Controlling the ultimate state of projective synchronization in chaotic systems of arbitrary dimension

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The ultimate state of projective synchronization is hardly predictable. A control algorithm is thus proposed to manipulate the synchronization in arbitrary dimension. The control law derived from the Lyapunov stability theory with the aid of slack variables is effective to any initial conditions. The method allows us to amplify and reduce the synchronized dynamics in any desired scale with tiny control inputs. Applications are illustrated for seven- and ten-dimensional chaotic systems.

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Adjacent chaotic trajectories governed by the same nonlinear systems may evolve into a state utterly uncorrelated, but could be synchronized through a coupling [1]. The concept of chaos synchronization attracts considerable attention [2–13]. It may lead to some potential applications in secure communication [2], ecological systems [3], and system identification [4].

Different forms of synchronization phenomena have been observed in a variety of chaotic systems, such as identical synchronization [1], phase synchronization [5], and generalized synchronization [6]. In partially linear chaotic systems, such as the Lorenz system, projective synchronization was noticed [7] with the characteristic that the states of two coupled systems synchronize up to a constant ratio known as scaling factor [8]. Further investigation [9] revealed that it occurs with a negative trace of the Jacobian matrix in three-dimensional systems. A recent study [10] derived a general condition for projective synchronization in arbitrary-dimensional systems. The early report [11] showed that the ultimate state of the synchronization is usually unpredictable. Thus a control algorithm [12] was developed to manipulate projective synchronization in three-dimensional systems. And the technique is extended to realize projective synchronization in nonpartially linear systems [13]. However, for the general case of arbitrary-dimensional systems, especially for high-dimensional systems, control of projective synchronization poses a challenge. In this paper, we present a general control method that can be used to create and manipulate projective synchronization in arbitrary-dimensional systems.

Projective synchronization results from the partial linearity of coupled systems. A partially linear system refers to an autonomous system in which the state vector $\mathbf{u}$ associates linearly with its time derivatives $\dot{\mathbf{u}}$ through Jacobian matrix $\mathbf{M}(z)$, where $\mathbf{M}(z)\in\Re^{n\times n}$ contains a variable z that is nonlinearly related to the state vector $\mathbf{u}$. A coupled system consists of a master system (denoted by subscript m) and a slave system (denoted by subscript s). The two subsystems linked with a coupling variable z can be expressed in the form as

$$\dot{\mathbf{u}}_m=\mathbf{M}(z)\mathbf{u}_m,\quad \dot{z}=g(\mathbf{u}_m,z),\quad \dot{\mathbf{u}}_s=\mathbf{M}(z)\mathbf{u}_s. \tag{1}$$

In the coupled system (1), the master system evolves independently, while the slave system is driven by the coupling variable z that is governed by the master system. Once the Jacobian $\mathbf{M}(z)$ satisfies the criteria stated in the report [10], the states of the master and slave systems will be synchronized up to a common scaling factor in all corresponding dimensions. Projective synchronization leads to a proportional relationship between the master and slave states, expressed as

$$\lim_{t\to\infty}\|\alpha\mathbf{u}_m-\mathbf{u}_s\|=0, \tag{2}$$

where the scaling factor α is defined by the limit of the state ratio

$$\alpha=\lim_{t\to\infty}\alpha(t)=\lim_{t\to\infty}\|\mathbf{u}_s\|/\|\mathbf{u}_m\|, \tag{3}$$

Here $\|\bullet\|$ denotes a norm of a vector. The state ratio $\alpha(t)$ may vary at any particular instant before the occurrence of projective synchronization. Note that the scaling factor depends on the initial conditions and chaotic variables of the underlying system [11]. Consequently, the ultimate state of synchronization is hardly estimated.

We wish to generate projective synchronization in coupled partially linear systems of arbitrary dimension. We wish to control the ultimate state of synchronized dynamics in a favorable manner. Projective synchronization allows us to duplicate a chaotic system in distinct scales with the same topological characteristics (such as Lyapunov exponents and fractal dimensions) [11]. It can also be utilized to amplify or reduce the response of the driven system.

We introduce a control method for projective synchronization in arbitrary dimension. By incorporating a controller to the master system, the general form of the controlled system is given as

$$\dot{\mathbf{u}}_m=\mathbf{M}(z)\mathbf{u}_m+\boldsymbol{\xi},\quad \dot{z}=g(\mathbf{u}_m,z),\quad \dot{\mathbf{u}}_s=\mathbf{M}(z)\mathbf{u}_s, \tag{4}$$

where $\mathbf{u}_m=(x_1,x_2,\ldots x_n)^T$ and $\mathbf{u}_s=(y_1,y_2,\ldots,y_n)^T$, and $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_n)^T$ is a control vector. We intend to develop a control algorithm with global stability that enables the control to be effective to any initial conditions of the coupled system. Therefore, the Lyapunov stability theory is employed for this intention.

To construct a proper Lyapunov function, we first look into an error dynamics of projective synchronization by inspecting the error vector

$$\mathbf{e}=(e_1,e_2,\ldots,e_n)^T=\alpha^*\mathbf{u}_m-\mathbf{u}_s\,, \qquad (5)$$

where $e_i=\alpha^* x_i-y_i$ for $i=1,2,...,n$ and α^* is a desired scaling factor. Obviously, if the error (5) tends to be zero, projective synchronization takes place with the desired scaling factor accordingly. We thus consider a Lyapunov function in the form

$$V(\mathbf{e})=\frac{1}{2}\sum_{i=1}^{n} e_i^2. \qquad (6)$$

According to the Lyapunov stability theory, if the function (6) satisfies the first condition: $V(\mathbf{e})>\mathbf{0}$ when $\mathbf{e}\neq\mathbf{0}$, $V(\mathbf{e})=\mathbf{0}$ when $\mathbf{e}=\mathbf{0}$, and the second condition: $\dot{V}(\mathbf{e})<\mathbf{0}$ when $\mathbf{e}\neq\mathbf{0}$, $\dot{V}(\mathbf{e})=\mathbf{0}$ when $\mathbf{e}=\mathbf{0}$, the error vector $\mathbf{e}(t)$ asymptotically tends to zero leading to $\lim_{t\to\infty}\|\alpha^*\mathbf{u}_m-\mathbf{u}_s\|=\mathbf{0}$. Surely, the employed Lyapunov function (6) satisfies the first condition. For the second condition, the time derivation of (6) must be negative, given as

$$\dot{V}(\mathbf{e})=\sum_{i=1}^{n} e_i\dot{e}_i<0 \quad \text{for } \mathbf{e}\neq\mathbf{0}. \qquad (7)$$

Insert $\dot{e}_i=\alpha^*\dot{x}_i-\dot{y}_i$ into inequality (7) and rewrite the inequality as

$$\dot{V}(\mathbf{e})=\sum_{i=1}^{n} e_i(\alpha^*\dot{x}_i-\dot{y}_i)<0 \quad \text{for } \mathbf{e}\neq\mathbf{0}. \qquad (8)$$

Substituting $\dot{x}_i=\mathbf{m}_i\mathbf{u}_m+\xi_i$ and $\dot{y}_i=\mathbf{m}_i\mathbf{u}_s$ into Eq. (8), where $\mathbf{m}_i$ is the ith row of the Jacobian matrix $\mathbf{M}$, we obtained the condition $\dot{V}(\mathbf{e})=\Sigma_{i=1}^{n}[e_i(\alpha^*(\mathbf{m}_i\mathbf{u}_m+\xi_i)-\mathbf{m}_i\mathbf{u}_s)]<0$ that can be rearranged into the form as

$$\dot{V}(\mathbf{e})=\sum_{i=1}^{n} [e_i\alpha^*\xi_i+e_i\mathbf{m}_i\mathbf{e}_i]<0, \quad \text{for } \mathbf{e}\neq\mathbf{0}. \qquad (9)$$

The inequality (9) carries the control function ξ_i. If the control functions are selected in such a way that the condition (9) is satisfied, the control leads to $\lim_{t\to\infty}\mathbf{e}(t)=\mathbf{0}$ and consequently $\lim_{t\to\infty}\|\alpha^*\mathbf{u}_m-\mathbf{u}_s\|=\mathbf{0}$. Thus projective synchronization is realized with the specified scaling factor α^*.

To derive the control functions, we convert the condition (9) into an equality form by introducing a *slack vector*, $\mathbf{S}=(K_1,K_2,\ldots,K_n)$ where K_i is a slack variable. Slack variable is frequently used in optimization theory [14] to convert a "less than or equal to" type of inequality into an equality form in dealing with constrained conditions. A slack variable is defined as

$$K_i=k_ie_ie_i \quad \text{for } i=1,2,...,n \qquad (10)$$

where $k_i>0$ is called the *slack constant*, which can be any real positive value. The value of k_i affects the convergence rate of the control, which will be discussed later in numerical applications. Add non-negative slack variable (10) into the left side of inequality (9). The inequality (9) is thus converted into an equality form as

$$e_1\alpha^*\xi_1+e_1\mathbf{m}_1\mathbf{e}+k_1e_1e_1+e_2\alpha^*\xi_2+e_2\mathbf{m}_2\mathbf{e}+k_2e_2e_2+\cdots$$
$$+e_n\alpha^*\xi_n+e_n\mathbf{m}_n\mathbf{e}+k_ne_ne_n=0. \qquad (11)$$

Examining equation (11), a possible solution of the condition (11) is that each component corresponding to error e_i can be set to zero, i.e.,

$$e_1(\alpha^*\xi_1+\mathbf{m}_1\mathbf{e}+k_1\mathbf{e}_1)=0, \quad e_2(\alpha^*\xi_2+\mathbf{m}_2\mathbf{e}+k_2e_2)=0,...,$$
$$e_n(\alpha^*\xi_n+\mathbf{m}_n\mathbf{e}+\mathbf{k}_ne_n)=0. \qquad (12)$$

Thus the control functions can be formulated as

$$\xi_i=-[\mathbf{m}_i\mathbf{e}+k_ie_i]/\alpha^* \quad \text{for } i=1,2,...,n \qquad (13)$$

Note that the control functions contain the error terms. Maintaining a controlled projective synchronization only needs tiny control inputs because the errors in the control functions (13) tend to zero after the synchronization is realized. Therefore, the controlled system preserves the dynamical characteristic of the original systems. In what follows, we shall apply the control method to newly explored high-dimensional chaotic systems.

Example 1 is to direct the ultimate state of projective synchronization of a coupled system to a desired state ratio. The system used here is a seven-dimensional chaotic system recently explored by the authors. Linking such two systems together with the coupling variable z, the coupled partially linear system is given as

$$\dot{\mathbf{u}}_m=\mathbf{M}(z)\cdot\mathbf{u}_m\,,$$
$$\dot{z}=3x_1x_3-14z\,, \qquad (14)$$
$$\dot{\mathbf{u}}_s=\mathbf{M}(z)\cdot\mathbf{u}_s\,,$$

with the 6×6 Jacobian matrix,

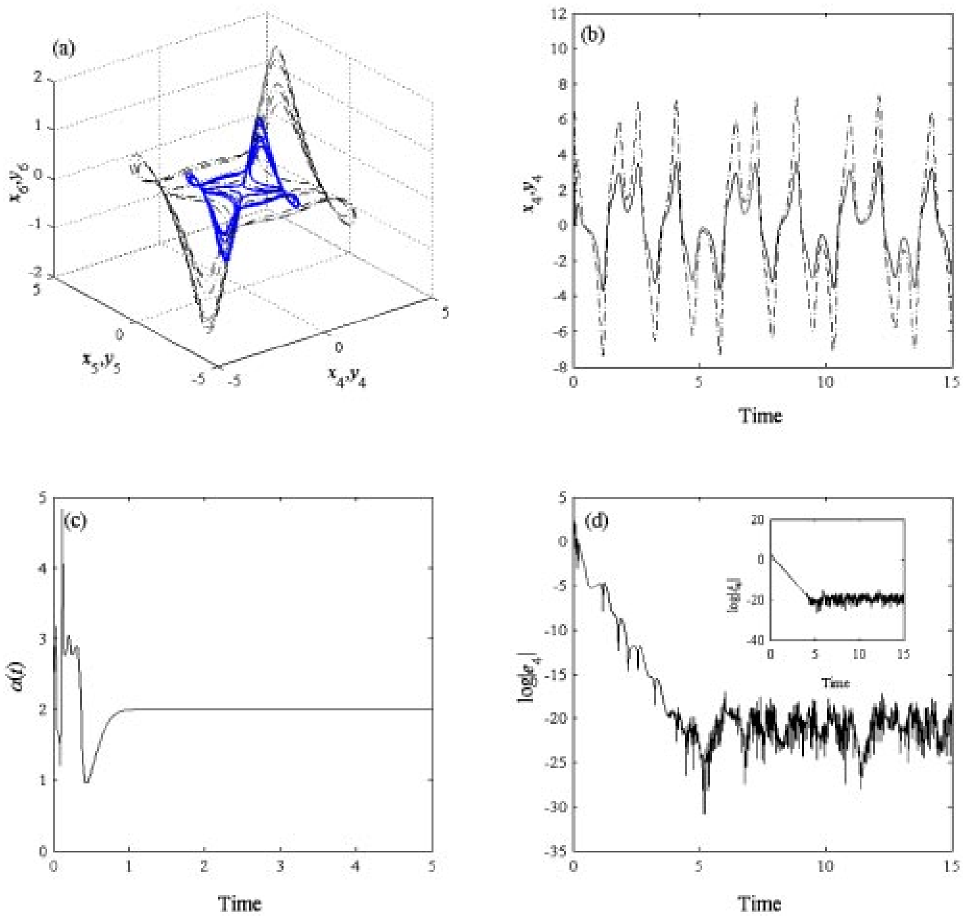

FIG. 1. (a) The synchronized chaotic attractors (master: solid line, slave: dashed line) of the coupled system (14). (b) The history of synchronized state. (c) The variation of scaling factor against time. (d) The error of the synchronization and the control input (inset). The initial condition $\{x_1,\ldots,x_6,z,y_1,\ldots,y_6\} = \{1,...,6,7,8,...,13\}$ and the desired scaling factor, $\alpha^* = 2$.

$$\mathbf{M}(z)=\begin{bmatrix} -14 & 8 & -6 & 1 & 2 & 3 \\ -8 & 12 & -(8+3z) & 2 & 3 & 4 \\ 6 & (8+3z) & -22 & 3 & 4 & 5 \\ -1 & -2 & -3 & -13 & 5 & 6 \\ -2 & -3 & -4 & -5 & -14 & (8+7z) \\ -3 & -4 & -5 & -6 & -(8+7z) & -15 \end{bmatrix}.$$

The initial condition of the coupled system (14) is $\{x_1,\ldots,x_6,z,y_1,\ldots,y_6\} = \{1,2,...,6,7,8,...,13\}$. The desired scaling factor α^* is set to 2 and the slack constant is set at $k_i = 2$ for $i = 1$, 2, ..., 6. The dynamical behavior of controlled system is illustrated in Fig. 1. Figure 1(a) shows the chaotic attractors of the master and slave systems in a three-dimensional subspace. The dynamics of the master system is traced by solid line while the dynamics of the slave system is traced by dashed line. The time-history diagram in Fig. 1(b) shows that the responses of two subsystems tend to be proportional with the scaling factor of 2. The state ratio $\alpha(t)$ converges to the specified scaling factor, $\alpha^* = 2$ as $t\to\infty$ [see Fig. 1(c)]. The error (5) of the synchronization decreases exponentially to a level about 10^{-20} as shown in Fig. 1(d). The inset in Fig. 1(d) displays a control input $\ln|\xi_4|$ against the time. It can be seen that once the system is directed to a desired synchronous state, the control input tends to be zero. The numerical experiment shows that the control algorithm works very well in the manipulation of the outcome of projective synchronization.

In example 2, we explore the effectiveness of the control for a sharp change of the scaling factor and the effects of the selection of the slack constants in the control. The control algorithm (13) is applied to a coupled nineteen-dimensional system that was explored by the authors according to the criteria [10]. In the coupled system, the master and slave systems are ten-dimensional linked by the variable z. The coupled chaotic system is given as

$$\dot{\mathbf{u}}_m = \mathbf{M}(z)\cdot\mathbf{u}_m,$$
$$\dot{z} = 3x_1x_3 - 7z, \qquad (15)$$
$$\dot{\mathbf{u}}_s = \mathbf{M}(z)\cdot\mathbf{u}_s,$$

with the 9×9 Jacobian matrix

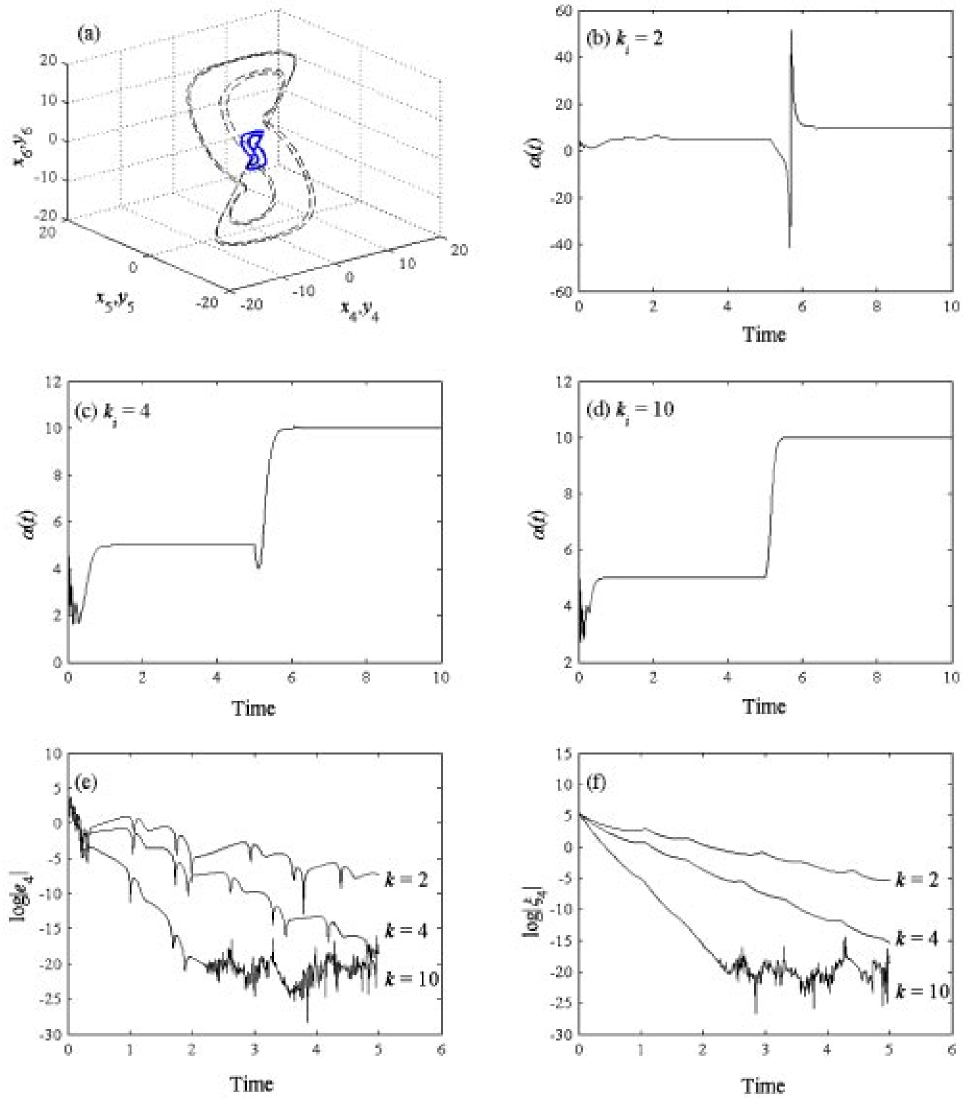

FIG. 2. (a) A view of the synchronized chaotic attractors (master: solid line, slave: dashed line) of the coupled system (15). (b) The variation of scaling factor with the control $k_i = 2$. (c) The variation of scaling factor with the control $k_i = 4$. (d) The variation of scaling factor with the control $k_i = 10$. (e) The errors associated with the slack constants. (f) The control inputs. The initial condition $\{x_1,\ldots,x_9,z,y_1,\ldots,y_9\} = \{1,...,9,10,11,...,19\}$ and the desired scaling factor, $\alpha^* = 5$ for $0 < t \leqslant 5$ and $\alpha^* = 10$ for $5 < t \leqslant 10$.

$$\mathbf{M}(z) = \begin{bmatrix} -14 & 8 & -6 & 1 & 2 & 3 & 4 & 5 & 6 \\ -8 & 12 & (8+3z) & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & (8+3z) & -22 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -13 & 5 & 6 & 7 & 8 & (9+7z) \\ -2 & -3 & -4 & -5 & -14 & 7 & 8 & 9 & 10 \\ -3 & -4 & -5 & -6 & -7 & -15 & 9 & 10 & 11 \\ -4 & -5 & -6 & -7 & -8 & -9 & -16 & 11 & 12 \\ -5 & -6 & -7 & -8 & -9 & -10 & -11 & -17 & 13 \\ -6 & -7 & -8 & -(9+7z) & -10 & -11 & -12 & -13 & -12 \end{bmatrix}.$$

The couple system (15) can naturally (without control) synchronize up to a scaling factor of $\lim_{t\to\infty}\alpha(t) = 9.4$ when the initial condition is taken at $\{x_1 \ldots x_9,\ z,\ y_1 \ldots y_9\} = \{1, 2, 3,..., 19\}$. Figure 2(a) shows the chaotic attractors of the master and slave systems in a three-dimension subspace. The response of the master system is traced by solid line while the dynamics of the slave system is traced by dashed line.

Three control experiments are carried out using different values of slack constants for the coupled system (15). In each control experiment, all the slack constants are the same. The scaling factor will be directed from a specified value

$\alpha^*=5$ to a new desired value $\alpha^*=10$ with a sharp increment of 5. Each experiment runs for 10 time units in which the control for each scaling factor lasts 5 time units. Each control is conducted continuously in the time interval from $t=0$ to 10. All the parameters and initial conditions are the same in the experiments except the slack constant used.

To view the effect of the selection of the slack constants on the convergence of the control, we use three slack constants $k_i=2$, $k_i=4$, and $k_i=10$ for $i=1,2,...,9$ in the control respectively. The results of each experiment are shown, respectively, in Fig. 2. In Fig. 2(b), by using the slack constant $k_i=2$, the control directs the scaling factor to $\alpha^*=5$ after a transient period about 3 time-units. At $t=5$, the control still remains but the desired scaling factor is sharply adjusted to $\alpha^*=10$. A large fluctuation of the scaling factor is observed in the transient period of $t=5\sim7$, before the scaling factor settles down to the new desired value. In Fig. 2(c), the slack constant $k_i=4$ is used in the control. The control steers the state ratio to the desired scaling factor with shorter transient period and smaller fluctuation. In Fig. 2(d), a smooth transition from one desired scaling factor to the other is observed with the control of the larger slack constant $k_i=10$. Obviously, the three experiments show that the transient period of controlling the scaling factor to a target value is generally reduced with an increase of the value of the slack constant. Clearly, the larger slack constant leads to the faster convergence rate in the control. Figure 2(e) illustrates the variation of the error $\ln|e_4|$ against time for the three control cases. The error of the synchronization decreases at the different rates corresponding to the different slack constants used. The larger value of the slack constant leads to smaller error in projective synchronization and faster convergent rate in the control. Figure 2(f) shows the three control-input signals corresponding to the different slack constants. The result shows that the required control input is relatively smaller in the control when the larger slack constant is used.

In short, we provide a control algorithm to reorganize the dynamical scale of synchronized state for coupled partially linear chaotic systems of any dimension. The control law, derived from Lyapunov stability theory with the aid of slack variables, has the feature of global stability such that the control is effective to any initial conditions of coupled systems. This control method could be employed to enforce a nonsynchronous system to be synchronized, and manipulate the ultimate state of projective synchronization to any desired ratio. It allows us to use tiny control inputs to amplify or reduce the response of the driven system to any scale in a short transient period. Numerical experiments have indicated the effectiveness of the control method for high-dimensional chaotic systems.

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