mathbf{f}_3$ will provide new contributions, whereas the terms $(\mathbf{f}_1\mathbf{f}_2)$ and $(\mathbf{f}_1)^3$ are directly deduced from the results of $Q_1(\mathbf{A})$ and $Q_2(\mathbf{A})$. Using Eq. (13), the term $\mathbf{f}_3$ is of the form

$$\mathbf{f}_3=\mathrm{Tr}(\mathbf{A}^3)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\sum_{m=0}^{\infty}\mathbf{A}_{pq}\mathbf{A}_{qm}\mathbf{A}_{mp}. \tag{74}$$

Inserting the definition of the matrix elements (2), $\mathbf{f}_3$ reads

$$\mathbf{f}_3=-\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{m=0}^{\infty}\frac{\gamma_q}{4}(-1)^p(\mathcal{S}_p-1)\frac{\gamma_m}{4}\times(-1)^q(\mathcal{S}_q-1)\frac{\gamma_p}{4}(-1)^m(\mathcal{S}_m-1)Z(p,q,m), \tag{75}$$

where

$$Z(p,q,m)=[H^{(1)}_{p-q}(kd)+(-1)^qH^{(1)}_{p+q}(kd)]\times[H^{(1)}_{q-m}(kd)+(-1)^mH^{(1)}_{q+m}(kd)]\times[H^{(1)}_{m-p}(kd)+(-1)^pH^{(1)}_{m+p}(kd)]. \tag{76}$$

The three sums over the integers p,q,m are replaced, thanks to the Watson transformation (18), by three contour integrals in the ν_1,ν_2,ν_3 complex planes, thus

$$\mathbf{f}_3=\frac{i}{64}\int_{C_1}\frac{\mathcal{S}_{\nu_1}-1}{\sin(\pi\nu_1)}\left[\int_{C_2}\frac{\mathcal{S}_{\nu_2}-1}{\sin(\pi\nu_2)}\left(\int_{C_3}\frac{\mathcal{S}_{\nu_3}-1}{\sin(\pi\nu_3)}Z(\nu_1,\nu_2,\nu_3)d\nu_3\right)d\nu_2\right]d\nu_1, \tag{77}$$

and $Z(\nu_1,\nu_2,\nu_3)$ reduces to

$$Z(\nu_1,\nu_2,\nu_3)\equiv 4H^{(1)}_{\nu_1-\nu_2}(kd)H^{(1)}_{\nu_2+\nu_3}(kd)H^{(1)}_{\nu_3+\nu_1}(kd)e^{i\pi(\nu_1+\nu_3)}+4H^{(1)}_{\nu_1+\nu_2}(kd)H^{(1)}_{\nu_2+\nu_3}(kd)H^{(1)}_{\nu_3+\nu_1}(kd)e^{i\pi(\nu_1+\nu_2+\nu_3)}. \tag{78}$$

We define

$$F_{\nu_3}(\nu_1,\nu_2)=\int_{C_3}\frac{\mathcal{S}_{\nu_3}-1}{\sin(\pi\nu_3)}Z(\nu_1,\nu_2,\nu_3)d\nu_3, \tag{79}$$

$$F_{\nu_2}(\nu_1)=\int_{C_2}\frac{\mathcal{S}_{\nu_2}-1}{\sin(\pi\nu_2)}F_{\nu_3}(\nu_1,\nu_2)d\nu_2, \tag{80}$$

$$\mathbf{f}_3=\frac{i}{64}\int_{C_1}\frac{\mathcal{S}_{\nu_1}-1}{\sin(\pi\nu_1)}F_{\nu_2}(\nu_1)d\nu_1, \tag{81}$$

and with the following properties:

$$Z(\pm\nu_1,\pm\nu_2,\pm\nu_3)=Z(\nu_1,\nu_2,\nu_3), \tag{82}$$

$$F_{\nu_3}(\pm\nu_1,\pm\nu_2)=F_{\nu_3}(\nu_1,\nu_2), \tag{83}$$

$$F_{\nu_2}(\pm\nu_1)=F_{\nu_2}(\nu_1), \tag{84}$$

the method described in Appendix A can be successively applied to $F_{\nu_3}(\nu_1,\nu_2)$, $F_{\nu_2}(\nu_1)$, and $\mathbf{f}_3$. After simplifications, we obtain four different contributions,

$$\mathbf{f}_3=\mathbf{f}_{ddd,3}+3\mathbf{f}_{gdd,3}+3\mathbf{f}_{\mathrm{dif},3}+\mathbf{f}_{g,3}, \tag{85}$$

with

$$\mathbf{f}_{ddd,3}=\frac{i\pi}{16}\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}4\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}4\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}Z(\nu_n,\nu_n,\nu_n), \tag{86}$$

$$\mathbf{f}_{gdd,3}=\frac{1}{4}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1)\left[\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}\pi\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}Z(\nu_1,\nu_n,\nu_n)\right]d\nu_1, \tag{87}$$

$$\mathbf{f}_{\mathrm{dif},3}=-\frac{i\pi}{16}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1)\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}\exp(-i\pi\nu_2)\left(\sum_{\nu_n} r_{\nu_n}\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)}Z(\nu_1,\nu_2,\nu_n)\right)d\nu_2\right]d\nu_1, \tag{88}$$

$$\mathbf{f}_{g,3}=-\frac{1}{64}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\exp(-i\pi\nu_1)\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}\exp(-i\pi\nu_2)\left(\int_{\Gamma_3}\mathcal{S}_{\nu_3}\exp(-i\pi\nu_3)Z(\nu_1,\nu_2,\nu_3)d\nu_3\right)d\nu_2\right]d\nu_1. \tag{89}$$

The previous contributions have now to be evaluated.

1. Contributions deduced from previous results

$\mathbf{f}_{ddd,3}$ is a purely diffractive contribution. Using relations (22) and (76), Eq. (86) reads

$$\mathbf{f}_{ddd,3}=(\mathbf{f}_{\mathrm{dif},1})^3. \tag{90}$$

$\mathbf{f}_{gdd,3}$ can be written from previous results. Indeed from Eqs. (22), (43), (61), and (76), we deduce

$$\mathbf{f}_{gdd,3}=\mathbf{f}_{\mathrm{dif},1}\mathbf{f}_{\mathrm{dif},2}. \tag{91}$$

2. Composite contribution

$\mathbf{f}_{\mathrm{dif},3}$ given by Eq. (88) is a composite contribution composed by a diffractive part and two geometrical parts evaluated using twice the method of steepest descent. We obtain

$$\mathbf{f}_{\mathrm{dif},3}=\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},3}+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},3}, \tag{92}$$

where

$$\begin{aligned}\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},3}=&-\sqrt{2i\pi}\sum_{\nu_n} r_{\nu_n}\frac{\exp\left\{2i\nu_n\left[\pi-\arccos\left(\frac{\nu^{\mathrm{I}}_{1,3}+\nu_n}{y}\right)\right]\right\}}{1-\exp(2i\pi\nu_n)}\{y[y^2-(\nu_n+\nu^{\mathrm{I}}_{1,3})^2]\}^{-1/2}\\&\times R(\nu^{\mathrm{I}}_{1,3},ka)^2\exp[2i\sqrt{y^2-(\nu_n+\nu^{\mathrm{I}}_{1,3})^2}-4i\sqrt{x^2-(\nu^{\mathrm{I}}_{1,3})^2}+iy]\\&\times\left[\left(\frac{1}{\sqrt{y^2-(\nu_n+\nu^{\mathrm{I}}_{1,3})^2}}-\frac{2}{\sqrt{x^2-(\nu^{\mathrm{I}}_{1,3})^2}}+\frac{1}{y}\right)^2-\left(\frac{1}{y}\right)^2\right]^{-1/2}\quad\text{with}\quad \nu^{\mathrm{I}}_{1,3}=\nu_n\frac{a}{2d-a},\end{aligned} \tag{93}$$

$$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},3}=-\sqrt{2i\pi}\sum_{\nu_n} r_{\nu_n}\frac{\exp\left\{2i\nu_n\left[\pi-\arccos\left(\frac{\nu^{\mathrm{II}}_{1,3}+\nu_n}{y}\right)\right]\right\}}{1-\exp(2i\pi\nu_n)}\{\sqrt{y^2-(2\nu^{\mathrm{II}}_{1,3})^2}[y^2-(\nu_n+\nu^{\mathrm{II}}_{1,3})^2]\}^{-1/2}$$
$$\times R(\nu^{\mathrm{II}}_{1,3},ka)^2\exp[2i\sqrt{y^2-(\nu_n+\nu^{\mathrm{II}}_{1,3})^2}-4i\sqrt{x^2-(\nu^{\mathrm{II}}_{1,3})^2}+i\sqrt{y^2-(2\nu^{\mathrm{II}}_{1,3})^2}]$$
$$\times\left[\left(\frac{1}{\sqrt{y^2-(\nu_n+\nu^{\mathrm{II}}_{1,3})^2}}-\frac{2}{\sqrt{x^2-(\nu^{\mathrm{II}}_{1,3})^2}}+\frac{1}{\sqrt{y^2-(2\nu^{\mathrm{II}}_{1,3})^2}}\right)^2-\left(\frac{1}{\sqrt{y^2-(2\nu^{\mathrm{II}}_{1,3})^2}}\right)^2\right]^{-1/2}\quad\text{with}\quad \nu^{\mathrm{II}}_{1,3}=\nu_n\frac{a}{2d-3a}. \tag{94}$$

In the previous relations we have introduced the saddle point notations which will be used later in the generalized formulas (see the following section). For instance, $\nu^{I}_{1,3}$ stands for the first (and the only one at this order) saddle point for a third-order composite contribution of type I, and $\nu^{\mathrm{II}}_{m,q}$ stands for the mth saddle point for a q-order composite contribution of type II.

3. Purely geometrical contribution

The term $\mathbf{f}_{g,3}$ given by Eq. (89) is a purely geometrical contribution. We successively apply the method of steepest descent three times, following the method used at the second order for the evaluation of $\mathbf{f}_{g,2}$ (see Appendix B). Alternatively, we can also apply the multiple integrals formula (C2) given in Appendix C. The two methods lead to the same results,

$$\mathbf{f}_{g,3}=\mathbf{f}^{\mathrm{I}}_{g,3}+\mathbf{f}^{\mathrm{II}}_{g,3}, \tag{95}$$

with

$$\mathbf{f}^{\mathrm{I}}_{g,3}=\tfrac{1}{2}R(0,ka)^3\sqrt{\frac{a}{2d}}\frac{a}{2d-3a}\times\exp[3ik(d-2a)], \tag{96}$$

$$\mathbf{f}^{\mathrm{II}}_{g,3}=\tfrac{1}{2}R(0,ka)^3\sqrt{\frac{a}{2(d-2a)}}\frac{a}{2d-a}\times\exp[3ik(d-2a)]. \tag{97}$$

Finally the third-order cumulant $Q_3(\mathbf{A})$ is given, after simplification, by

$$Q_3(\mathbf{A})=\frac{1}{3}\left[\mathbf{f}_{g,3}-\frac{3}{2}\mathbf{f}_{g,1}\mathbf{f}_{g,2}+\frac{1}{2}(\mathbf{f}_{g,1})^3\right]-\frac{1}{2}[\mathbf{f}_{g,2}-(\mathbf{f}_{g,1})^2]\mathbf{f}_{\mathrm{dif},1}-\mathbf{f}_{g,1}\mathbf{f}_{\mathrm{dif},2}+\mathbf{f}_{\mathrm{dif},3}. \tag{98}$$

As for $Q_2(\mathbf{A})$, only purely geometrical contributions $\mathbf{f}_{g,q}$ and diffractive contributions $\mathbf{f}_{\mathrm{dif},q}$ are involved in $Q_3(\mathbf{A})$.

The main results obtained in this section are summarized in the two following points.

(i) All the contributions of the cumulant expansion up to the order $q=3$ have been determined.

(ii) Each new cumulant provides only two purely geometrical contributions and two diffractive contributions, as mentioned in Ref. [9].

We can note that some results have direct counterparts in Refs. [6,7] in the Dirichlet case, especially the purely geometrical contributions up to the third order and the first-order diffractive contribution.

III. GENERALIZATION

In the preceding section, we have extracted all the contributions from the first three orders of the cumulant expansion. We derive here the generalization of the purely geometrical and the diffractive (or composite) contributions that provide scattering resonances. Consequently, we obtain an asymptotic approximation of det $\mathbf{M}$, for any truncation order q of the cumulant expansion (10).

A. Purely geometrical contributions

To evaluate q-order geometrical contributions, the method of steepest descent is q times successively applied. Thus, we generalize the procedure used for the first three orders $\mathbf{f}_{g,1}$, $\mathbf{f}_{g,2}$, $\mathbf{f}_{g,3}$. The corresponding generalized formula reads

$$\mathbf{f}_{g,q}=\tfrac{1}{2}(G^{\mathrm{I}}_q+G^{II}_q)R(0,ka)^q\exp[qik(d-2a)]\quad\text{for}\quad q\geqslant 1, \tag{99}$$

where we have defined the following recurrence relations (l stands for I or II):

$$G^l_q=g^l_qG^l_{q-2}, \tag{100}$$

$$g^l_q=\frac{a\mathfrak{D}[g^l_{q-2}]}{2(d-a)\mathfrak{D}[g^l_{q-2}]-a\mathfrak{N}[g^l_{q-2}]}. \tag{101}$$

The functions $\mathfrak{N}[z]$ and $\mathfrak{D}[z]$, respectively, correspond to the numerator and to the denominator of z. a is the radius of the cylinders and d is the center-to-center separation distance. The initial coefficients are given by

$$G^{\mathrm{I}}_1=\sqrt{\frac{a}{2d}},\quad g^{\mathrm{I}}_3=\frac{a}{2d-3a},$$

$$G_1^{\mathrm{II}}=\sqrt{\frac{a}{2(d-2a)}},\quad g_3^{\mathrm{II}}=\frac{a}{2d-a},$$

$$G_2^{\mathrm{I}}=\frac{a}{2(d-a)},\quad g_4^{\mathrm{I}}=\frac{a(d-a)}{2(d-a)^2-a^2},$$

$$G_2^{\mathrm{II}}=\frac{a}{2\sqrt{d(d-2a)}},\quad g_4^{\mathrm{II}}=\frac{a}{2(d-a)}. \tag{102}$$

Therefore, each purely geometrical q-order contribution is given by formula (99). For example, in the case of $q=3$, we deduce from Eqs. (100) and (102)

$$G_3^{\mathrm{I}}=g_3^{\mathrm{I}}G_1^{\mathrm{I}}=\sqrt{\frac{a}{2d}}\frac{a}{2d-3a},$$

$$G_3^{\mathrm{II}}=g_3^{\mathrm{II}}G_1^{\mathrm{II}}=\sqrt{\frac{a}{2(d-2a)}}\frac{a}{2d-a}, \tag{103}$$

thus we obtain with Eq. (99)

$$\mathbf{f}_{g,3}=\frac{1}{2}\left(\sqrt{\frac{a}{2d}}\frac{a}{2d-3a}+\sqrt{\frac{a}{2(d-2a)}}\frac{a}{2d-a}\right)\times R(0,ka)^3\exp[3ik(d-2a)]. \tag{104}$$

This result is in good agreement with the one obtained from Eq. (95) in Sec. II.

B. Composite contributions

The diffractive (or composite) contributions $\mathbf{f}_{\mathrm{dif},q}$ are composed by one diffractive part and $(q-1)$ geometrical parts. Their evaluation is carried out applying $(q-1)$ times the method of steepest descent. We give different formulas whether q is even or odd, generalizing the results for all the diffractive contributions.

First of all, we give some coefficient definitions that are valid for q even or odd. We define the determinant C_q^l of order $(q-1)$ given by

$$C_q^l=\begin{vmatrix} a_{1,q}^l & b_{1,q}^l & 0 & 0 & \cdots & \cdots & 0\\ b_{1,q}^l & a_{2,q}^l & b_{2,q}^l & 0 & \cdots & \cdots & 0\\ 0 & b_{2,q}^l & a_{3,q}^l & b_{3,q}^l & \cdots & \cdots & 0\\ 0 & 0 & b_{3,q}^l & a_{4,q}^l & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & a_{q-2,q}^l & b_{q-2,q}^l\\ 0 & 0 & 0 & 0 & \cdots & b_{q-2,q}^l & a_{q-1,q}^l \end{vmatrix}, \tag{105}$$

which can be written as

$$C_q^l=\prod_{m=1}^{q-1}a_{m,q}^l\times\left[1-\sum_{j=1}^{q-2}\frac{(b_{j,q}^l)^2}{a_{j,q}^l a_{j+1,q}^l}\left(1-\sum_{k=j+2}^{q-2}\frac{(b_{k,q}^l)^2}{a_{k,q}^l a_{k+1,q}^l}\right)\right], \tag{106}$$

with

$$a_{m,q}^l=\frac{1}{\sqrt{y^2-(\nu_{m-1,q}^l+\nu_{m,q}^l)^2}}-\frac{2}{\sqrt{x^2-(\nu_{m,q}^l)^2}}+\frac{1}{\sqrt{y^2-(\nu_{m,q}^l+\nu_{m+1,q}^l)^2}}, \tag{107}$$

$$b_{m,q}^l=\frac{1}{\sqrt{y^2-(\nu_{m,q}^l+\nu_{m+1,q}^l)^2}}. \tag{108}$$

Here $x=ka$, $y=kd$ and the superscript l denotes I or II. The saddle points $\nu_{m,q}^l$ are defined at the q order by

$$\nu_{m,q}^l=\nu_n c_{m,q}^l, \tag{109}$$

where ν_n stands for the poles of the $\mathcal{S}_\nu$ function. The coefficients $c_{m,q}^l$ are given by the following recurrence relations:

$$c_{1,q}^l=\frac{a\mathfrak{D}[c_{1,q-2}^l]}{2(d-a)\mathfrak{D}[c_{1,q-2}^l]-a\mathfrak{N}[c_{1,q-2}^l]}, \tag{110}$$

$$c_{m,q}^l=\frac{a\mathfrak{N}[c_{m-1,q-2}^l]}{2(d-a)\mathfrak{D}[c_{1,q-2}^l]-a\mathfrak{N}[c_{1,q-2}^l]},$$

$$\text{for } 1<m\leqslant p \quad \text{with } q=2p \text{ or } q=2p+1. \tag{111}$$

At the q order, q saddle points are involved in C_q^l but only p of them are different in modulus.

1. Even truncation order q = 2p

Diffractive contributions for $q=2p$ are given by the generalized formula

$$\mathbf{f}_{\mathrm{dif},2p}=\mathbf{f}_{\mathrm{dif},2p}^{\mathrm{I}}+\mathbf{f}_{\mathrm{dif},2p}^{\mathrm{II}}, \tag{112}$$

where

$$\mathbf{f}^{l}_{\mathrm{dif},2p}=-\sqrt{2i\pi}\sum_{\nu_n} r_{\nu_n}\frac{\exp(2i\nu_n\{\pi-\arccos[(\nu^{l}_{1,2p}+\nu_n)/y]\})}{1-\exp(2i\pi\nu_n)}\frac{R(\nu^{l}_{p,2p},ka)}{\sqrt{-C^{l}_{2p}}}$$
$$\times\frac{\exp[2i\sqrt{y^2-(\nu^{l}_{p-1,2p}+\nu^{l}_{p,2p})^2}-2i\sqrt{x^2-(\nu^{l}_{p,2p})^2}]}{\sqrt{y^2-(\nu^{l}_{p-1,2p}+\nu^{l}_{p,2p})^2}}$$
$$\times\prod_{m=1}^{p-1}\left[R(\nu^{l}_{m,2p},ka)^2\frac{\exp[2i\sqrt{y^2-(\nu^{l}_{m-1,2p}+\nu^{l}_{m,2p})^2}-4i\sqrt{x^2-(\nu^{l}_{m,2p})^2}]}{\sqrt{y^2-(\nu^{l}_{m-1,2p}+\nu^{l}_{m,2p})^2}}\right],\quad \text{for }\ p\geqslant 1\quad \text{with }\ l=\mathrm{I,II}. \tag{113}$$

We use the following relations for the saddle points:

$$\nu^{l}_{2p-j,2p}=\nu^{l}_{j,2p}\quad \text{for }\ 1\leqslant j\leqslant p-1,$$
$$\nu^{l}_{0,2p}=\nu_n,$$
$$\nu^{l}_{2p,2p}=\nu_n, \tag{114}$$

and the initial coefficients are

$$c^{\mathrm{I}}_{1,2}=0\equiv\frac{0}{1},\qquad c^{\mathrm{II}}_{1,2}=\frac{a}{d-a}, \tag{115}$$

so we deduce with Eqs. (110) and (111)

$$c^{\mathrm{I}}_{1,4}=\frac{a}{2(d-a)},\qquad c^{\mathrm{I}}_{2,4}=0,$$
$$c^{\mathrm{II}}_{1,4}=\frac{a(d-a)}{2(d-a)^2-a^2},\qquad c^{\mathrm{II}}_{2,4}=\frac{a^2}{2(d-a)^2-a^2}. \tag{116}$$

2. Odd truncation order $q=2p+1$

Diffractive contributions for $q=2p+1$ are given by the generalized formula

$$\mathbf{f}_{\mathrm{dif},2p+1}=\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2p+1}+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2p+1}, \tag{117}$$

where

$$\mathbf{f}^{l}_{\mathrm{dif},2p+1}=-\sqrt{2i\pi}\sum_{\nu_n} r_{\nu_n}\frac{\exp(2i\nu_n\{\pi-\arccos[(\nu^{l}_{1,2p+1}+\nu_n)/y]\})}{1-\exp(2i\pi\nu_n)}\frac{R(\nu^{l}_{p,2p+1},ka)^2}{\sqrt{C^{l}_{2p+1}}}$$
$$\times\frac{\exp[2i\sqrt{y^2-(\nu^{l}_{p-1,2p+1}+\nu^{l}_{p,2p+1})^2}-4i\sqrt{x^2-(\nu^{l}_{p,2p+1})^2}+i\sqrt{y^2-(\nu^{l}_{p,2p+1}+\nu^{l}_{p+1,2p+1})^2}]}{\sqrt{\sqrt{y^2-(\nu^{l}_{p,2p+1}+\nu^{l}_{p+1,2p+1})^2}[y^2-(\nu^{l}_{p-1,2p+1}+\nu^{l}_{p,2p+1})^2]}}$$
$$\times\prod_{m=1}^{p-1}\left[R(\nu^{l}_{m,2p+1},ka)^2\frac{\exp[2i\sqrt{y^2-(\nu^{l}_{m-1,2p+1}+\nu^{l}_{m,2p+1})^2}-4i\sqrt{x^2-(\nu^{l}_{m,2p+1})^2}]}{\sqrt{y^2-(\nu^{l}_{m-1,2p+1}+\nu^{l}_{m,2p+1})^2}}\right],\quad \text{for }\ p\geqslant 1\quad \text{with }\ l=\mathrm{I,II}. \tag{118}$$

TABLE I. Generalization of the purely geometrical and diffractive contributions for the four irreducible representations of the $\mathcal{C}_{2v}$ symmetry group ($q=1,2,3,\ldots,\infty$).

A_1	A_2	B_1	B_2
$\mathbf{f}^{\mathrm{I}}_{g,q}$	$-(-1)^q\mathbf{f}^{\mathrm{I}}_{g,q}$	$-\mathbf{f}^{\mathrm{I}}_{g,q}$	$(-1)^q\mathbf{f}^{\mathrm{I}}_{g,q}$
$\mathbf{f}^{\mathrm{II}}_{g,q}$	$+(-1)^q\mathbf{f}^{\mathrm{II}}_{g,q}$	$+\mathbf{f}^{\mathrm{II}}_{g,q}$	$(-1)^q\mathbf{f}^{\mathrm{II}}_{g,q}$
$\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},q}$	$-(-1)^q\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},q}$	$-\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},q}$	$(-1)^q\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},q}$
$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},q}$	$+(-1)^q\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},q}$	$+\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},q}$	$(-1)^q\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},q}$

We use the following relations for the saddle points:

$$\nu^{\mathrm{I}}_{2p+1-j,2p+1}=-\nu^{\mathrm{I}}_{j,2p+1} \quad \text{for } 0\leqslant j\leqslant p,$$

$$\nu^{\mathrm{I}}_{0,2p+1}=\nu_n\,, \tag{119}$$

$$\nu^{\mathrm{II}}_{2p+1-j,2p+1}=\nu^{\mathrm{II}}_{j,2p+1} \quad \text{for } 0\leqslant j\leqslant p,$$

$$\nu^{\mathrm{II}}_{0,2p+1}=\nu_n\,. \tag{120}$$

The initial coefficients are

$$c^{\mathrm{I}}_{1,3}=\frac{a}{2d-a}, \quad c^{\mathrm{II}}_{1,3}=\frac{a}{2d-3a}. \tag{121}$$

Equations (113) and (118) provide all the diffractive contributions $\mathbf{f}_{\mathrm{dif},q}$ for any truncation order $q\geqslant 2$. The first-order diffractive contributions are given in Sec. II.

C. Semiclassical approximation of det M

Generalizing previous results concerning the cumulants determined up to the third order [see Eqs. (39), (73), and (98)], a q-order cumulant can be approximated by

$$Q_q(\mathbf{A})\simeq Q_{g,q}+\sum^{q}\ m=1(-1)^{m+1}Q_{g,q-m}\mathbf{f}_{\mathrm{dif},m}\,, \tag{122}$$

where $\mathbf{f}_{\mathrm{dif},m}$ is the m-order composite contribution given by Eqs. (113) and (118) and we define $Q_{g,q}$ as the "geometrical" cumulants

$$Q_{g,0}=1, \tag{123}$$

$$Q_{g,q}=\frac{1}{q}\sum_{m=1}^{q}(-1)^{m+1}Q_{g,q-m}\mathbf{f}_{g,m} \quad \text{for } q\geqslant 1, \tag{124}$$

and $\mathbf{f}_{g,m}$ stands for the m-order purely geometrical contribution given by relation (99). Finally, inserting Eq. (122) in Eq. (10), det **M** is approximated by

$$\det\mathbf{M}\simeq\sum_{q=0}^{+\infty}\left[Q_{g,q}+\sum_{m=1}^{q}(-1)^{m+1}Q_{g,q-m}\mathbf{f}_{\mathrm{dif},m}\right]. \tag{125}$$

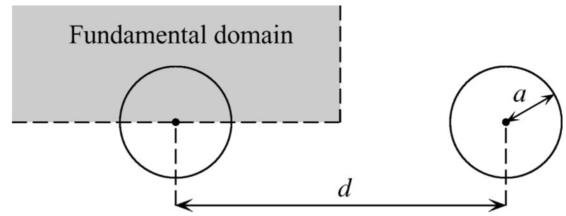

FIG. 1. Fundamental domain of the $\mathcal{C}_{2v}$ symmetry group.

As a result, the previous relation (125) permits one to semiclassically evaluate det **M** for the A_1 representation of the $\mathcal{C}_{2v}$ symmetry group for any truncation order q.

D. Extension to the A_2,B_1,B_2 irreducible representations

All the studies have been realized in the case of the A_1 representation of the $\mathcal{C}_{2v}$ symmetry group. The method described in the case of the A_1 representation of the $\mathcal{C}_{2v}$ symmetry group can be easily extended to the three other irreducible representations A_2, B_1, and B_2. The corresponding results are deduced by introducing in the expression of det **M** (125) the simple modifications reported in Table I. In this section, we have given the generalized formulas that allow us to evaluate det **M** for any truncation order q in the cumulant expansion and for the four irreducible representations of $\mathcal{C}_{2v}$. Furthermore, all the scattering resonances of the two impenetrable cylinders system are determined and interpreted in the following section.

IV. NUMERICAL RESULTS AND PHYSICAL INTERPRETATION OF RESONANCES

The aim of this section is to provide a physical interpretation for all the scattering resonances of the two-cylinder system as periodic paths. We use the expressions obtained in Secs. II and III for the A_1 representation and we particularly focus on the exponential terms yielding the periodic orbit interpretation. The contributions of the three other representations A_2,B_1,B_2 of $\mathcal{C}_{2v}$ do not provide new geometrical paths because the exponential terms are identical for all the representations. The periodic paths should be displayed in the fundamental domain of the scatterer (see Fig. 1) but they are presented in the entire domain where they appear more clearly.

A. Purely geometrical contributions

Using the results of Sec. II, the geometrical contributions $\mathbf{f}_{g,q}$ up to $q=3$ are given by

$$\mathbf{f}_{g,1}=\frac{1}{2}\left(\sqrt{\frac{a}{2d}}+\sqrt{\frac{a}{2(d-2a)}}\right)\times R(0,ka)\exp[ik(d-2a)], \tag{126}$$

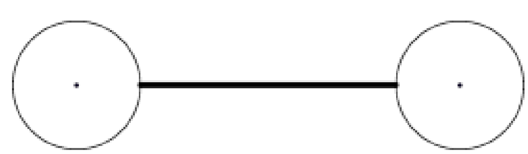

FIG. 2. Periodic orbit of the geometrical contributions $\mathbf{f}_{g,q}$.

$$\mathbf{f}_{g,2}=\frac{1}{2}\left(\frac{a}{2(d-a)}+\frac{a}{2\sqrt{d(d-2a)}}\right)\times R(0,ka)^2\exp[2ik(d-2a)], \tag{127}$$

$$\mathbf{f}_{g,3}=\frac{1}{2}\left(\sqrt{\frac{a}{2d}}\frac{a}{2d-3a}+\sqrt{\frac{a}{2(d-2a)}}\frac{a}{2d-a}\right)\times R(0,ka)^3\exp[3ik(d-2a)], \tag{128}$$

and from formula (99), we can write

$$\mathbf{f}_{g,q}\propto\exp[qik(d-2a)]\quad\text{for}\quad q\geqslant 1, \tag{129}$$

where the exponential term provides the periodic orbit interpretation. These contributions are obviously associated with the closed geometrical path described in Fig. 2. More precisely, q corresponds to the number of reflections on the cylinder in the fundamental domain.

The prefactors of the geometrical contributions involved in Eqs. (126)–(128) are the well-known stability factors given in Ref. [7]. It is important to note that they are different from those obtained following Ref. [26], in the case of acoustic scattering by two spheres, where the geometrical theory of diffraction involving a single scatterer is directly applied to a system composed by two objects. This latter method only provides the first-order stability factors.

B. Composite contributions

From the results obtained in Secs. II and III, we can write that all composite contributions $\mathbf{f}^{l}_{\mathrm{dif},q}$ ($l=$I or II) are of the form

$$\mathbf{f}^{l}_{\mathrm{dif},q}\propto\exp(ikt)\exp(i\nu_n\beta), \tag{130}$$

where t denotes the geometrical path between the two cylinders and β stands for the angle of the creeping section. The reflection angles are directly deduced from the saddle-point values. We display the interpretation of the diffractive contributions as periodic paths up to third order.

For instance, Fig. 3 displays the periodic orbit deduced from the first-order diffractive contribution $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}$ given by Eq. (27). In this case, the geometrical path between the two cylinders is $t=d$ and the angle of the creeping section is $\beta=\pi$. Figure 4 shows the path deduced from the other first-order diffractive contribution $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}$ given by Eq. (28) with $t=\sqrt{d^2-4a^2}$ and $\beta=2\pi-2\arccos(2a/d)$.

Figures 5, 6, 7, and 8 display the periodic orbits, respectively, deduced from the composite contributions $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2}$, $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2}$, $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},3}$, and $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},3}$ given by Eqs. (68), (69), (93), and (94). These periodic orbits present creeping sections around the cylinders

FIG. 3. First-order periodic orbit deduced from $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1}$.

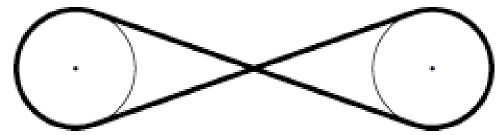

FIG. 4. First-order periodic orbit deduced from $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1}$.

FIG. 5. Second-order periodic orbit deduced from $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2}$.

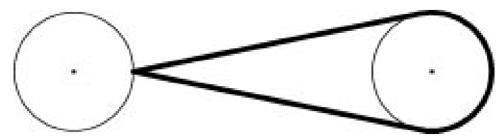

FIG. 6. Second-order periodic orbit deduced from $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2}$.

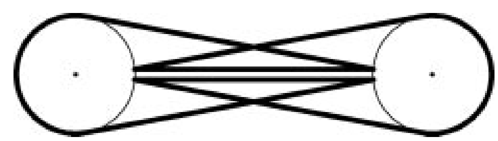

FIG. 7. Third-order periodic orbit deduced from $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},3}$.

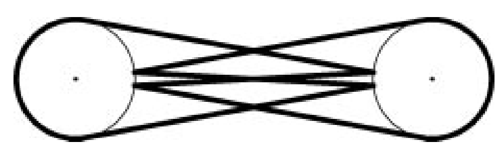

FIG. 8. Third-order periodic orbit deduced from $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},3}$.

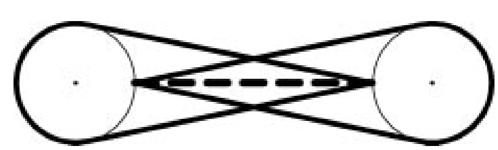

FIG. 9. Limit periodic orbit of the composite contributions $\mathbf{f}^{l}_{\mathrm{dif},q}$ ($l=$I or II).

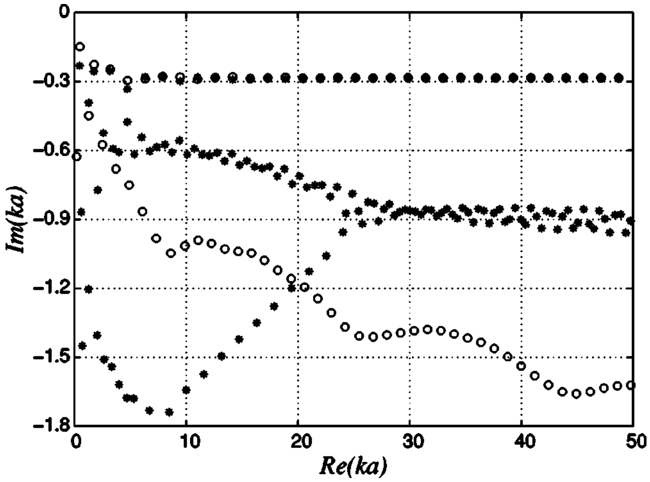

FIG. 10. Exact resonances (*) and first-order asymptotic resonances (○) in the complex ka plane. (Neumann BC, separation distance $d=6a$, A_1 representation.)

and a number of reflections growing up with the order q of the composite contribution. In the limit of high q value, the composite contributions go to the limit periodic path displayed in Fig. 9. Note that the periodic orbits deduced from the two first-order contributions (Figs. 2–6) already appear in Refs. [6,7].

It should be noted that all the composite contributions contain the denominator $[1-\exp(2i\pi\nu_n)]$. This term results from the additional creeping paths, which correspond to sections of length $2p\pi a$ $(p\geqslant 1)$ around one cylinder, in addition to the primary creeping paths (see also Refs. [6–9,24]).

C. Exact versus asymptotic scattering resonances

We present here a comparison between the exact resonances and the asymptotic resonances calculated from our semiclassical theory (see Sec. II). Neumann, Dirichlet, and impedance BC are investigated for the center-to-center distance $d=6a$ in the complex ka plane.

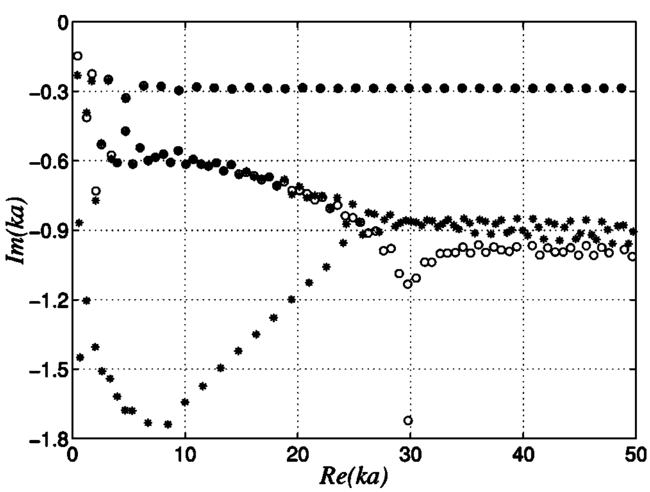

FIG. 11. Exact resonances (*) and second-order asymptotic resonances (○) in the complex ka plane. (Neumann BC, separation distance $d=6a$, A_1 representation.)

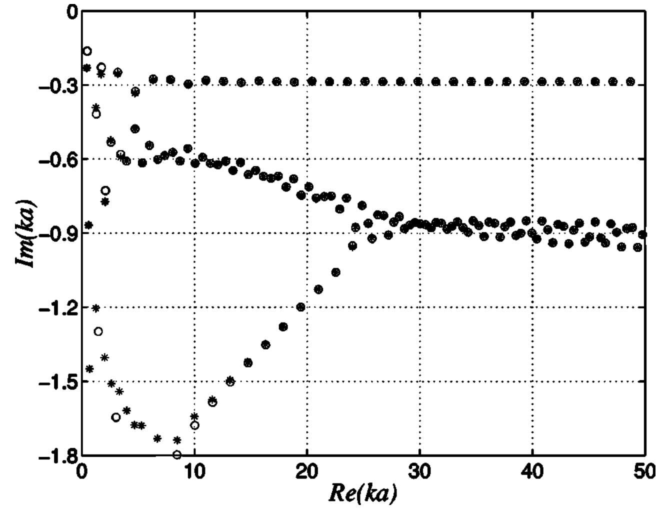

FIG. 12. Exact resonances (*) and third-order asymptotic resonances (○) in the complex ka plane. (Neumann BC, separation distance $d=6a$, A_1 representation.)

The exact scattering resonances are the zeros of $\det \mathbf{M}^{(\alpha)}$ with $\alpha=A_1,A_2,B_1,B_2$. Exact numerical calculations have been performed by replacing the infinite matrices $\mathbf{M}^{(\alpha)}$ by the associated matrices of rank N, with

$$N=\sup[8,(ka+4(ka)^{1/3}+1)]. \tag{131}$$

The above truncation order N has been chosen from the numerical discussions of Young and Bertrand [27] and Nüssenzveig [28], and it has been numerically tested. The scattering resonances have been determined in the restricted domain $0\leqslant \mathrm{Re}(ka)\leqslant 50$ and $-1.8\leqslant \mathrm{Im}(ka)\leqslant 0$ using the "argument principle" [29].

The asymptotic formulas obtained in Sec. II are inserted in the truncated cumulant expansion (10) in order to calculate the asymptotic resonances. The function $\mathcal{S}_\nu$ and the reflection coefficient $R(\nu,ka)$ involved in the asymptotic calculus are, respectively, given by Eqs. (7), (8), and (34)–(36)

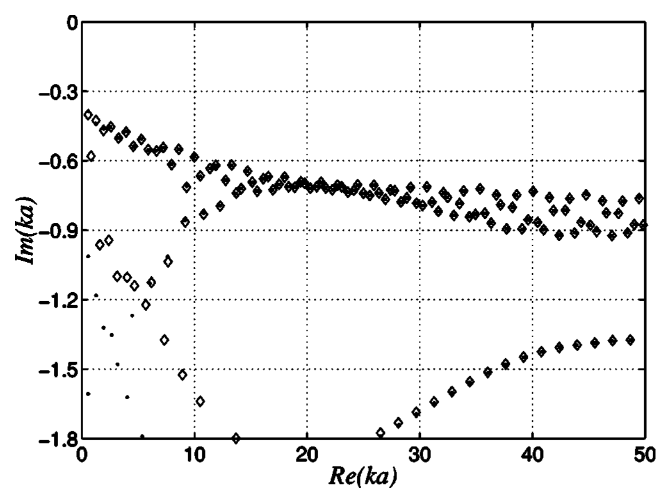

FIG. 13. Exact resonances (·) and third-order asymptotic resonances (◇) in the complex ka plane. (Neumann BC, separation distance $d=6a$, A_2 representation.)

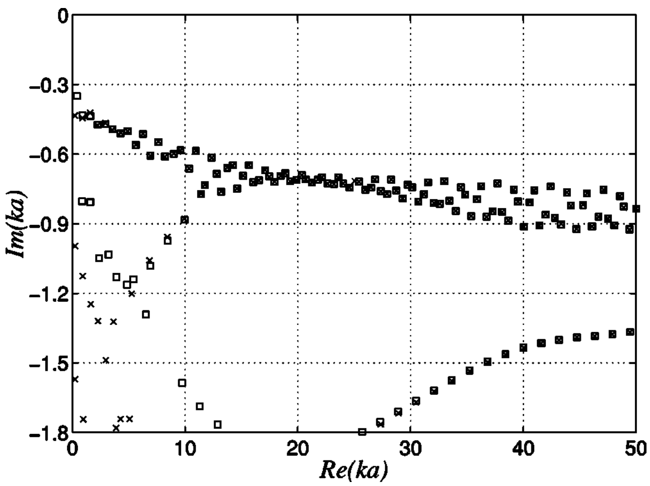

FIG. 14. Exact resonances (×) and third-order asymptotic resonances (□) in the complex ka plane. (Neumann BC, separation distance $d=6a$, B_1 representation.)

according to the BC. The poles ν_n and the residues r_{ν_n} of the $\mathcal{S}_\nu$ function are given in Appendix D. We only take into account the first pole of the $\mathcal{S}_\nu$ function in the asymptotic calculus. We can note that a comparison of the exact and semiclassical resonances in the Dirichlet case has been performed in Refs. [9,10] using "the periodic orbit theory of diffraction," in a more restricted frequency domain. Similar comparisons have been performed in Refs. [8,11] for the three-disk system.

1. Neumann boundary condition

The exact resonances are compared to resonances obtained with our semiclassical approach for the first three cumulants in the case of Neumann BC. At the first, second, and third truncation orders, the expansion of $\det \mathbf{M}$ reads

$$\det \mathbf{M}_{(1)} \simeq Q_0(\mathbf{A}) + Q_1(\mathbf{A}), \tag{132}$$

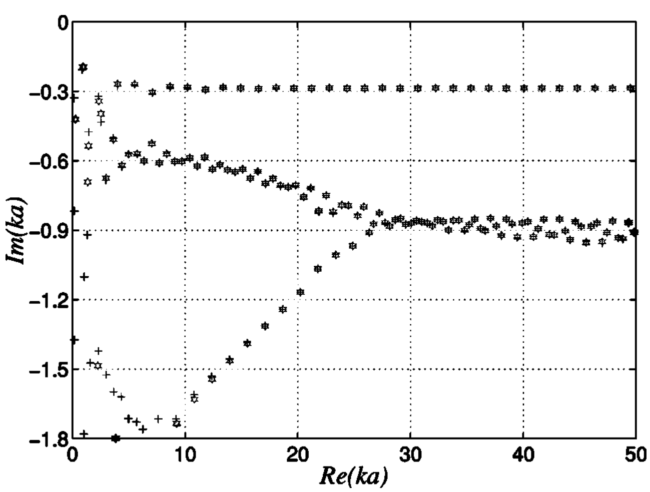

FIG. 15. Exact resonances (+) and third-order asymptotic resonances (⋆) in the complex ka plane. (Neumann BC, separation distance $d=6a$, B_2 representation.)

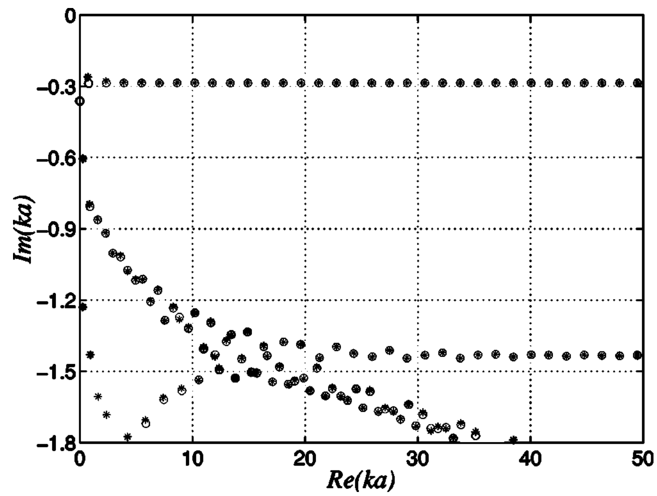

FIG. 16. Exact resonances (*) and third-order asymptotic resonances (○) in the complex ka plane. (Dirichlet BC, separation distance $d=6a$, A_1 representation.)

$$\det \mathbf{M}_{(2)} \simeq Q_0(\mathbf{A}) + Q_1(\mathbf{A}) + Q_2(\mathbf{A}), \tag{133}$$

$$\det \mathbf{M}_{(3)} \simeq Q_0(\mathbf{A}) + Q_1(\mathbf{A}) + Q_2(\mathbf{A}) + Q_3(\mathbf{A}), \tag{134}$$

where $Q_0(\mathbf{A})=1$ and $Q_1(\mathbf{A})$, $Q_2(\mathbf{A})$, and $Q_3(\mathbf{A})$ are, respectively, given by Eqs. (39), (73), and (98).

Figure 10 displays the comparison between exact and first-order asymptotic resonances [the zeros of Eq. (132)]. We observe a good agreement for the resonances lying on the line close to the real ka axis. They are associated with the first-order geometrical contribution $\mathbf{f}_{g,1}$ presented in Fig. 2. The first-order approximation provides a second asymptotic line whose resonances do not match the exact ones. They are associated with the purely diffractive contribution $\mathbf{f}_{\mathrm{dif},1}$ plotted in Figs. 3 and 4. It should also be noted that the second asymptotic line is located deeper inside the ka plane and contains fewer resonances than the exact second line. The first approximation of $\det \mathbf{M}$ does not provide the complete location of exact resonances in the studied region. We must therefore take into account the second cumulant.

Figure 11 displays a comparison between exact and second-order asymptotic resonances [the zeros of Eq. (133)]. The first asymptotic line still matches the exact data. The second resonances line is well approximated up to $\mathrm{Re}(ka) \simeq 25$. The corresponding asymptotic resonances are associated with the diffractive contributions $\mathbf{f}_{\mathrm{dif},1}$, $\mathbf{f}_{\mathrm{dif},2}$ (Figs. 3–6), and with the second-order geometrical contribution $\mathbf{f}_{g,2}$ (Fig. 2). A third exact line is not displayed by the second-order expansion of $\det \mathbf{M}$. We therefore take into account the third-order cumulant.

Figure 12 displays comparison between exact and third-order asymptotic resonances [the zeros of Eq. (134)]. A very good agreement is obtained in the whole studied domain. Nevertheless, a weak discrepancy is observed in the region $\mathrm{Re}(ka) \lesssim 8$ and $\mathrm{Im}(ka) \lesssim -1.2$ where the asymptotic expansions used are not very efficient. Moreover, the second pole of the $\mathcal{S}_\nu$ function should be taken into account. The third line, coming from $\mathrm{Re}(ka) \simeq 8$, $\mathrm{Im}(ka) \simeq -1.7$, and joining

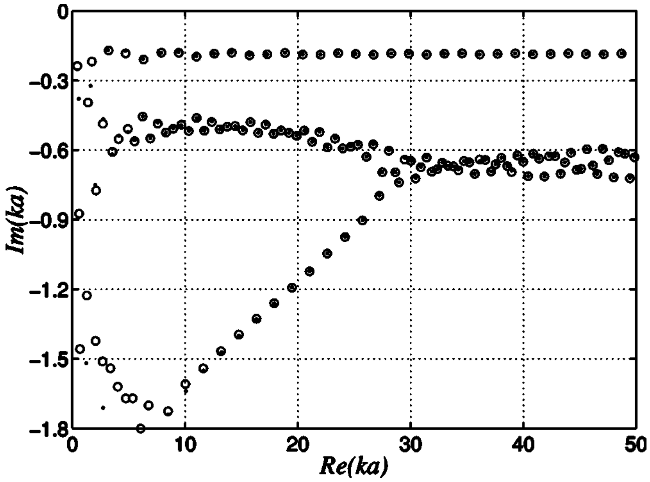

FIG. 17. Exact resonances (○) and third-order asymptotic resonances (·) in the complex ka plane. (Impedance BC, $\zeta=-5$, separation distance $d=6a$, A_1 representation.)

the second line near $\mathrm{Re}(ka)\simeq 25$, $\mathrm{Im}(ka)\simeq -0.9$, is associated with the third-order geometrical contribution $\mathbf{f}_{g,3}$ (Fig. 2) and with the diffractive contribution $\mathbf{f}_{\mathrm{dif},3}$ (Figs. 7 and 8).

Similar results are obtained for the three other irreducible representations A_2, B_1, B_2 in the case of the Neumann BC (see Figs. 13–15). The third-order asymptotic resonances match the exact ones except in the small region of high negative ka imaginary part and low ka real part.

2. Dirichlet and impedance boundary conditions

The exact resonances are compared to those obtained with our semiclassical approach for the third-order expansion of $\det \mathbf{M}$ (134) in the cases of Dirichlet and impedance BC. Figures 16 and 17, respectively display the results for Dirichlet and for impedance BC with the reduced impedance $\zeta=-5$. A good agreement is observed. For the Dirichlet case, Fig. 16 can be compared with Fig. 2 of Ref. [9].

D. Additional periodic orbits in the case of a particular impedance

As mentioned by Keller and Karal [30], in order for the surface to support a surface wave, it is necessary that ζ satisfy the conditions

$$\mathrm{Re}(\zeta)\geqslant 0,\mathrm{Im}(\zeta)>0. \tag{135}$$

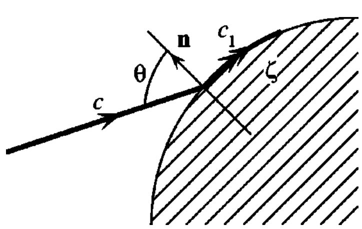

FIG. 18. Excitation of a surface wave on a circular cylinder. The angle θ between the incident complex ray and the exterior normal $\mathbf{n}$ is defined by Eq. (138).

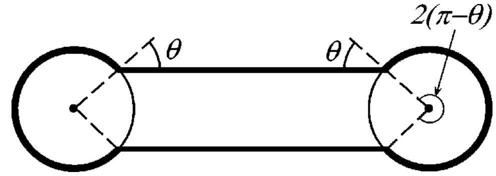

FIG. 19. First-order periodic orbit deduced from $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1S}$.

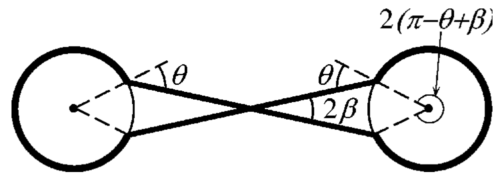

FIG. 20. First-order periodic orbit deduced from $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1S}$.

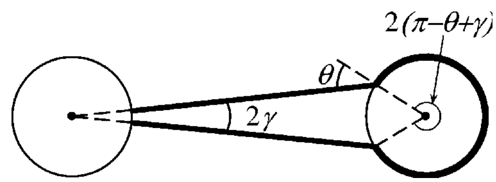

FIG. 21. Second-order periodic orbit deduced from $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2S}$.

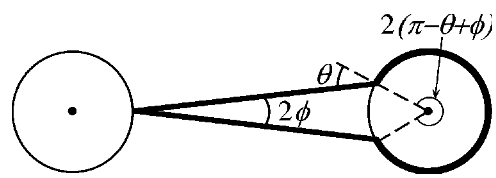

FIG. 22. Second-order periodic orbit deduced from $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2S}$.

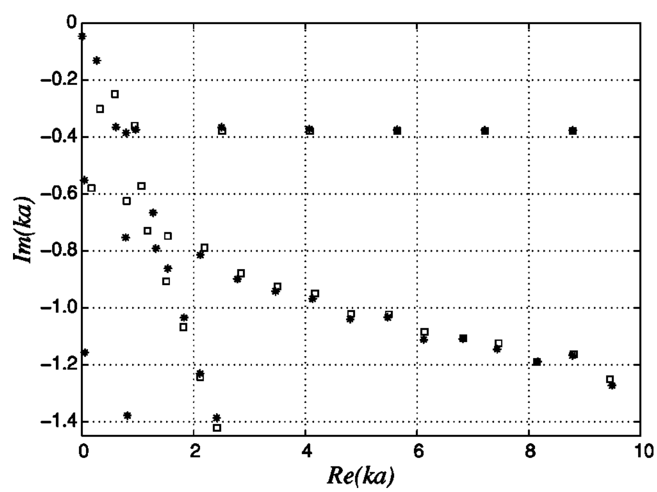

FIG. 23. Exact resonances (*) and second-order asymptotic resonances (□) in the complex ka plane. (Impedance BC, $\zeta=0.2+0.3i$, separation distance $d=6a$, A_1 representation.)

In this particular case, a new pole ν_S appears on the right-hand side of ka ($|\nu_S|>ka$) in the first quadrant of the complex ν plane. In order to take into account the contributions due to this supplementary pole ν_S, it suffices to make the substitution

$$\sum_{\nu_n} \to \sum_{\nu_n+\nu_S} \tag{136}$$

that corresponds to

$$\mathbf{f}^{l}_{\mathrm{dif},q} \to \mathbf{f}^{l}_{\mathrm{dif},q} + \mathbf{f}^{l}_{\mathrm{dif},qS}, \quad \text{for } q \geqslant 1 \text{ with } l=\mathrm{I},\mathrm{II} \tag{137}$$

in the generalized formulas given previously for the composite contributions (see Sec. III). The characteristic impedance ζ is related to the complex angle θ, the propagation velocity c in the medium, and the phase velocity c_1 of the wave along the cylinder surface (see Fig. 18) by

$$\cos\theta = -\zeta^{-1}, \sin\theta = \sqrt{1-\zeta^{-2}} = \frac{c}{c_1}. \tag{138}$$

Using the appropriate Debye asymptotic expansions and the approximation

$$\nu_S \sim ka \sin\theta, \tag{139}$$

the residue r_{ν_S} at the pole ν_S reads

$$r_{\nu_S} \sim -2ika \frac{\cos^2\theta}{\sin\theta} \exp(-2ika\cos\theta) \exp[i\nu_S(\pi-2\theta)]. \tag{140}$$

Inserting Eqs. (139), and (140) in Eqs. (24) and (25), taking into account the substitution (136), and following the method described in Sec. II A 1, we then obtain two new first-order composite contributions associated with the pole ν_S,

$$\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1S} = -2ka \frac{\cos^2\theta}{\sin\theta} \sqrt{\frac{2\pi}{ikd}} \frac{\exp[2i\nu_S(\pi-\theta)]}{1-\exp[2i\pi\nu_S]} \times \exp[ik(d-2a\cos\theta)], \tag{141}$$

$$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1S} = -2ka \frac{\cos^2\theta}{\sin\theta} \sqrt{\frac{2\pi}{ik\sqrt{d^2-(2a\sin\theta)^2}}} \times \frac{\exp[2i\nu_S(\pi-\theta+\beta)]}{1-\exp[2i\pi\nu_S]} \times \exp\{ik[\sqrt{d^2-(2a\sin\theta)^2}-2a\cos\theta]\}$$

$$\text{with } \beta = \arcsin\frac{2a\sin\theta}{d}. \tag{142}$$

Similarly, inserting Eqs. (139) and (140) in Eqs. (66) and (67) and taking into account the substitution (136), we then obtain two new second-order composite contributions associated with the pole ν_S,

$$\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2S} = -2ka \frac{\cos^2\theta}{\sin\theta} \sqrt{\frac{\pi a}{ik\sqrt{d^2-(a\sin\theta)^2}[\sqrt{d^2-(a\sin\theta)^2}-a]}} R(0,ka) \frac{\exp[2i\nu_S(\pi-\theta+\gamma)]}{1-\exp[2i\pi\nu_S]} \times \exp\{2ik[\sqrt{d^2-(a\sin\theta)^2}-a(1+\cos\theta)]\}, \quad \text{with } \gamma = \arcsin\frac{a\sin\theta}{d}, \tag{143}$$

$$\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2S} = -2ka \frac{\cos^2\theta}{\sin\theta} \sqrt{\frac{\pi a}{ikd\sqrt{(d-a)^2-(a\sin\theta)^2}}} R\left(\nu_S\frac{a}{d-a},ka\right) \frac{\exp[2i\nu_S(\pi-\theta+\phi)]}{1-\exp[2i\pi\nu_S]} \times \exp\{2ik[\sqrt{(d-a)^2-(a\sin\theta)^2}-a\cos\theta]\}, \quad \text{with } \phi = \arcsin\frac{a\sin\theta}{d-a}. \tag{144}$$

The complex periodic orbits associated with these four composite contributions (141)–(144) are displayed in Figs. 19–22.

All the composite periodic orbits obtained here in the case of impenetrable objects are associated with external surface waves. We can note that the periodic orbits involving an excitation angle also exist in the case of scattering by penetrable cylinders. They correspond then to internal surface waves associated with Wait poles (mixed or fluid BC) [31], Rayleigh, and Whispering Gallery poles (scattering by solid elastic cylinders immersed in a fluid) [32].

Figure 23 displays a comparison between exact and second-order asymptotic resonances [the zeros of Eq. (133) including the substitution (137)] in the case of the impedance

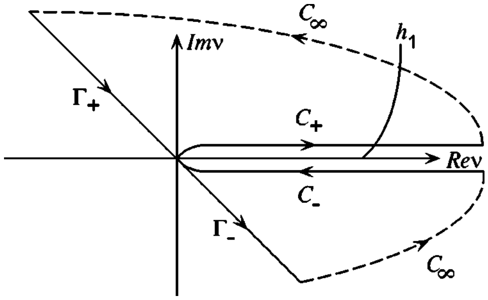

FIG. 24. Contour deformation in the complex ν plane where $C=C_++C_-$ and $\Gamma=\Gamma_++\Gamma_-$.

BC with $\zeta=0.2+0.3i$. The two resonances lines located in the region $-1.2\leqslant\mathrm{Im}(ka)\leqslant 0$ have been previously interpreted and they are associated with the contributions $\mathbf{f}_{g,1}$, $\mathbf{f}_{g,2}$, $\mathbf{f}_{\mathrm{dif},1}$, and $\mathbf{f}_{\mathrm{dif},2}$ (see Figs. 2–6). A new resonances line, located deeper inside the ka plane and extended to $\mathrm{Re}(ka)\simeq 2.5$ and $\mathrm{Im}(ka)\simeq -1.4$, is associated with the new composite contributions $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},1S}$, $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},1S}$, $\mathbf{f}^{\mathrm{I}}_{\mathrm{dif},2S}$, and $\mathbf{f}^{\mathrm{II}}_{\mathrm{dif},2S}$ (Figs. 19–22). A good agreement is obtained in the studied domain, except in the region $\mathrm{Re}(ka)\lesssim 2$ where the asymptotic expansions are not very efficient.

V. CONCLUSION

In this paper, we have entirely solved the two-cylinders scattering problem for Dirichlet, Neumann, and impedance boundary conditions. All the scattering resonances for the first three terms of the cumulant expansion have been extracted. Generalized formulas have been derived at any truncation order for all the contributions that are purely geometrical or composite. We have then obtained a semiclassical approximation of the characteristic determinant for each irreducible representation of the $\mathcal{C}_{2v}$ symmetry group. All the contributions have been interpreted in terms of periodic orbits. Moreover, our semiclassical approach provides scattering resonances in excellent agreement with the exact results. We can then postulate that, in the scalar case, scattering of waves and particles by two identical, impenetrable cylinders is a canonical problem.

The semiclassical formalism developed in this paper is actually extended to the scattering problem by two penetrable cylinders (scattering of a transverse electric wave by dielectric cylinders in electromagnetism, fluid BC in acoustics, or mixed BC in quantum physics). In this case, the poles associated with the internal waves (or with the interior potential) must be taken into account. The main difficulty comes from the slow convergence of the Debye series expansion introduced to evaluate the geometrical contributions (each incident ray gives rise to an infinite series of multiple internal reflections). It should be noted that multiple scattering problems by penetrable objects have never been semiclassically treated.

ACKNOWLEDGMENT

The authors would like to thank Antoine Folacci for useful discussions.

APPENDIX A: CONTOUR DEFORMATION IN THE COMPLEX PLANE

We consider the integral

$$F(\nu,ka)=\int_C \frac{\mathcal{S}_\nu-1}{\sin(\pi\nu)}H(\nu,ka)\,d\nu, \tag{A1}$$

where the contour C encircles the real positive axis in the clockwise sense (see Fig. 24). The function $H(\nu,ka)$ must satisfy the symmetry property

$$H(-\nu,ka)=H(\nu,ka), \tag{A2}$$

and the function $\mathcal{S}_\nu$ is given by Eq. (6), Eq. (7), or Eq. (8) according to the boundary condition. Let us write

$$C=C_++C_-, \tag{A3}$$

and introduce the expansions

$$\frac{1}{\sin(\pi\nu)}=-2i\sum_{p=0}^{+\infty} e^{i\pi\nu(2p+1)} \quad \text{for } \mathrm{Im}(\nu)>0, \tag{A4}$$

$$\frac{1}{\sin(\pi\nu)}=2i\sum_{p=0}^{+\infty} e^{-i\pi\nu(2p+1)} \quad \text{for } \mathrm{Im}(\nu)<0, \tag{A5}$$

respectively, on C_+ and C_-, so

$$\begin{aligned}F(\nu,ka)=&-2i\sum_{p=0}^{+\infty}\int_{C_+}(\mathcal{S}_\nu-1)e^{i\nu(2p+1)\pi}H(\nu,ka)\,d\nu\\&+2i\sum_{p=0}^{+\infty}\int_{C_-}(\mathcal{S}_\nu-1)e^{-i\nu(2p+1)\pi}H(\nu,ka)\,d\nu.\end{aligned} \tag{A6}$$

The residue theorem permits one to write

$$\int_{C_+}\mathcal{I}_1 d\nu+\int_{C_\infty}\mathcal{I}_1 d\nu+\int_{\Gamma_+}\mathcal{I}_1 d\nu=2i\pi \text{ residue } (\mathcal{I}_1)_{|\nu=\nu_n}, \tag{A7}$$

$$\int_{C_-}\mathcal{I}_2 d\nu+\int_{\Gamma_-}\mathcal{I}_2 d\nu+\int_{C_\infty}\mathcal{I}_2 d\nu=0, \tag{A8}$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$, respectively, denote the integrands of the first integral and of the second one in Eq. (A6). ν_n are the poles of the $\mathcal{S}_\nu$ function in the complex ν plane. They are symmetrically distributed with respect to the origin, so we have to only consider them in the right half-plane located close to the curve h_1. This curve h_1 cuts the real ν axis at $\nu=ka$, at an angle of $\pi/3$. The tangent to this curve tends to the vertical direction for $|\nu|\to\infty$ (see Fig. 24). This poles distribution is valid in the three cases treated in the paper, i.e., for the Dirichlet, Neumann, and impedance boundary conditions. The dominant behavior in Eq. (A6) is dictated by $(\mathcal{S}_\nu-1)e^{i\pi\nu}$ in the first integral and by $(\mathcal{S}_\nu-1)e^{-i\pi\nu}$ in the second one. Following the methods of Nüssenzveig [33], we obtain

$$\int_{C_\infty}(\mathcal{S}_\nu-1)e^{i\pi\nu}H(\nu,ka)d\nu\to 0 \quad \text{for Im}(\nu)>0, \tag{A9}$$

$$\int_{C_\infty}(\mathcal{S}_\nu-1)e^{-i\pi\nu}H(\nu,ka)d\nu\to 0 \quad \text{for Im}(\nu)<0, \tag{A10}$$

as $|\nu|\to\infty$, so using Eqs. (A7) and (A8), Eq. (A6) reads

$$\begin{aligned}F(\nu,ka)=&2i\sum_{p=0}^{+\infty}\int_{\Gamma_+}(\mathcal{S}_\nu-1)e^{i\pi\nu(2p+1)}H(\nu,ka)d\nu\\&+4\pi\sum_{p=0}^{+\infty}\sum_{\nu_n}r_{\nu_n}e^{i\pi\nu_n(2p+1)}H(\nu_n,ka)\\&-2i\sum_{p=0}^{+\infty}\int_{\Gamma_-}(\mathcal{S}_\nu-1)e^{-i\pi\nu(2p+1)}H(\nu,ka)d\nu,\end{aligned} \tag{A11}$$

where r_{ν_n} is the residue of the $\mathcal{S}_\nu$ function at the poles $\nu=\nu_n$. We change the sign of ν in the second integral over the contour Γ_-. Using the symmetry properties (A2) and

$$\mathcal{S}_{-\nu}=e^{-2i\pi\nu}\mathcal{S}_\nu, \tag{A12}$$

Eq. (A11) becomes

$$\begin{aligned}F(\nu,ka)=&-2i\int_{\Gamma_+}\mathcal{S}_\nu e^{-i\pi\nu}H(\nu,ka)d\nu\\&+4\pi\sum_{p=0}^{+\infty}\sum_{\nu_n}r_{\nu_n}e^{i\pi\nu_n(2p+1)}H(\nu_n,ka).\end{aligned} \tag{A13}$$

Finally, after a last contour modification in Eq. (A13), an integral of the form (A1) with the symmetry property (A2) is written as a sum of a geometrical contribution $F_g(\nu,ka)$ and a residue-series contribution $F_d(\nu,ka)$,

$$F(\nu,ka)=F_g(\nu,ka)+F_d(\nu,ka), \tag{A14}$$

with

$$F_g(\nu,ka)=-i\int_\Gamma \mathcal{S}_\nu e^{-i\pi\nu}H(\nu,ka)d\nu, \tag{A15}$$

$$F_d(\nu,ka)=4\pi\sum_{\nu_n}r_{\nu_n}\frac{e^{i\pi\nu_n}}{1-e^{2i\pi\nu_n}}H(\nu_n,ka). \tag{A16}$$

The contour $\Gamma=\Gamma_++\Gamma_-$ displayed in Fig. 24 is chosen in order to apply the method of steepest descent.

APPENDIX B: THE PURELY GEOMETRICAL CONTRIBUTIONS OF THE SECOND ORDER

We consider here the purely geometrical contribution, given by Eq. (62), extracted from the second cumulant,

$$\mathbf{f}_{g,2}=\frac{1}{16}\int_{\Gamma_1}\mathcal{S}_{\nu_1}e^{-i\pi\nu_1}\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}e^{-i\pi\nu_2}X(\nu_1,\nu_2)d\nu_2\right]d\nu_1.$$

Replacing $X(\nu_1,\nu_2)$ by relation (48), two different terms have to be evaluated

$$\mathbf{f}_{g,2}=\mathbf{f}^{\mathrm{I}}_{g,2}+\mathbf{f}^{\mathrm{II}}_{g,2}, \tag{B1}$$

with

$$\begin{aligned}\mathbf{f}^{\mathrm{I}}_{g,2}=&\frac{1}{8}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}e^{-i\pi\nu_2}\right.\\&\left.\times H^{(1)}_{\nu_2+\nu_1}(kd)H^{(1)}_{\nu_1-\nu_2}(kd)d\nu_2\right]d\nu_1,\end{aligned} \tag{B2}$$

$$\mathbf{f}^{\mathrm{II}}_{g,2}=\frac{1}{8}\int_{\Gamma_1}\mathcal{S}_{\nu_1}\left[\int_{\Gamma_2}\mathcal{S}_{\nu_2}[H^{(1)}_{\nu_1+\nu_2}(kd)]^2d\nu_2\right]d\nu_1. \tag{B3}$$

Equations (B3) and (B3) are composed by an integration with respect to the variable ν_1, which contains another integration with respect to the variable ν_2. Thus, we successively twice apply the method of steepest descent: the first time to perform the ν_2 integration and the second one to perform the ν_1 integration. We present the detailed resolution of $\mathbf{f}^{\mathrm{I}}_{g,2}$ and the main results concerning $\mathbf{f}^{\mathrm{II}}_{g,2}$. The notations $x=ka$ and $y=kd$ are used.

1. Evaluation of $\mathbf{f}^{\mathrm{I}}_{g,2}$

a. Integration with respect to ν_2

In Eq. (B2), we define the inner integral

$$F_{\nu_2}=\int_{\Gamma_2}\mathcal{S}_{\nu_2}e^{-i\pi\nu_2}H^{(1)}_{\nu_2+\nu_1}(kd)H^{(1)}_{\nu_1-\nu_2}(kd)d\nu_2. \tag{B4}$$

Using the Debye asymptotic expansion for the Hankel functions (32) and for $\mathcal{S}_{\nu_2}$ (33), F_{ν_2} is approximated by

$$F_{\nu_2}\sim-\frac{2}{\pi}\int_{\Gamma_2}\frac{R(\nu_2,ka)}{[y^2-(\nu_1+\nu_2)^2]^{1/4}[y^2-(\nu_1-\nu_2)^2]^{1/4}}\exp\left[-2i\sqrt{x^2-\nu_2^2}+2i\nu_2\arccos\frac{\nu_2}{x}-i\pi\nu_2\right]$$
$$\times\exp\left[i\sqrt{y^2-(\nu_1+\nu_2)^2}-i(\nu_1+\nu_2)\arccos\frac{\nu_1+\nu_2}{y}+i\sqrt{y^2-(\nu_1-\nu_2)^2}-i(\nu_1-\nu_2)\arccos\frac{\nu_1-\nu_2}{y}\right]d\nu_2. \quad \text{(B5)}$$

We define

$$f(\nu_2)=-\frac{2}{\pi}\frac{R(\nu_2,ka)}{[y^2-(\nu_1+\nu_2)^2]^{1/4}[y^2-(\nu_1-\nu_2)^2]^{1/4}}, \quad \text{(B6)}$$

$$g(\nu_2)=\frac{i}{x}\left[-2\sqrt{x^2-\nu_2^2}+2\nu_2\arccos\frac{\nu_2}{x}-\pi\nu_2+\sqrt{y^2-(\nu_1+\nu_2)^2}-(\nu_1+\nu_2)\arccos\frac{\nu_1+\nu_2}{y}\right.$$
$$\left.+\sqrt{y^2-(\nu_1-\nu_2)^2}-(\nu_1-\nu_2)\arccos\frac{\nu_1-\nu_2}{y}\right]. \quad \text{(B7)}$$

We calculate the first and second derivatives of $g(\nu_2)$ with respect to ν_2 (labeled by $\bullet$ and $\bullet\bullet$),

$$g^{\bullet}(\nu_2)=\frac{\partial g(\nu_2)}{\partial\nu_2}=\frac{i}{x}\left[2\arccos\frac{\nu_2}{x}-\arccos\frac{\nu_1+\nu_2}{y}+\arccos\frac{\nu_1-\nu_2}{y}-\pi\right], \quad \text{(B8)}$$

$$g^{\bullet\bullet}(\nu_2)=\frac{\partial^2 g(\nu_2)}{\partial\nu_2^2}=\frac{i}{x}\left[-\frac{2}{\sqrt{x^2-\nu_2^2}}+\frac{1}{\sqrt{y^2-(\nu_1+\nu_2)^2}}+\frac{1}{\sqrt{y^2-(\nu_1-\nu_2)^2}}\right]. \quad \text{(B9)}$$

The saddle point $\overline{\nu_2}$ is determined by

$$g^{\bullet}(\overline{\nu_2})=0\Leftrightarrow\overline{\nu_2}=0. \quad \text{(B10)}$$

F_{ν_2} is approximated by (see Refs. [22,23])

$$F_{\nu_2}\simeq\sqrt{-2\pi}f(\overline{\nu_2})\frac{\exp[xg(\overline{\nu_2})]}{[xg^{\bullet\bullet}(\overline{\nu_2})]^{1/2}}, \quad \text{(B11)}$$

so the integration with respect to ν_2 gives the result

$$F_{\nu_2}=-2R(0,ka)\sqrt{\frac{x}{i\pi(\sqrt{y^2-\nu_1^2}-x)\sqrt{y^2-\nu_1^2}}}$$
$$\times\exp\left[2i\sqrt{y^2-\nu_1^2}-2i\nu_1\arccos\frac{\nu_1}{y}-2ix\right]. \quad \text{(B12)}$$

b. Integration with respect to ν_1

Equation (B2) can be written as

$$\mathbf{f}^{\mathrm{I}}_{g,2}=\frac{1}{8}\int_{\Gamma_1}\mathcal{S}_{\nu_1}F_{\nu_2}d\nu_1 \quad \text{(B13)}$$

and we replace F_{ν_2} by Eq. (B12) and $\mathcal{S}_{\nu_2}$ by its Debye asymptotic expansion (33), so

$$\mathbf{f}^{\mathrm{I}}_{g,2}\sim\frac{1}{4}\int_{\Gamma_1}R(0,ka)R(\nu_1,ka)$$
$$\times\sqrt{\frac{ix}{\pi(\sqrt{y^2-\nu_1^2}-x)\sqrt{y^2-\nu_1^2}}}$$
$$\times\exp\left[-2ix-2i\sqrt{x^2-\nu_1^2}+2i\nu_1\arccos\frac{\nu_1}{x}\right]$$
$$\times\exp\left[2i\sqrt{y^2-\nu_1^2}-2i\nu_1\arccos\frac{\nu_1}{y}\right]d\nu_1. \quad \text{(B14)}$$

We define

$$f(\nu_1)=\frac{1}{4}R(0,ka)R(\nu_1,ka)\times\sqrt{\frac{ix}{\pi(\sqrt{y^2-\nu_1^2}-x)\sqrt{y^2-\nu_1^2}}},\tag{B15}$$

$$g(\nu_1)=\frac{2i}{x}\left[-x-\sqrt{x^2-\nu_1^2}+\nu_1\arccos\frac{\nu_1}{x}+\sqrt{y^2-\nu_1^2}-\nu_1\arccos\frac{\nu_1}{y}\right].\tag{B16}$$

We calculate the first and second derivatives of $g(\nu_1)$ with respect to ν_1 labeled by $'$ and $''$. The saddle point is determined by

$$g'(\overline{\nu_1})=0\Leftrightarrow\overline{\nu_1}=0.\tag{B17}$$

$\mathbf{f}^{\mathrm{I}}_{g,2}$ is approximated by

$$\mathbf{f}^{\mathrm{I}}_{g,2}\simeq\sqrt{-2\pi}f(\overline{\nu_1})\frac{\exp[xg(\overline{\nu_1})]}{[xg''(\overline{\nu_1})]^{1/2}}.\tag{B18}$$

Finally the first purely geometrical contribution of the second order is

$$\mathbf{f}^{\mathrm{I}}_{g,2}=\tfrac{1}{2}R(0,ka)^2\frac{a}{2(d-a)}\exp[2ik(d-2a)].\tag{B19}$$

2. Evaluation of $\mathbf{f}^{\mathrm{II}}_{g,2}$

The second geometrical contribution is given by Eq. (B3). We apply the same procedure as for $\mathbf{f}^{\mathrm{I}}_{g,2}$. We call

$$F_{\nu_2}=\int_{\Gamma_2}\mathcal{S}_{\nu_2}[H^{(1)}_{\nu_1+\nu_2}(kd)]^2d\nu_2,\tag{B20}$$

and the integration with respect to ν_2 gives us

$$F_{\nu_2}=-\frac{2}{\sqrt{i\pi}}R(\overline{\nu_2},ka)\times\frac{1}{\sqrt{y^2-(\nu_1+\overline{\nu_2})^2}\sqrt{\frac{1}{\sqrt{x^2-\overline{\nu_2}^2}}-\frac{1}{\sqrt{y^2-(\nu_1+\overline{\nu_2})^2}}}}\times\exp\left[-2i\sqrt{x^2-\overline{\nu_2}^2}+2i\sqrt{y^2-(\nu_1+\overline{\nu_2})^2}\right]\exp\left[-2i\nu_1\arccos\frac{\nu_1+\overline{\nu_2}}{y}\right],\tag{B21}$$

with

$$\overline{\nu_2}=\nu_1\frac{a}{d-a}.\tag{B22}$$

Afterwards the integration with respect to ν_1 of the expression

$$\mathbf{f}^{\mathrm{II}}_{g,2}=\frac{1}{8}\int_{\Gamma_1}\mathcal{S}_{\nu_1}F_{\nu_2}d\nu_1\tag{B23}$$

is carried out taking into account the ν_1 dependence of ν_2. Finally, the second purely geometrical contribution of the second order reads

$$\mathbf{f}^{\mathrm{II}}_{g,2}=\tfrac{1}{2}R(0,ka)^2\frac{a}{2\sqrt{d(d-2a)}}\exp[2ik(d-2a)].\tag{B24}$$

APPENDIX C: MULTIPLE INTEGRALS

In this appendix, the real multiple integrals formula, given by Felsen [22], is extended to the case of integration in the complex plane. Let us consider the multiple integration with respect to q complex variables $(\nu_1,\nu_2,\ldots,\nu_q)$ for $q\geqslant 1$,

$$I_q=\int\int\cdots\int f(\nu_1,\nu_2,\ldots,\nu_q)\times\exp[xg(\nu_1,\nu_2,\ldots,\nu_q)]d\nu_1d\nu_2\cdots d\nu_q,\tag{C1}$$

which can be approximated by

$$I_q\simeq\left(\frac{2\pi}{x}\right)^{q/2}f(\overline{\nu_1},\overline{\nu_2},\ldots,\overline{\nu_q})\frac{\exp[xg(\overline{\nu_1},\overline{\nu_2},\ldots,\overline{\nu_q})]}{[(-1)^qD_q]^{1/2}},\tag{C2}$$

where the saddle points $\overline{\nu_i}$ are the roots of

$$\frac{\partial g(\nu_1,\nu_2,\ldots,\nu_q)}{\partial\nu_i}\Big|_{(\nu_i=\overline{\nu_i})}=0.\tag{C3}$$

D_q is the following q determinant:

$$D_q=\begin{vmatrix} \frac{\partial^2 g}{\partial \nu_1^2} & \frac{\partial^2 g}{\partial \nu_1 \partial \nu_2} & 0 & \cdots & 0 & \frac{\partial^2 g}{\partial \nu_1 \partial \nu_q} \\ \frac{\partial^2 g}{\partial \nu_2 \partial \nu_1} & \frac{\partial^2 g}{\partial \nu_2^2} & \frac{\partial^2 g}{\partial \nu_2 \partial \nu_3} & \cdots & 0 & 0 \\ 0 & \frac{\partial^2 g}{\partial \nu_3 \partial \nu_2} & \frac{\partial^2 g}{\partial \nu_3^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{\partial^2 g}{\partial \nu_{q-1}^2} & \frac{\partial^2 g}{\partial \nu_{q-1} \partial \nu_q} \\ \frac{\partial^2 g}{\partial \nu_q \partial \nu_1} & 0 & 0 & \cdots & \frac{\partial^2 g}{\partial \nu_q \partial \nu_{q-1}} & \frac{\partial^2 g}{\partial \nu_q^2} \end{vmatrix} \tag{C4}$$

defined for $\nu_i=\overline{\nu_i}(i=1,2,3,\ldots,q)$. It should be noted that for $q=1$, Eq. (C2) is equivalent to the formula (B11) used in the case of simple integration.

The resolution of multiple integrals such as $\mathbf{f}_{g,2}^{\mathrm{II}}$ has been performed using the method of steepest descent successively on ν_2 and afterwards on ν_1 (see Appendix B). We apply here the multiple integral formula (C2) with $q=2$ to the contribution $\mathbf{f}_{g,2}^{\mathrm{II}}$ given by Eq. (B3). Using the Debye asymptotic expansions (32) and (33), $\mathbf{f}_{g,2}^{\mathrm{II}}$ reads

$$\mathbf{f}_{g,2}^{\mathrm{II}}\sim-\frac{1}{4}\int_{\Gamma_1}\int_{\Gamma_2}\frac{R(\nu_1,ka)R(\nu_2,ka)}{i\pi\sqrt{y^2-(\nu_1+\nu_2)^2}}\exp\left[-2i\sqrt{x^2-\nu_1^2}+2i\nu_1\arccos\frac{\nu_1}{x}-2i\sqrt{x^2-\nu_2^2}+2i\nu_2\arccos\frac{\nu_2}{x}\right]$$
$$\times\exp\left[2i\sqrt{y^2-(\nu_1+\nu_2)^2}-2i(\nu_1+\nu_2)\arccos\frac{\nu_1+\nu_2}{y}\right]d\nu_1 d\nu_2. \tag{C5}$$

We define

$$f(\nu_1,\nu_2)=-\frac{1}{4}\frac{R(\nu_1,ka)R(\nu_2,ka)}{i\pi\sqrt{y^2-(\nu_1+\nu_2)^2}}, \tag{C6}$$

$$g(\nu_1,\nu_2)=\frac{i}{x}\left[-2\sqrt{x^2-\nu_1^2}+2\nu_1\arccos\frac{\nu_1}{x}-2\sqrt{x^2-\nu_2^2}+2\nu_2\arccos\frac{\nu_2}{x}+2\sqrt{y^2-(\nu_1+\nu_2)^2}-2(\nu_1+\nu_2)\arccos\frac{\nu_1+\nu_2}{y}\right]. \tag{C7}$$

We evaluate the first and second derivatives of $g(\nu_1,\nu_2)$ with respect to ν_1 and ν_2,

$$\frac{\partial g}{\partial \nu_1}=\frac{2i}{x}\left[\arccos\frac{\nu_1}{x}-\arccos\frac{\nu_1+\nu_2}{y}\right], \tag{C8}$$

$$\frac{\partial g}{\partial \nu_2}=\frac{2i}{x}\left[\arccos\frac{\nu_2}{x}-\arccos\frac{\nu_1+\nu_2}{y}\right], \tag{C9}$$

$$\frac{\partial^2 g}{\partial \nu_1^2}=\frac{2i}{x}\left[-\frac{1}{\sqrt{x^2-\nu_1^2}}+\frac{1}{\sqrt{y^2-(\nu_1+\nu_2)^2}}\right], \tag{C10}$$

$$\frac{\partial^2 g}{\partial \nu_1\partial \nu_2}=\frac{\partial^2 g(\nu_1,\nu_2)}{\partial \nu_2\partial \nu_1}=\frac{2i}{x}\left[\frac{1}{\sqrt{y^2-(\nu_1+\nu_2)^2}}\right], \tag{C11}$$

$$\frac{\partial^2 g}{\partial \nu_2^2}=\frac{2i}{x}\left[-\frac{1}{\sqrt{x^2-\nu_2^2}}+\frac{1}{\sqrt{y^2-(\nu_1+\nu_2)^2}}\right], \tag{C12}$$

and $\overline{\nu_1}$, $\overline{\nu_2}$ are determined as follows:

$$\frac{\partial g}{\partial \nu_1}\Big|_{(\nu_1=\overline{\nu_1})}=0\Leftrightarrow\overline{\nu_1}=\nu_2\frac{a}{d-a}, \tag{C13}$$

$$\frac{\partial g}{\partial \nu_2}\Big|_{(\nu_2=\overline{\nu_2})}=0\Leftrightarrow\overline{\nu_2}=\nu_1\frac{a}{d-a}. \tag{C14}$$

We obtain

$$\overline{\nu_1}=\overline{\nu_2}=0 \tag{C15}$$

and

$$g(\overline{\nu_1},\overline{\nu_2})=\frac{2i}{x}(y-2x). \tag{C16}$$

The second-order determinant reads

$$D_2=\begin{vmatrix}\frac{\partial^2 g}{\partial\nu_1^2} & \frac{\partial^2 g}{\partial\nu_1\partial\nu_2}\\ \frac{\partial^2 g}{\partial\nu_2\partial\nu_1} & \frac{\partial^2 g}{\partial\nu_2^2}\end{vmatrix}_{(\overline{\nu_1},\overline{\nu_2})}=\left(\frac{2i}{x}\right)^2\left(\frac{y-2x}{xy}\right)\left(\frac{1}{x}\right). \tag{C17}$$

Finally, $\mathbf{f}^{\mathrm{II}}_{g,2}$ is given by formula (C2) with $q=2$,

$$\mathbf{f}^{\mathrm{II}}_{g,2}\simeq\frac{2\pi}{x}f(\overline{\nu_1},\overline{\nu_2})\frac{\exp[xg(\overline{\nu_1},\overline{\nu_2})]}{[D_2]^{1/2}}, \tag{C18}$$

therefore

$$\mathbf{f}^{\mathrm{II}}_{g,2}=\tfrac{1}{2}R(0,ka)^2\frac{a}{2\sqrt{d(d-2a)}}\exp[2ik(d-2a)]. \tag{C19}$$

The result is the same as the one obtained in Eq. (B24) of Appendix B by applying twice the method of steepest descent. Consequently, we consider that the formula (C2) can be applied for any truncation order q.

APPENDIX D: ASYMPTOTIC FORMULAS

In this appendix, we give the asymptotic formulas, used in Sec. V, for the poles ν_n and the residues r_{ν_n} of the $\mathcal{S}_\nu$ function (see, for example, Refs. [21,24,33]).

1. Dirichlet boundary conditions

The poles of the $\mathcal{S}_\nu$ function are the zeros of $\mathrm{H}^{(1)}_\nu(ka)$ and are given by

$$\nu_n=ka-\mu_n e^{i\pi/3}\left(\frac{ka}{2}\right)^{1/3}+\frac{\mu_n^2}{60}e^{2i\pi/3}\left(\frac{ka}{2}\right)^{-1/3}. \tag{D1}$$

The residue of the $\mathcal{S}_\nu$ function at the poles $\nu=\nu_n$ is approximated by

$$r_{\nu_n}\sim\frac{\exp(-i\pi/6)}{2\pi\,\mathrm{Ai}'(\mu_n)^2}\left(\frac{ka}{2}\right)^{1/3}, \tag{D2}$$

where μ_n $(n\in N^*)$ is the nth zero of the Airy function $\mathrm{Ai}(x)$.

2. Neumann boundary conditions

The poles of the $\mathcal{S}_\nu$ function are the zeros of $\mathrm{H}^{(1)'}_\nu(ka)$ and are given by

$$\nu_n=ka-\eta_n e^{i\pi/3}\left(\frac{ka}{2}\right)^{1/3}+\left(\frac{\eta_n^2}{60}-\frac{1}{10\eta_n}\right)e^{2i\pi/3}\left(\frac{ka}{2}\right)^{-1/3}. \tag{D3}$$

The residue of the $\mathcal{S}_\nu$ function at the poles $\nu=\nu_n$ is approximated by

$$r_{\nu_n}\sim-\frac{\exp(-i\pi/6)}{2\pi\eta_n\mathrm{Ai}(\eta_n)^2}\left(\frac{ka}{2}\right)^{1/3}, \tag{D4}$$

where η_n $(n\in N^*)$ is the nth zero of the derivative of the Airy function $\mathrm{Ai}'(x)$.

3. Impedance boundary conditions

Here, the poles of the $\mathcal{S}_\nu$ function are the zeros of $[\zeta\mathrm{H}^{(1)'}_\nu(ka)+i\mathrm{H}^{(1)}_\nu(ka)]$ and are given by two different asymptotic formulas whether $|\zeta|>1$ or $|\zeta|<1$.

For $\underline{|\zeta|>1}$,

$$\begin{aligned}\nu_n=&ka-i\frac{14\zeta^2(\eta_n^3-1)-1}{140\zeta^3\eta_n^3}-\frac{11+90\zeta^2}{90\zeta^3\eta_n}e^{i\pi/6}\left(\frac{ka}{2}\right)^{2/3}\\&+\frac{8(3\zeta^2-1)\eta_n^3-15}{48\zeta^4\eta_n^6}\left(\frac{ka}{2}\right)\\&-\frac{3+2\eta_n^3+280\zeta^4\eta_n^6+14\zeta^2(3+2\eta_n^3)}{280\zeta^4\eta_n^5}e^{i\pi/3}\left(\frac{ka}{2}\right)^{1/3}\\&+\frac{7\zeta^2(\eta_n^3-6)-3}{420\zeta^2\eta_n}e^{2i\pi/3}\left(\frac{ka}{2}\right)^{-1/3}.\end{aligned} \tag{D5}$$

For $\underline{|\zeta|<1}$,

$$\begin{aligned}\nu_n=&ka-i\zeta-\mu_n e^{i\pi/3}\left(\frac{ka}{2}\right)^{1/3}+\frac{\mu_n^2}{60}e^{2i\pi/3}\left(\frac{ka}{2}\right)^{-1/3}\\&-\frac{2\zeta^3-\zeta}{6}\mu_n e^{5i\pi/6}\left(\frac{ka}{2}\right)^{-2/3}-\frac{3\zeta^2}{20}\left(\frac{ka}{2}\right)^{-1}.\end{aligned} \tag{D6}$$

These two asymptotic formulas (D5) and (D6), providing the location of the poles ν_n for the impedance BC, have been established following the method of Streifer and Kodis [34]. The residue of the $\mathcal{S}_\nu$ function at the poles $\nu=\nu_n$ is approximated by (for $|\zeta|>1$ or $|\zeta|<1$)

$$r_{\nu_n}\sim\frac{\exp(-i\pi/6)}{2\pi[\mathrm{Ai}'(z)^2-z\,\mathrm{Ai}'(z)^2]}\left(\frac{ka}{2}\right)^{1/3}, \tag{D7}$$

where z is defined by

$$z=\left(\frac{ka}{2}\right)^{-1/3}e^{-i\pi/3}(ka-\nu_n). \tag{D8}$$

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