Manipulation of dynamical systems by symmetry breaking

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(Received 20 December 2001; published 7 October 2002)

We propose an approach to manipulate and control transport in Hamiltonian systems which are characterized by a mixed phase space. The approach is based on symmetry breaking of the phase space structure by applying a *zero-mean* periodic force for a finite duration of time. This induces time and space reversal asymmetry, which modifies the internal dynamics of the system and leads to directed transport. It is shown that our strategy allows to perform manipulations both with individual particles and with statistical ensembles of particles.

DOI: 10.1103/PhysRevE.66.046203 PACS number(s): 05.45.Ac, 02.30.Yy, 05.60.Cd

The origin of transport in dynamical systems has been an active research area for a long time [1]. Recent investigations in this field have been motivated by the revival of the *ratchet* idea that leads to directed transport in the absence of bias [2]. Most studies have focused on noisy dissipative systems in the overdamped limit, which appear suitable for microbioligical examples [3]. The opposite limit, in which inertia plays a essential role, has attracted less attention, although it offers some new directions in current rectification [4].

Fundamental to questions of transport in general, and rectification in particular, are Hamiltonian systems which exhibit rich and varied behavior, ranging from regular to anomalous motion which can be related to a mixed structured phase space [5]. Kinetics in Hamiltonian systems has been shown to provide a foundation for problems of importance such as the link between stochastic processes and the nature of dynamical systems [6]. It has been demonstrated [7] that the necessary conditions for current rectification in Hamiltonian systems is time reversal asymmetry. Namely, in order to obtain a current, symmetry breaking (SB) needs to be imposed on the system [7,8].

In this paper, we show how understanding the SB mechanism for directed motion [9] naturally leads to a new tool to manipulate classical Hamiltonian systems. The approach we propose aims at cases for which imposing global gradients is impossible. One can easily imagine a situation where remote control is required, or when a global bias is not desirable. Here we demonstrate how the SB approach helps to manipulate both single particles and statistical ensembles of particles. In the case of a statistical ensemble, SB opens new possibilities for handling some fraction of the particles, a process which cannot be performed using standard bias techniques.

Let us consider the example of a particle which moves in the periodic nonlinear potential $U(x)=\cos(x)$ under the influence of a periodic train of δ kicks with an amplitude E_1 and period T [1,5],

$$H_s(p,x,t)=\frac{p^2}{2}+E_1\cos(x)\sum_{n=-\infty}^{\infty}\delta(t-nT). \tag{1}$$

The Hamiltonian (1) is symmetric with respect to time and space reversal transformation $\{t\to -t, x\to -x\}$, so a particle, whose dynamics obey Eq. (1), performs a diffusive motion with zero drift. The symmetry can be broken by switching on a second source of kicks which are shifted, in time and space domains, with respect to the first one,

$$H(p,x,t)=H_s(p,x,t)+\Phi(t,t_{on},t_{off})E_2\cos(x+\phi)\times\sum_{l=-\infty}^{\infty}\delta(t-nT+\tau), \tag{2}$$

where ϕ and τ are spatial and temporal shift constants, $\Phi(t,t_{on},t_{off})=\Theta(t-t_{on})\Theta(t_{off}-t)$ is a square pulse function, and t_{on} and t_{off} are the switching on and switching off times. The duration time for the SB force is, therefore, $t_{\mathrm{SB}}=t_{off}-t_{on}$. Notice that this SB method is different from the usually used "two-harmonics" ratchet approach [4,7,9].

Let us start from the case of a constant SB ($t_{on}=-\infty$, $t_{off}=\infty$). For nonequidistant kicks, $\tau\neq mT/2$, and $\phi\neq s\pi$ ($m,s=\cdots -1,0,1,\ldots$), all relevant symmetries are broken and we fulfill the necessary conditions for the appearance of a current [7]. The evolution of the Hamiltonian system (2) can be described by a pair of consecutive maps in position x and momentum p,

$$x'_{n+1}=x_n+p_n\tau,$$
$$p'_{n+1}=p_n+U'(x_{n+1}), \tag{3}$$

$$x_{n+1}=x'_{n+1}+p'_{n+1}(T-\tau),$$
$$p_{n+1}=p'_{n+1}+\Phi(t,t_{on},t_{off})U'(x_{n+1}+\phi), \tag{4}$$

where Eqs. (3) correspond to kicks from the main source, and Eqs. (4) correspond to kicks from the SB source.

The Hamiltonian system in Eqs. (1)–(4) has a complicated phase space with coexisting and interwoven sets of invariant manifolds with different drift velocity values and directions. It is characterized by the presence of chaotic layers, which originate from nonlinear resonance separatrixes, and regular regions, consisting of KAM-tori with complex *sticky* barriers between the chaotic and regular regions [5]. Due to these barriers a trajectory can be trapped near regular islands for a long time. This leads to the appearance of flights in the case of nonzero winding numbers v, or localized rotating modes with $v=0$. For some hierarchy of ballistic islands this trapping time can be anomalously long, resulting in Lévy flights [10]. In Refs. [9,11], it has been shown that

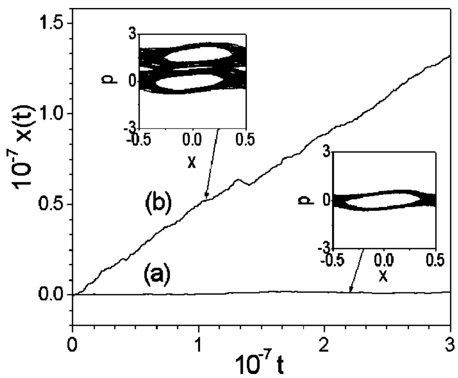

FIG. 1. Dependence of $x(t)$ versus t for (a) the symmetric Hamiltonian, Eq. (1) ($E_1 = 0.24$, $T = 0.6$) and (b) the Hamiltonian with the additional SB source (2), Eq. (2) ($E_1 = 0.24$, $T = 0.6$, $E_2 = 0.11$, $\phi = 0.8$, $\tau = 0.4$). Insets show the corresponding Poincaré sections.

the generation of directed currents in Hamiltonian systems is determined by breaking the symmetry of Lévy flights due to asymmetry in the structure of the hierarchy of regular islands in phase space.

Here we are interested in dynamics inside the main stochastic layer near $p = 0$, which corresponds to the ground state. Switching on the second kicking source with amplitude E_2 and phase shifts ϕ and τ results in asymmetric overlapping of the main chaotic layer with the layer of ballistic islands (see Fig. 1). This overlapping produces a current. Current inversion (mirroring the layers overlap) can be obtained by a simple shift inversion $\phi \to -\phi$ or $\tau \to T - \tau$.

For the analysis of the dynamics we use the propagator $P(x,t)$, i.e., the probability density of a particle to be at x at time t [12]. In Fig. 2 we show the propagator for time $t = 1000T$ obtained by averaging over 10^5 trajectories, starting in a chaotic area of the main layer. The peaks in the propagator correspond to flights that a particle performs when it sticks to ballistic islands. The locations of the peaks are determined by the corresponding winding numbers. From the structure of the propagator it is clearly seen that the large scale particle displacements, when compared to the period of

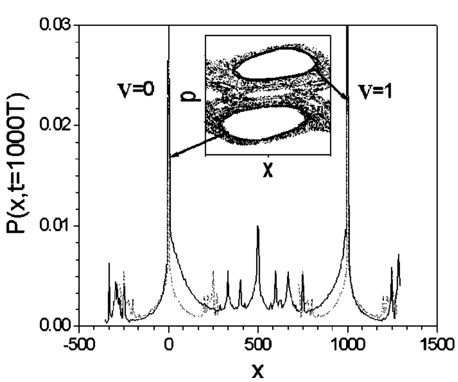

FIG. 2. The propagator for a fixed time ($t = 1000T$) for (a) the symmetric Hamiltonian, Eq. (1) (dotted line) and (b) the Hamiltonian with constant SB, Eq. (2) (solid line). Inset displays sticky islands correspond to the main peaks in the propagator shape. Parameters same as in Fig. 1.

the potential $U(x)$, are a result of the flights. Directed particle drift stems from asymmetry in the structure of the ballistic islands with positive and negative winding numbers. Moreover, most of the contribution to the particle's transport in the positive direction comes directly from the main ballistic islands with $v = 1$ (see inset in Fig. 2).

Based on the above, we expect that controlling the manifold overlap in phase space, one can control the directed transport by tuning the value of the velocity. We now describe a possible way to manipulate a particle through SB during a finite time interval t_{SB}. In order to do this we use two features of the system: (i) the possibility to temporarily remove the barriers in phase space (formed by invariant KAM-tori) between different invariant manifolds, and (ii) the sticky nature of the regular islands. Namely, one can remove the barriers from the phase space during a time interval t_{SB} and then restore them. This can be viewed as an act of a *demon* [5,13]. Here the demon removes the barrier (''opens a door'') at time t_{on} and restores the barrier (''closes the door'') at time t_{off}. Due to the stickness property, the information required for the control the particle is its velocity only. This means that the ''door'' closes when the velocity of the particle is close to a desired winding number. The most efficient manipulation can be achieved using the ''stickiest'' islands, which are present in both Hamiltonians, the symmetric one, H_s, in Eq. (1), and the SB one, H, in Eq. (2). In this way the time duration t_{SB} needed for the manipulation decreases, and the accuracy of the procedure increases. Below we outline an example of the SB strategy of our demon.

Let us start from the situation in which a particle is located inside the main stochastic layer. If the demon wants to move the particle in the positive direction then the particle must be shifted into the upper ballistic layer. In this case, the demon must switch on the second source of δ kicks, that leads to overlapping of the main layer with the upper one. The demon has to follow the velocity of the particle. When, for duration of about $t_{contr} = 10T$, the velocity is close enough to the winding number of stickiest island, it indicates that the particle is trapped in the upper layer with a high probability. At this stage the demon switches off the second source. Now the particle remains locked inside this layer and moves approximately with a constant velocity in the positive direction. After some time, when the particle reaches a required region in space and the demon would like to stop it, he should again switch on the second source and follow the velocity. If the particle velocity is close enough to zero then it means that the particle sticks to a localized resonance and has returned to the main chaotic layer. The demon now switches off the second source and the particle is locked back inside the main layer. The mean energy returns to its value before the SB action.

In Fig. 3, we show the realization of the SB procedure for the system according to Eq. (2). We check the displacement of the particle after each time step $10T$. If this displacement is close enough to t_{contr} (about 10%), we take it as a sign that the particle is near the corresponding island. The direction of the motion is defined by the value of the time shift τ of the second source.

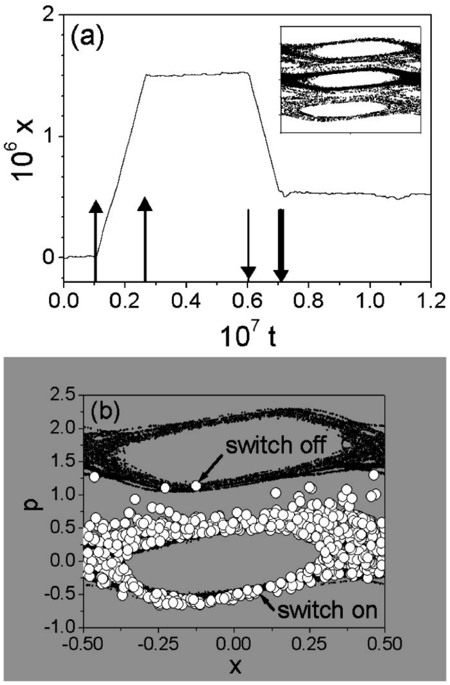

FIG. 3. A realization of the control approach. (a) Trajectory for the Hamiltonian in Eq. (2). The SB source acts for time windows marked by the bars. The widths of the bars equal the duration of the SB action. The arrows point to the resonance involved in the overlap (upper and lower ones correspondingly). The time phase shift is $\tau=0.4$ for the two first pulses and $\tau=0.2$ for the last ones. Inset shows the Poincaré section for the manipulation period. (b) The Poincaré section (white circles) of the system in Eq. (2) after every time step $t=10T$ (see text for details) during the first SB pulse. The parameters as in Fig. 1.

The mean time needed for the demon's SB action can be evaluated using the distribution of times between consecutive sticking events in the case of a permanent SB action, Eq. (2) [10]. The time needed to moves particle to the flying mode can be estimated as the first moment of the probability distribution function (PDF) for the ballistic island. For the same parameters as in Fig. 3, this procedure gives $\simeq 35t_{contr}$. The time needed for return the particle back to the nondrift diffusive mode can be estimated as first moment of the PDF for the central localized island. This gives $\simeq 48t_{contr}$.

The SB strategy can be also used in the case of a statistical ensemble of particles. In this case, the SB can change populations of particles on the different manifolds through the control of the KAM-tori barriers. Let us consider the example of a continuous system with a Hamiltonian, which describes the motion of particles in a standing wave with a modulating amplitude [1]:

$$H_s(p,x,t)=\frac{p^2}{2}+E_1\cos(x)\cos^2(\omega t). \tag{5}$$

Such a Hamiltonian system has been realized in atomic optics experiments probing motion in a wave produced by a laser field [14]. Here we investigate the classical limit.

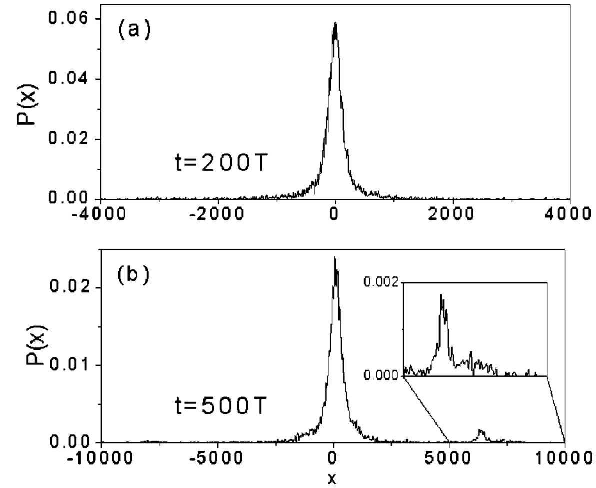

FIG. 4. The spatial distribution of an ensemble of particle $N=10^4$ (see text) for the Hamiltonian in Eq. (6) ($E_1=0.5$, $E_2=0.1$, $T=2\pi$, $\phi=1.2$, $\tau=0.8$) (a) just before and (b) after the action of the SB force ($t_{on}=200T$, $t_{off}=220T$). The inset is an enlargement of the additional peak that corresponds to the chipped fraction of the directed particles.

Following steps analogous to those applied above, we can brake all relevant symmetries by switching on a second standing wave, shifted with respect to the main one,

$$H(p,x,t)=H_s(p,x,t)+\Phi(t,t_{on},t_{off})E_2\cos(x+\phi)\times\cos^2(\omega t+\tau), \tag{6}$$

where ϕ and τ are spatial and temporal shift constants and E_2 is the amplitude of the second standing wave.

We consider the dynamics of an ensemble of particles with an initial Maxwellian distribution in p, and homogeneous in x, inside one spatial period of the potential

$$\rho(p,x,0)=\frac{1}{2\pi}\sqrt{\frac{\beta}{2\pi}}e^{-(\beta/2)p^2}\Theta(x)\Theta(2\pi-x) \tag{7}$$

with $\beta=10$.

Under the influence of the main standing wave, $E_1\cos(x)\cos^2(\omega t)$, the ensemble performs diffusive spreading with no drift. Now we show that using SB for a finite duration t_{SB}, i.e., a pulse of a second force, Eq. (6), it is possible to chip off a small fraction of the particles from the an initial "cloud" and transport it in a preasiggned direction. Namely, a small fraction (compared to the initial ensemble) of particles can be locked into the manifold with a nonzero drift. After switching off the pulse of the second standing wave the particles move in the prescribed direction with a velocity of the corresponding manifold. This is a kind of tweezers, which chip off a some fraction of particles. In Fig. 4 we show an example of the realization of this strategy. SB induces an overlap of the main chaotic layer with the thin upper ballistic layer. The number of chipped particles can be controlled by variation of t_{SB}. For example, for $t_{SB}=10T$, the chipped fraction is about 3% and for $t_{SB}=20T$ is about 7%. It should be mentioned that this manipulation cannot be achieved by standard technique using an external bias, since this will lead to the total displacement of the ensemble only. The SB approach, proposed here, provides, therefore, a new possibility to perform a nontrivial manipulation with statistical ensembles using zero-mean external fields.

In summary, we have shown that SB provides a new tool for manipulating and directing dynamical systems. This approach can be also realized for systems with an additive driving force [9,11]. In dissipative systems, the roles of manifolds with different currents are expected to be played by limit cycles with different winding numbers [1]. Symmetry implies here that ballistic cycles with opposite winding numbers always appear as pairs and have equal (in volume) basins of attraction. The SB can lead to a situation where one of these cycles loses stability and disappears from the phase space [7]. Namely, as in the Hamiltonian case discussed above, it is possible to redirect particles from one cycle to another using SB and thereby change the cycle populations in the ensemble case.

S.D. thanks S. Flach (MPIPKS) for helpful discussions. Financial support from the Israel Science Foundation, the USA Israeli BSF, DIP, and SISITOMAS grants are gratefully acknowledged.

[1] R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics: From the Pendulum to Turbulence and Chaos* (Harwood, NY, 1985).

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. 1.

[3] F. Julicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).

[4] P. Hänggi and R. Bartussek, in *Nonlinear Physics of Complex Systems*, Lecture Notes in Physics Vol. 476, edited by J. Parisi, S. C. Muller, and W. Zimmermann (Springer, Berlin, 1996), p. 294; P. Jung, J. G. Kissner, and P. Hänggi, Phys. Rev. Lett. **76**, 3436 (1996); J. L. Mateos, Phys. Rev. Lett. **84**, 258 (2000).

[5] G. M. Zaslavsky, *Physics of Chaos in Hamiltonian Systems* (Imperial College Press, London, 1998).

[6] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993).

[7] S. Flach, O. Yevtushenko, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 2358 (2000).

[8] H. Schanz, M.-F. Otto, R. Ketzmerick, and T. Dittrich, Phys. Rev. Lett. **87**, 070601 (2001).

[9] S. Denisov and S. Flach, Phys. Rev. E **64**, 056236 (2001).

[10] J. Klafter, G. Zumofen, and M. F. Shlesinger, in *Lévy Flights and Related Topics in Physics*, Lecture Notes in Physics Vol. 450 (Springer, Berlin, 1994).

[11] S. Denisov, J. Klafter, M. Urbakh, and S. Flach, Pysica D **170**, 131 (2002).

[12] J. Klafter and G. Zumofen, Phys. Rev. E **49**, 4873 (1994).

[13] G. M. Zaslavsky and M. Edelman, Phys. Rev. E **56**, 5310 (1997).

[14] J. R. Robinson *et al.*, Phys. Rev. Lett. **76**, 3304 (1996); F. L. Monte *et al.*, *ibid.* **75**, 4598 (1995); B. G. Klappauf, W. H. Oskay, D. A. Steck, and M. G. Raizen, *ibid.* **81**, 4044 (1998).