Fractional Langevin model of memory in financial markets

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The separation of the microscopic and macroscopic time scales is necessary for the validity of ordinary statistical physics and the dynamical description embodied in the Langevin equation. When the microscopic time scale diverges, the differential equations on the macroscopic level are no longer valid and must be replaced with fractional differential equations of motion; in particular, we obtain a fractional-differential stochastic equation of motion. After decades of statistical analysis of financial time series certain "stylized facts" have emerged, including the statistics of stock price fluctuations having "fat tails" and their linear correlations in time being exceedingly short lived. On the other hand, the magnitude of these fluctuations and other such measures of market volatility possess temporal correlations that decay as an inverse power law. One explanation of this long-term memory is that it is a consequence of the time-scale separation between "microscopic" and "macroscopic" economic variables. We propose a fractional Langevin equation as a dynamical model of the observed memory in financial time series.

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I. INTRODUCTION

Statistical features of financial time series, which appear to command sufficiently wide consensus among investigators and practitioners alike, are generally known as "stylized facts" [1]. Notable among them is the exponential decay of the two-time correlation function of asset returns, with a time constant on the order of a few minutes for liquid markets. This fact, known for decades [2] and confirmed recently by sophisticated analyses [3], is often interpreted as compelling evidence that asset prices are essentially unpredictable. This means that asset prices cannot be predicted beyond a few minutes using their past behavior [4].

However, the autocorrelation function of a stochastic variable is only sensitive to its linear temporal correlations, and thus the rapid exponential decay of such correlations does not preclude the possibility of more subtle nonlinear long-range correlations among the data. In fact, some nonlinear functions of the returns are known to be long-range correlated [1], which indicates that the stochastic processes underlying financial time series do possess long-range memory [5]. The aim of this paper is to propose a model of such memory effects based on a fractional Langevin equation.

Perhaps the simplest nonlinear function of the returns to exhibit long-range correlations is the absolute value, as, of course, do all the powers of the absolute value [1]. Such a function, or its square, is often used as a quantitative measure of the *volatility*, one of the most important parameters for risk-management purposes [1,6,7], and one that has been the object of numerous theoretical models in the literature [8].

A. Econophysics

In finance, volatility is generally understood as a measure of the size and frequency of the fluctuations of asset prices, thus leading to the standard deviation σ of the corresponding probability distribution function (PDF) as a natural candidate for such a measure. However, in the absence of a satisfactory theoretical model of price dynamics, the underlying PDF's are not known and their relevant parameters can at best only be estimated from the data.

Financial time series are notoriously erratic, but not quite structureless: even a cursory glance at a sufficiently detailed chart of price movements reveals that asset prices appear to suffer periods of relatively low variability interspersed with periods of much higher variability. This stylized fact is widely referred to as "volatility clustering" and intriguingly, seems to be independent of the specific nature of the asset. A constant value of σ would fail to capture this aspect of market data and thus would be of limited utility for the purposes of long-term risk management. A suitable compromise is to allow for a time-dependent σ, to be properly defined, thus giving rise to a class of models known as "stochastic volatility models." The operational definition of "local" volatility we consider here is the one adopted in Ref. [6].

Let the return at time t be

$$g(t)=\ln\left[\frac{p(t+\Delta t)}{p(t)}\right], \tag{1.1}$$

where $p(t)$ is the price of the asset at time t and Δt is the interval at which prices are sampled. Given the total time $T=n\Delta t$, with n an integer, $v(t)$, the volatility at time t, is defined as the average of the absolute value of $g(t)$ over the time window T,

$$v_T(t)\equiv\frac{1}{n}\sum_{t'=t}^{t+n-1}|g(t')|. \tag{1.2}$$

One is to choose T long enough so that the averages are statistically meaningful but not so long as to lose the temporal "resolution." The authors of Ref. [6] reported their results with time windows varying between tens of minutes to several days, with their data analysis confirming long-range correlations in the volatility.

Herein we are primarily interested in identifying a mechanism that may be responsible for such long-term memory in financial time series, rather than merely devising an algorithm capable of reproducing the observed statistical features. In another recent empirical study [3] compelling statistical evidence is presented that long-range correlations in volatility are due to corresponding long-range correlations in *market activity*, as measured by the time-dependent number of transactions per unit time of a given stock [3,9]. The size of individual transactions, on the other hand, turns out to be essentially immaterial to memory effects. With such empirical evidence in mind, we therefore choose to focus on market activity, as the variable that by and large incorporates the memory contents manifested in financial time series through the volatility. This is also the viewpoint adopted in Ref. [10], in which a microscopic mechanism is sought for long-range correlations in the volatility, even though market activity is the quantity that appears explicitly in the mathematical model proposed therein.

B. Statistical physics models

Market activity is a stochastic process and in statistical physics there have been two approaches to describing stochastic phenomena. One uses dynamical variables, as did Langevin. The existence of a separation between the microscopic and macroscopic time scales leads to a stochastic differential equation to describe the macroscopic dynamics. This is the Heisenberg representation in which the focus is on the time evolution of the physical observables [11]. The second approach uses the Schrödinger perspective corresponding to the time evolution of the Liouville density in the phase space for the system. In the former case, the usual outcome is the ''derivation'' from mechanics of an ordinary Langevin equation [12]. In the latter case, the evolution of the system is described by a master equation for the probability density. The latter approach usually leads to the conventional diffusion equation, with the diffusion process described by a second-order derivative in the phase space variables and first-order derivative in time [12].

In the Heisenberg perspective, after averaging over an ensemble of realizations of the stochastic force, the relaxation of a physical observable is described by an exponential function. In the Schrödinger perspective, the mathematical representation of the diffusion process is given, as we have said, by a second-order spatial derivative of a probability density function. Therefore, the mathematical description rests on either ordinary analytical functions (exponential functions) describing the dynamics, or on conventional differential operators (first-and second-order partial derivatives) describing the phase space evolution.

There is a relation between the nondifferentiability of microscopic processes, the differentiability of macroscopic processes and the conditions of the central limit theorem (CLT). Recall that in the CLT the quantities being added together are statistically independent, or at most weakly dependent, in order for the theorem to be applicable. When there is a large number of statistically independent, identically distributed random variables, with a finite variance, added together, Gaussian statistics emerge for the sum variable. In a dynamical system the CLT applies if the time scales for the microscopic processes are much smaller than the time scales for the macroscopic processes. This separation of time scales implies that the microscopic dynamics are stable, since dynamical instabilities can have arbitrarily long time scales. Once a condition of time-scale separation between the microscopic and macroscopic is established, in the long-time limit, the memory of the details of microscopic dynamics is lost, and Gaussian statistics result. This separation of time scales also means that the macroscopic dynamics can be described by the ordinary differential calculus, even if the microscopic dynamics are incompatible with the methods of ordinary calculus [13]. It is useful to point out here that the data from financial markets indicates that the price statistics, the dynamical process of interest, are not Gaussian [14,15].

On the other hand, in the case where a time-scale separation between the macroscopic and the microscopic level of description does not exist, the memory of the nondifferentiable nature of the phenomenon at the microscopic level of description is not suppressed. In this case the transport equations cannot be expressed in terms of ordinary differential calculations, even if we limit our observation to the macroscopic level. This inability to use the ordinary calculus at the macroscopic level forces the time derivative in the Langevin equation to be replaced with a fractional time derivative. Thus, we obtain a fractional stochastic equation to describe the dynamics of the physical observables. Another consequence of this nondifferentiability is that the Laplacian operator of normal diffusion is replaced with a fractional Laplacian, yielding a fractional diffusion equation in the phase space for the system. The arguments leading to these equations in a physical context have been developed by a number of investigators, see, for example, Refs. [16,17], and for a review [18].

In the present case, the stochastic variable constituting the macroscopic process is market activity, whereas the microscopic process (noise term) driving the latter, can represent the flow of information made available to agents. Uncertainty is a fact of life, and controlling its influence or, more technically, managing risk, is the ultimate motivation for trading activity to occur at all. The very''natural'' desire for making profit can be viewed as an extended, less defensive, form of risk management. Agents respond not to uncertainty itself, which is an ever-present background in everyday life, but rather to perceived variations in its intensity, which are ultimately triggered by information, to be intended in the broadest sense of the word. Individual trades take place over a time scale of minutes, and the time scale for the flow of information, however one would choose to quantify it, is unlikely to be much smaller than that. It seems to us, therefore, that market activity constitutes a case in which the time scales for the macroscopic and microscopic processes cannot be clearly separated, thus leading, according to our preceding discussion, to the propagation of the nondifferentiable aspects of the noise term all the way to the macroscopic variable, and ultimately to the need for a fractional stochastic differential equation to model that variable.

In Sec. II we provide physical motivations for a differential equation with a fractional, rather than integer, index as our mathematical instrument of choice. We introduce *fractional random walks*, and illustrate how even short-range correlated noise can yield a long-range correlated process provided we adjust the fractional index appropriately. By taking the continuum limit of a fractional random walk we obtain a fractional differential equation that can serve as a model of anomalous diffusion. In Sec. III we introduce our fractional Langevin model of memory, and elucidate the physical meaning of each term of the equation by first examining particular cases of it and deriving their solutions. We then proceed to solve the complete equation and interpret the absolute value of the solution as the quantity describing market activity. Finally we calculate the autocorrelation function of the latter variable and show how it can agree quite favorably with empirical data by a suitable choice of the fractional index. In Sec. IV we draw some conclusions and based on a physical interpretation of the model, comment on the possible underlying causes of long-range memory in market data.

II. FRACTIONAL STOCHASTIC EQUATIONS

We mentioned the possibility that the fractional calculus can be of value in describing the changes in fractal processes over time [19], and that the dynamics of market activity might be described by such a process. However we should be cognizant of the fact that there is not just one fractional calculus, rather there is a collection of fractional differentials and fractional integrals that have been found to reduce to the standard calculus when the appropriate fractional index becomes integer and the functions being acted upon have the specified properties. We use the Riemann-Liouville fractional operators, which are, by far, the most popular formalism among those that use the fractional calculus to describe complex phenomena, see, for example, Refs. [18,20].

A. Fractional random walk

We find that in order to model long-term memory in complex, nondifferentiable, phenomena we need to generalize the concept of differencing to include fractional values. In the same spirit as the random walk model, this approach to modeling long-time memory provides us with a conceptually straightforward mathematical representation of rather complex processes. This kind of random walk was introduced into economics by Hosking [21].

Let us define the discrete shift operator B such that its operation on a discrete data set Y, shifts the index by one unit,

$$BY_j = Y_{j-1}, \tag{2.1}$$

thereby shifting the data value to one unit of time earlier. A simple random walk can be written in terms of the shift operator as

$$(1-B)Y_j = \xi_j, \tag{2.2}$$

where each step size has the value ξ_j [22,23]. We generalize this simple random walk by considering the fractional difference equation

$$(1-B)^{\epsilon} Y_j = \xi_j, \tag{2.3}$$

where ϵ is not an integer.

Now we must find the proper interpretation of Eq. (2.3) and to do this we follow, in part, the discussion of the operator $(1-B)^{\epsilon}$ given by West [24], based on the work of Hosking [21]. The solution to the fractional difference equation (2.3) can be written as

$$Y_j = (1-B)^{-\epsilon}\xi, \tag{2.4}$$

which in terms of the binomial expansion, for $|\epsilon|<1$, becomes

$$\begin{aligned} Y_j &= \sum_{k=0}^{\infty} \binom{-\epsilon}{k} (-1)^k B^k \xi_j, \\ &= \sum_{k=0}^{\infty} \binom{-\epsilon}{k} (-1)^k \xi_{j-k}. \end{aligned} \tag{2.5}$$

The difference between Eq. (2.5) and the standard random walk is that the memory extends infinitely far back in time. In the ordinary random walk, where ϵ is an integer, the Γ functions have simple poles and the binomial coefficient vanishes after $\epsilon+1$ time steps, thereby cutting off the sum. In the present case this does not happen, and using some identities among Γ functions we obtain

$$\binom{-\epsilon}{k} = \frac{\Gamma(1-\epsilon)}{\Gamma(k+1)\Gamma(-\epsilon-k+1)} = (-1)^k \frac{\Gamma(k+\epsilon)}{\Gamma(k+1)\Gamma(\epsilon)}. \tag{2.6}$$

The solution to the fractional-difference stochastic equation (2.3) given by Eq. (2.5) clearly couples the present response of the system Y_j to fluctuations that occurred infinitely far back in time through ξ_{j-k} as $k\to\infty$. The size of the influence of these infinitely remote fluctuations is determined by the magnitude of the binomial coefficients, since these coefficients are essentially the coupling strengths of the fluctuations to the system. We can estimate the strength of the system-environment coupling using Stirling's approximation for Γ functions

$$\frac{\Gamma(k+\epsilon)}{\Gamma(k+\beta)} \propto k^{\epsilon-\beta}, \quad k \gg \epsilon, \beta. \tag{2.7}$$

so that the coupling strength in Eq. (2.6) becomes

$$(-1)^k \binom{-\epsilon}{k} \propto \frac{k^{\epsilon-1}}{\Gamma(\epsilon)} \tag{2.8}$$

for $k\to\infty$ since $k \gg |\epsilon|$. Thus, the strength of the contribution to Eq. (2.5) decreases with increasing time lag as an inverse power law as long as $|\epsilon|<1$.

We see from the infinite series representation of the fractional-difference process that, since Eq. (2.5) is linear, when the statistics of the ξ fluctuations are assumed to be Gaussian, so too are the statistics of the observed process. However, whereas the ξ spectrum is flat, characteristic of white noise, the Y spectrum is an inverse power law, characteristic of fractal stochastic processes. From these analytic results we conclude that the process defined by the fractional-difference stochastic equation is analogous to fractional Gaussian noise. The analogy is complete if we set $\epsilon = H-\frac{1}{2}$ so that the spectrum reads [21,25]

$$S(\omega) \approx \frac{1}{\omega^{2H-1}} \quad \text{as} \quad \omega \to 0. \tag{2.9}$$

In the language of random walks the inverse power law (2.9) for $1 \geqslant H > \frac{1}{2}$, or equivalently for $0 < \epsilon \leqslant \frac{1}{2}$, implies persistence. In the same way for $\frac{1}{2} \geqslant H > 0$, or equivalently for $-\frac{1}{2} \leqslant \epsilon < 0$, the spectrum increases as a power law in frequency and the process is antipersistent. In 1981 Hosking [21] recognized that fractional-difference processes exhibit long-term persistent and antipersistent behavior. Thus, the long-time memory that was *assumed* in the preceding section is here a consequence of the fractional dynamics describing the evolution of the process.

B. Fractional stochastic equations

Let us consider a continuum version of the fractional-difference stochastic equation (2.3),

$$D_t^{\alpha}[Y(t)] = \xi(t); \quad 0 < \alpha \leqslant 1. \tag{2.10}$$

The proper interpretation of this fractional stochastic equation, is actually an integral equation of the form

$$Y_{\alpha}(t) \equiv D_t^{-\alpha}[\xi(t)],$$

which can be written explicitly in terms of the Riemann-Liouville fractional integral [18,20]

$$Y_{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\xi(\tau)\, d\tau}{(t-\tau)^{1-\alpha}}. \tag{2.11}$$

Using the power-law index $H = \alpha - \frac{1}{2}$ we write Eq. (2.11) as

$$Y_H(t) = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-\tau)^{H-1/2} \xi(\tau)\, d\tau, \tag{2.12}$$

which is one choice for the continuum analog of the fractional-difference stochastic process. Note that our choice of Eq. (2.12) differs from the one made by Mandelbrot and van Ness [26] to describe fractional Brownian motion.

The distribution function is the same as that of the random force, since the integral operator (2.11) is linear. Thus, if $\xi(t)$ is a δ-function correlated Gaussian process, the system response will also be Gaussian, but with a variance given by

$$\sigma_H^2(t) = \sigma_H^2 t^{2H} \tag{2.13}$$

and

$$\sigma_H^2 \equiv \frac{\langle \xi^2 \rangle}{2H\Gamma(H+1/2)^2}. \tag{2.14}$$

Consequently, the statistics of the solution to the above fractional-differential stochastic equation, driven by a Wiener process, are Gaussian with a variance that increases as a power law in time. For $H > \frac{1}{2}$ these fluctuations diffuse faster than a normal diffusion process and are persistent. For $H < \frac{1}{2}$ the fluctuations diffuse slower than normal diffusion and are antipersistent. The fractional integral therefore transforms a Wiener process into an anomalous diffusion process [24,26]. Recall that $H = \alpha - \frac{1}{2}$ and $1 \geqslant \alpha > 0$ so that the above process is always antipersistent when resulting from the solution to a fractional-differential stochastic equation.

III. FRACTIONAL STATISTICS

The modeling of complex phenomena using a simple random walk model of normal diffusion leads to Gaussian statistics and a mean-square displacement that increases linearly with time. The most complex phenomena we modeled above involved the limit of fractional differences becoming fractional derivatives, so that a stochastic process with long-term memory can be generated by taking the fractional integral of a Wiener process. We saw that such processes have Gaussian statistics, but they also have inverse power-law spectra. The system response is therefore a fractal function with fractal dimension given by $D = 2 - H$ [24,27,28].

We now want to shift our focus from random walks to non differential or more accurately fractional differential, stochastic phenomena. We generalize the standard approach for modeling complex, statistical, physical phenomena, first presented by Langevin. The Langevin equation for the simple one-dimensional Brownian motion of a unit mass particle is

$$\frac{dv(t)}{dt} + \lambda v(t) = \xi(t). \tag{3.1}$$

This is often referred to as an Ornstein-Uhlenbeck process, due to its dependence on the dissipation parameter λ, and the fact that these two scientists gave the first complete mathematical description of the solution to this equation [29]. Physically the dissipation parameter is a consequence of the Stokes drag on Brown's pollen mote. The proper interpretation of Eq. (3.1) is not as a differential equation, but as an integral equation of the form

$$dv(t) + \lambda v(t)\, dt = dB(t) \tag{3.2}$$

where $dB(t)$ is a differential Wiener process. We now generalize Eq. (3.2) to account for nonlocal influences, that is, for the kind of relaxation that occurs in polymers and in viscoelastic materials [30]. In an economic context the analogs of those influences are war, unemployment, inflation rates, political scandals, and so on.

A. Fractional Langevin equation

Note that the Langevin equation (3.2) is phenomenological in character, so that it is reasonable, in the case of physical phenomena with memory, to replace Newton's force law with a fractional derivative of the velocity. Physically, this replacement means that the force is only defined on a fractal set of points. To ensure the physical reasonability of this model, the fractional force law ought to include a dependence on the initial velocity to ensure a proper interpretation of the initial value problem. In addition the dissipation parameter should have the appropriate scaled units. This line of approach has been taken [18,30] in physical systems, so the fractional Langevin equation is

$$D_t^{\alpha}[v(t)]-v_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)}=-\lambda^{\alpha}v(t)+\xi(t), \quad a\geqslant\alpha>0 \tag{3.3}$$

where $\xi(t)$ is, for the moment, chosen to be a Wiener process and the initial value for the process is given by v_0. The question is: *Is Eq. (3.3) a reasonable generalization of the usual Langevin equation given by Eq. (3.1) to provide a dynamical model of the temporal evolution of financial market activity*?

Here we adapt the above physical arguments to financial markets and write the fractional-dynamical equation of motion for the normalized number of trades in a given interval of time, $n(t)$. Of course, with this definition of the dynamical variable the initial value vanishes in Eq. (3.3), but we shall not use that fact for a while. We first of all examine the solutions to equations of the form (3.3).

1. Stochastic fractional differential equation — no dissipation

Before we work on solving the full fractional Langevin equation (3.3), let us look at a somewhat simpler version of this equation, one without dissipation,

$$D_t^{\alpha}[v(t)]-v_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)}=\xi(t), \tag{3.4}$$

and for the moment we disregard the fact that $v_0=0$ for the market variable. The solution to Eq. (3.4) can be written in terms of a fractional integral operator

$$v(t)-v_0=D_t^{-\alpha}[\xi(t)]. \tag{3.5}$$

We also know that the statistics of the solution to this equation are Gaussian when $\xi(t)$ is a Wiener process and the spectrum of the solution is an inverse power law, as we found in the preceding section. However, we did not explicitly calculate the correlation properties of the system response. Let us now evaluate the two-point correlation function using the formal properties of the Riemann-Liouville fractional integral to obtain

$$\langle[v(t_1)-v_0][v(t_2)-v_0]\rangle =\frac{1}{\Gamma(\alpha)^2}\int_0^{t_1}dt_1'\int_0^{t_2}dt_2'\frac{\langle\xi(t_1')\xi(t_2')\rangle}{(t_1-t_1')^{1-\alpha}(t_2-t_2')^{1-\alpha}}, \tag{3.6}$$

where the fluctuations are assumed to be δ-function correlated in time

$$\langle\xi(t_1')\xi(t_2')\rangle=2D\,\delta(t_1'-t_2'). \tag{3.7}$$

The integral (3.6) is completely symmetric in the times t_1 and t_2, but we know that the δ function will restrict the integration to the earlier of the two times, since this is where both variables can be equal. Therefore, we introduce the notation $t_>$ for the greater time and $t_<$ for the lesser time, and implementing the δ function (3.7) under the integral (3.6) yields

$$\langle[v(t_>)-v_0][v(t_<)-v_0]\rangle=\frac{4D}{\Gamma(\alpha)^2}\int_0^{t_<}dt(t_>-t)^{\alpha-1}\times(t_<-t)^{\alpha-1}. \tag{3.8}$$

Introducing the normalized variable $\zeta=t/t_<$ we obtain, after some algebra,

$$\langle[v(t_>)-v_0][v(t_<)-v_0]\rangle=\frac{4Dt_>^{\alpha-1}t_<^{\alpha}}{\Gamma(\alpha)^2}\int_0^1 d\zeta\left(1-\frac{t_<}{t_>}\zeta\right)^{\alpha-1}(1-\zeta)^{\alpha-1}. \tag{3.9}$$

Using the following integral representation of the *hypergeometric function*, see for example Miller and Ross (page 304) [20] or Ref. [31]

$${}_2F_1(a;b;c{:}z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 d\zeta\zeta^{\alpha-1}(1-\zeta)^{c-\alpha-1}\times(1-\zeta z)^{-b}, \tag{3.10}$$

where $\mathrm{Re}\,c>\mathrm{Re}\,a>0$, and equating coefficients in Eq. (3.10) with the terms in Eq. (3.9) we obtain

$$\langle[v(t_>)-v_0][v(t_<)-v_0]\rangle=\frac{4Dt_>^{\alpha-1}t_<^{\alpha}}{\alpha\Gamma(\alpha)^2}F\left(1;1-\alpha;1+\alpha{:}\frac{t_<}{t_>}\right), \tag{3.11}$$

where we have suppressed the suffixes on the hypergeometric function and which is only valid for $\alpha>0.5$ when $t_>=t_<$. Note that the statistics of the solution to Eq. (3.4) are nonstationary, since the correlation function depends on $t_>$ and $t_<$ separately, and not just on the difference $t_>-t_<$.

We know, from the linear nature of the differentiation procedure, that the statistics of the fractional-dynamical process described by Eq. (3.4) would be Gaussian, if the ξ fluctuations are assumed to have Gaussian statistics. On the other hand, we know from the data analysis of Gopikrishnan *et al.* [32], among others, that the dynamical financial variables do not have Gaussian statistics. In fact, market activity has a PDF with power-law tails [3], whereas the volatility is log-normal in the central region with an inverse power-law tail [6] and market activity also has inverse power-law time correlations with exponent $\nu=0.85\pm0.01$ [9,15].

Therefore we do not make the assumption that the ξ fluctuations are Gaussian, but we still assume them to have a δ-function correlated character. In this way, if we identify the

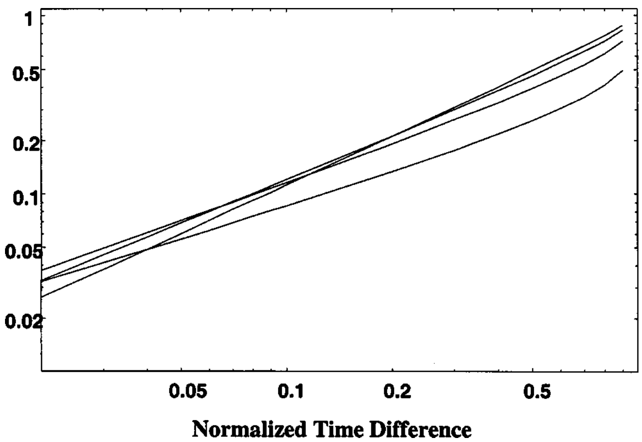

FIG. 1. We graph the logarithm of the autocorrelation function from Eq. (3.16) versus the logarithm of z for power-law indices $\alpha = 0.6$, 0.7, 0.8, and 0.9.

system variable with the normalized number of trades and introduce the normalized variable $z = t_< / t_>$, we obtain

$$\langle n(t_>)n(t_<)\rangle = \frac{4Dt_>^{2\alpha-1}}{\alpha\Gamma(\alpha)^2} z^\alpha F(1;1-\alpha;1+\alpha:z). \tag{3.12}$$

Of course, we can also use Eq. (3.11) to write the second moment at time $t = t_> = t_<$

$$\langle n(t)^2\rangle = \frac{4Dt^{2\alpha-1}}{\alpha\Gamma(\alpha)^2} F(1;1-\alpha;1+\alpha:1) \tag{3.13}$$

and using [31]

$$F(a;b;c:1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{3.14}$$

provided Re $c >$ Re$[a+b]$ and c is not a nonpositive integer, we obtain for the second moment

$$\langle n(t)^2\rangle = \frac{4D}{(2\alpha-1)\Gamma(\alpha)^2} t^{2\alpha-1}. \tag{3.15}$$

This result, Eq. (3.15), agrees with that obtained for anomalous diffusion, if we make the identification $H = \alpha - \frac{1}{2}$, but this identification can only be made for $1 > \alpha \geqslant 1/2$ in order to satisfy the condition on the hypergeometric function. In this case we have $\frac{1}{2} \geqslant H \geqslant 0$, corresponding to an antipersistent process. In Fig. 1 we graph the autocorrelation function as a function of the normalized variable for a variety of fractional-differential indices α,

$$C_\alpha(t_>, t_<) \equiv \frac{\langle n(t_>)n(t_<)\rangle}{\langle n(t_> + T)^2\rangle} = z^\alpha \frac{F(1;1-\alpha;1+\alpha:z)}{F(1;1-\alpha;1+\alpha:1)}. \tag{3.16}$$

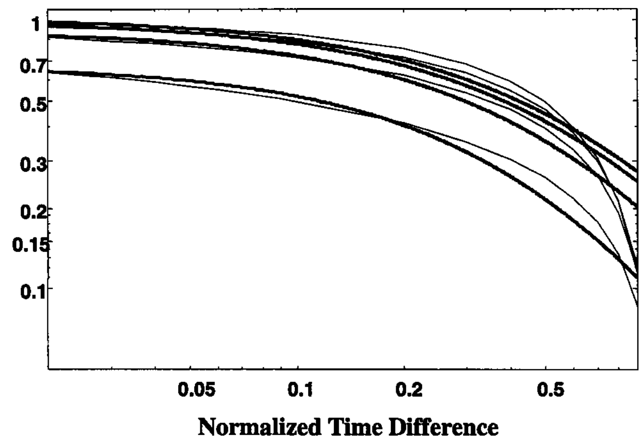

FIG. 2. The auto-correlation function for $\alpha = 0.6$, 0.7, 0.8, and 0.9 is fit with the empirical equation (3.18) using a least-squares fit to the parameters. The entire range of the data was used in the fitting.

We can see from the figure that each curve has a dominant power-law form, but with differing slopes. So we can write the empirical relation

$$C_\alpha(z) \propto z^\mu. \tag{3.17}$$

The exponent μ is a function of the fractional derivative parameter α, and we obtain $\mu \geqslant 1.5$ for $0.6 \leqslant \alpha \leqslant 0.9$.

However, data that have been processed and published in the literature are in a form that can be directly compared with the scaling result (3.17). Therefore we introduce $t_< = t_> - \tau$ and $t = t_>$ into the autocorrelation function and graph the resulting function versus the normalized time separation variable τ/t in Fig. 2. What we find is

$$C_\alpha(\tau,t) = \left(1 - \frac{\tau}{t}\right)^\alpha \frac{F\left(1;1-\alpha;1+\alpha:1-\frac{\tau}{t}\right)}{F(1;1-\alpha;1+\alpha:1)} \approx \frac{A(\alpha)}{\left(1+\frac{\tau}{t}\right)^{B(\alpha)}}, \tag{3.18}$$

where the empirical parameters A and B are functions of the fractional-derivative index α and are obtained by a least-squares fit of the indicated phenomenological equation to the autocorrelation function.

The values of the parameters in Eq. (3.18) for each value of α obtained by least-squares fitting the entire range of the correlation function is depicted in Fig. 2. The parameter values obtained by fitting the data are recorded in Table I. The values of the empirical power-law index recorded in Table I are fit to the linear equation in α,

$$B(\alpha) = -2.52 + 6.78\alpha. \tag{3.19}$$

TABLE I. The least-squares fit of the parameters in the phenomenological equation (3.18). A and B are recorded for each of the values of the order of the fractional derivative.

α	A	B
0.6	0.67	1.53
0.7	0.95	2.22
0.8	1.00	3.03
0.9	1.01	3.50

We can see that the range for the power-law index is $0.87 \leqslant B(\alpha) \leqslant 4.26$ for the fractional-derivative index interval $0.5 \leqslant \alpha \leqslant 1$. By comparing Eq. (3.19) with the data fit [32], should have

$$-2.52 + 6.78\alpha = 0.3,$$

which implies $\alpha = 0.42$. However, this value of the fractional-derivative index is outside the range of validity of our solution. Thus, the present model of the fractional Langevin equation does not satisfactorily describe the central moment properties of the price fluctuations in financial markets. However, it should be noted that we have not taken into account market retardation forces, that is, dissipation. We now turn our attention to the modeling of such forces.

2. Fractional differential equation—no fluctuations

Let us examine the solution to the homogeneous fractional differential equation, and once we understand that solution, consider the inhomogeneous case. The homogenous fractional Langevin equation does not contain ''thermal'' fluctuations,

$$D_t^{\alpha}[v(t)] - v_0 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} = -\lambda^{\alpha} v(t). \tag{3.20}$$

Equation (3.20) is mathematically well defined, but what does it mean physically? From statistical physics we know that the fluctuations in the equation of motion are intimately related to the dissipation, and that in fact they have the same source. This is what gives rise to the fluctuation-dissipation relation, relating the strength of the fluctuations to the ratio of the temperature to the dissipation parameter. However, in Eq. (3.20) we have a dissipation without a corresponding set of fluctuations. Since all the operators in Eq. (3.20) are linear we could interpret this equation in terms of the average velocity.

For now we treat Eq. (3.20) as a mathematical expression with the initial velocity given by v_0, the time dependence is included so as to have a well-defined initial value problem and the dissipation parameter is appropriately scaled to have the units corresponding to the order of the fractional derivative. The solution to this equation is obtained from the corresponding fractional integral equation

$$v(t) - v_0 = -\lambda^{\alpha} D_t^{-\alpha}[v(t)] \tag{3.21}$$

by taking the Laplace transform of the dynamical variable. Denoting the Laplace transform of a variable by a tilde over the function, $\tilde{v}(s) \equiv \mathcal{T}_L\{v(t);s\}$, we obtain after some algebra, from Eq. (3.21)

$$\tilde{v}(s) = \frac{v_0 s^{\alpha-1}}{\lambda^{\alpha} + s^{\alpha}}, \tag{3.22}$$

where s is the Laplace variable conjugate to the time. The inverse Laplace transform of the right-hand side of Eq. (3.22) is

$$\mathcal{T}_L^{-1}\left\{\frac{s^{\alpha-1}}{\lambda^{\alpha}+s^{\alpha}};t\right\} = \int_{C-i\infty}^{C+i\infty} e^{st} \frac{s^{\alpha-1} ds}{\lambda^{\alpha}+s^{\alpha}} \tag{3.23}$$

which we calculate using Fox functions to give [30,18]

$$v(t) = v_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k\alpha)} (\lambda t)^{k\alpha}. \tag{3.24}$$

The solution to the homogeneous fractional-differential equation is therefore given by the Mittag-Leffler function

$$v(t) = v_0 E_{\alpha}(-(\lambda t)^{\alpha}). \tag{3.25}$$

Thus, the fundamental process is not that of an exponential relaxation, as it is for the Ornstein-Uhlenbeck process; rather the relaxation properties are determined by the asymptotic properties of the Mittag-Leffler function. At early times it is not difficult to show that the Mittag-Leffler function has the form of a stretched exponential [30]

$$\lim_{t\to 0} E_{\alpha}(-(\lambda t)^{\alpha}) \approx e^{-(\lambda t)^{\alpha}}. \tag{3.26}$$

At late times it is also not difficult to show that the Mittag-Leffler function has the form of an inverse power law [30]

$$\lim_{t\to\infty} E_{\alpha}(-(\lambda t)^{\alpha}) \approx (\lambda t)^{-\alpha}. \tag{3.27}$$

The transition time between the two relaxation domains, stretched exponential, and inverse power law is determined by the parameter, λ.

3. Stochastic fractional-differential equation—with dissipation

Let us now look at the solution to the complete fractional Langevin equation. Again we begin by replacing this equation, Eq. (3.3), with the equivalent fractional integral equation

$$v(t) - v_0 = -\lambda^{\alpha} D_t^{-\alpha}[v(t)] + D_t^{-\alpha}[\xi(t)]. \tag{3.28}$$

The Laplace transform of this equation yields after some algebra

$$\tilde{v}(s) = \frac{v_0 s^{\alpha-1}}{\lambda^{\alpha}+s^{\alpha}} - \frac{\tilde{\xi}(s)}{\lambda^{\alpha}+s^{\alpha}}. \tag{3.29}$$

We note the difference in the s dependence of the two coefficients on the right-hand side of Eq. (3.29). The inverse

Laplace transform of the first term on the right-hand side of Eq. (3.29) is the Mittag-Leffler function that we found in the homogeneous case. The inverse Laplace transform of the second term is the convolution of the random fluctuations and a stationary kernel. The kernel is given in terms of a Fox function [18]

$$T_L^{-1}\left\{\frac{1}{\lambda^\alpha+s^\alpha}:t\right\}=\frac{(\lambda t)^{\alpha-1}}{\alpha}H_{12}^{11}\left(\lambda t\middle|\begin{matrix}(0,1/\alpha)\\(0,1/\alpha),(1-\alpha,1)\end{matrix}\right) \tag{3.30}$$

as we obtained for the second term in Eq. (3.29). The series expansion for this Fox function can be written as [30]

$$\frac{1}{\alpha}H_{12}^{11}\left(\lambda t\middle|\begin{matrix}(0,1/\alpha)\\(0,1/\alpha),(1-\alpha,1)\end{matrix}\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha+k\alpha)}(\lambda t)^{k\alpha}, \tag{3.31}$$

where the series is a representation of the generalized Mittag-Leffler function, and is defined in general by

$$E_{\alpha,\beta}(z)\equiv\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\quad \alpha>0,\beta>0. \tag{3.32}$$

The generalized Mittag-Leffler function reduces to the familiar form for $\beta=1$,

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}=E_\alpha(z), \tag{3.33}$$

so that both the homogeneous and inhomogeneous terms in the solution to Eq. (3.28) can be expressed in terms of Mittag-Leffler functions.

We now write the general solution to the fractional Langevin equation, using the inverse Laplace transform of Eq. (3.29), as

$$v(t)=v_0E_\alpha(-(\lambda t)^\alpha)+\int_0^t(t-t')^{\alpha-1}\times E_{\alpha,\alpha}(-(\lambda[t-t']^\alpha)\xi(t')dt'. \tag{3.34}$$

This result was obtained by Kobelev and Romanov [33] using standard techniques for solving Volterra integral equations. In the case $\alpha=1$, the Mittag-Leffler function becomes an exponential, so that the solution to the fractional Langevin equation becomes identical to that for an Ornstein-Uhlenbeck process

$$v(t)=v_0e^{-\lambda t}+\int_0^t e^{-\lambda(t-t')}\xi(t')dt' \tag{3.35}$$

as it should.

B. Market activity as a fractal processes

The traditional quantities calculated from the normalized number of trades time series are the autocorrelation function and the standard deviation of the time series. The latter is often referred to as the market activity as we mentioned [9]. We can calculate these quantities using the solution to the fractional Langevin equation (3.34). The autocorrelation function is

$$\langle n(t_1)n(t_2)\rangle=\int_0^t dt_1'(t_1-t_1')^{\alpha-1}\int_0^{t_2}dt_2'(t_2-t_2')^{\alpha-1}\times E_{\alpha,\alpha}(-\lambda^\alpha(t_1-t_1')^\alpha)E_{\alpha,\alpha}(-\lambda^\alpha(t_2-t_2')^\alpha)\times\langle\xi(t_1')\xi(t_2')\rangle, \tag{3.36}$$

where the meaning of this equation is tied to the statistics of the random force driving the system and we have set the initial value to zero as it would be for a financial market variable. The traditional assumption is that the random fluctuations have Gaussian statistics and no memory, that is, they are δ-function correlated in time, see Eq. (3.7), and D is the strength of the fluctuations. We make the latter assumption here, but not the former; that is, we assume δ-function correlated fluctuations, but we do not specify the statistics.

1. Evaluating the integral term

Here again we observe that the correlation integral is completely symmetric in the times t_1 and t_2, so that introducing the greater and lesser times, $t_>$ and $t_<$, and implementing the δ function, the integral term in Eq. (3.36) reduces to

$$I=2\int_0^{t_<}dt(t_>-t)^{\alpha-1}(t_<-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda^\alpha(t_>-t)^\alpha)\times E_{\alpha,\alpha}(-\lambda^\alpha(t_<-t)^\alpha). \tag{3.37}$$

Making use of the series expression for the generalized Mittag-Leffler function in Eq. (3.37) and changing the initial value on the sums yields

$$I=2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^\alpha)^{k+l-2}}{\Gamma(k\alpha)\Gamma(l\alpha)}I_{kl}, \tag{3.38}$$

where we have introduced the integral

$$I_{kl}=\int_0^{t_<}dt(t_>-t)^{k\alpha-1}(t_<-t)^{l\alpha-1}. \tag{3.39}$$

Factoring the times $t_>$ and $t_<$ out of the integral and introducing the scaled variable $\zeta=t/t_<$ allows us to write

$$I_{kl}=t_>^{k\alpha-1}t_<^{l\alpha}\int_0^1 d\zeta\left(1-\zeta\frac{t_<}{t_>}\right)^{k\alpha-1}(1-\zeta)^{l\alpha-1}, \tag{3.40}$$

so that we can again use the integral representation of the hypergeometric function, Eq. (3.10), to evaluate this integral as

$$I_{kl}=t_>^{k\alpha-1}t_<^{l\alpha}\frac{\Gamma(l\alpha)}{\Gamma(l\alpha+1)}F\left(1;1-k\alpha;1+l\alpha:\frac{t_<}{t_>}\right). \tag{3.41}$$

Thus, the integral term in the trade autocorrelation function becomes

$$I=2\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^{\alpha})^{k+l-2}t_{<}^{k\alpha-1}t_{<}^{l\alpha}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}F\left(1;1-k\alpha;1+l\alpha:\frac{t_{<}}{t_{>}}\right) \tag{3.42}$$

and the entire autocorrelation function is

$$\langle n(t_{>})n(t_{<})\rangle=4D\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^{\alpha})^{k+l-2}t_{>}^{k\alpha-1}t_{<}^{l\alpha}}{\Gamma(k\alpha)\Gamma(l\alpha+1)} \times F\left(1;1-k\alpha;1+l\alpha:\frac{t_{<}}{t_{>}}\right), \tag{3.43}$$

which clearly, is a nonstationary result, due to the dependence on both times, independently of one another. There is not much more that we can do analytically with Eq. (3.43) due to its generality; let us therefore simplify the expression somewhat.

2. The time dependence of the market activity

The second moment of the dynamical variable is obtained by setting $t_{>}=t_{<}=t$ in Eq. (3.43) to yield

$$\langle n(t)^2\rangle=4D\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^{\alpha})^{k+l-2}t^{(k+l)\alpha-1}}{\Gamma(k\alpha)\Gamma(l\alpha+1)} \times F(1;1-k\alpha;1+l\alpha:1), \tag{3.44}$$

where we can use Eq. (3.14) to replace the hypergeometric function by ratios of Γ functions. After some cancellation of terms, Eq. (3.44) reduces to

$$\langle n(t)^2\rangle=4D\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^{\alpha})^{k+l-2}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}t^{(k+l)\alpha-1}\frac{l\alpha}{l\alpha+k\alpha-1},$$

where, if the second term on the right-hand side of this equation is denoted by $\mathcal{I}$, we can write [33]

$$\frac{d\mathcal{I}}{dt}=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-\lambda^{\alpha})^{k+l-2}}{\Gamma(k\alpha)\Gamma(l\alpha)}t^{(k+1)\alpha-2},$$

which clearly integrates to

$$\mathcal{I}=\int_0^t\left[\sum_{k=1}^{\infty}\frac{(-\lambda^{\alpha}t'^{\alpha})^{k}}{\Gamma(k\alpha)}\right]^2\frac{dt'}{t'^2}.$$

We can also take the derivative of the Mittag-Leffler function

$$\frac{dE_{\alpha}(-\lambda t)^{\alpha})}{dt}=\sum_{k=1}^{\infty}\frac{(-\lambda^{\alpha}t^{\alpha})^{k}}{\Gamma(k\alpha)}\frac{1}{t}, \tag{3.45}$$

where the $k=0$ term vanishes due to the pole of the Γ function, so that the second moment of the velocity can be rewritten [33]

$$\langle n(t)^2\rangle=4D\int_0^t\left[\frac{dE_{\alpha}(-(\lambda t')^{\alpha})}{dt'}\right]^2dt'. \tag{3.46}$$

We can determine the early time properties of the second moment in Eq. (3.46), by keeping the lowest-order term in the series expansion (3.45). Thus, the leading terms in the early time analysis of the market activity as measured by the standard deviation is

$$\lim_{t\to 0}\sqrt{\langle n(t)^2\rangle}\approx\frac{2}{\Gamma(\alpha)}\sqrt{\frac{D}{2\alpha-1}}(\lambda t)^{\alpha-1/2}. \tag{3.47}$$

This is consistent with the results obtained earlier, see Eqs. (3.13)–(3.15).

3. The autocorrelation function

We can use the results of the Secs. III B 2 and III B 3 to define the autocorrelation function as the ratio of Eq. (3.43) to Eq. (3.44) with $t_{>}=t$ and $t_{<}=t-\tau$

$$C_{\alpha}(\tau,t)=\frac{\langle n(t)n(t-\tau)\rangle}{\langle n(t)^2\rangle}. \tag{3.48}$$

Here again we can plot the autocorrelation function versus the dimensionless time difference, $\eta=\tau/t$, to see the dependence of this quantity on the fractional derivative index. The form of the auto-correlation function is

$$C(\tau,t)=\frac{\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l-2}(\lambda t)^{k\alpha+l\alpha-1}(1-\eta)^{l\alpha}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}F(1{:}1-k\alpha;1+l\alpha{:}1-\eta)}{\displaystyle\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-|\lambda t|^{\alpha})^{k+l-2}}{\Gamma(k\alpha)\Gamma(l\alpha+1)}F(1{:}1-k\alpha{:}1+l\alpha{:}1)}, \tag{3.49}$$

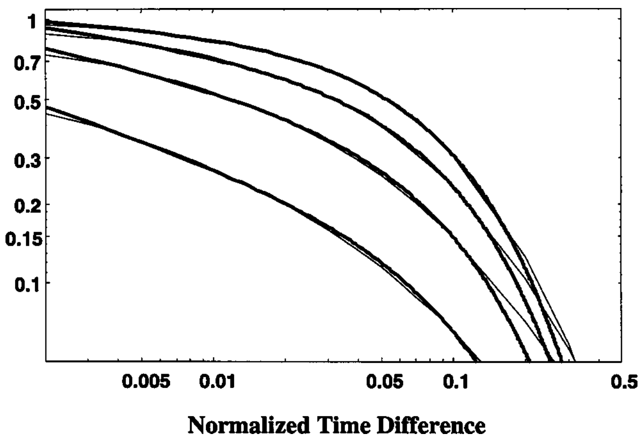

FIG. 3. The auto-correlation function (3.48) is plotted versus the dimensionless time interval τ/t on log-log graph paper and a least-squares fit to all the data using the phenomenological equation (3.50). The fits for the values of the fractional derivative index $\alpha = 0.9$, 0.8, 0.07, and 0.6 are shown.

which does not have the apparently simple form observed earlier.

In Fig. 3 we observe the decrease in the autocorrelation function. We attempted to fit this decrease, as we did earlier, with a simple inverse power-law equation, using $\lambda t = 10$, but this yielded power-law indices greater than 10. As an alternative, we decided on the phenomenological equation

$$C_\alpha(\eta) = \frac{A}{\eta^B} \exp|-\eta C|, \tag{3.50}$$

where the empirical parameters A, B and C are functions of the fractional-derivative index α. The parameters in Eq. (3.50) are obtained by a least-squares fit to the autocorrelation function (3.49), using $\lambda t = 10$ and restricting the range on the sums to $1 \leqslant k, l \leqslant 50$. The values of the parameters for four values of α are recorded in Table II using all the data to fit the coefficients. Again we can use the data [32] to fit the inverse power law and from this we find $\alpha = 0.05$ which is again outside the domain of our solution.

The problem is that we are fitting the theoretical correlation function over its entire domain, whereas the correlation function for the data is only fit over very early times. It is no wonder that the parameters we obtain from such a fit do not match with those obtained from the data. Therefore let us reduce the domain over which we fit the data. Using the above least-square fit, we write for a fixed length time series the autocorrelation function in the interval $0.001t \leqslant \tau \leqslant 0.02t$ as

$$C_\alpha(\eta) = \frac{A}{\eta^B}, \tag{3.51}$$

where again A and B are functions of α. Here we employ a lower limit on the range of the correlation function, as well as an upper limit, to simulate the discrete nature of the financial data. In Table III we record the values for the parameters obtained using Eq. (3.51) as the fitting function. In Fig. 4 we depict the fit of Eq. (3.51) to the theoretical correlation function.

Here we can use the data analysis of Gopikrishnan *et al.* [32] for the correlation function of the absolute value of the price returns in their Fig. 3(b) or that of Liu *et al.* [6] in their Fig. 8(b). By comparing the inverse power-law indices with those in their figures, we obtain

$$0.81 - 0.84\alpha = 0.30 \pm 0.08, \tag{3.52}$$

indicating a fractional-derivative index of $\alpha = 0.60 \pm 0.10$. This index is consistent with the fractional Langevin model and with an antipersistent random walk interpretation with $H = 0.40 \pm 0.10$. Further, using a Tauberian theorem, we conclude that the high-frequency form of the spectrum is given by the inverse power law

TABLE II. The least-squares fit of the parameters in the phenomenological equation (3.50). A, B and C to all the data are recorded for each value of the order of the fractional derivative.

α	A	B	C
0.6	0.23	0.12	23.42
0.7	0.52	0.06	17.48
0.8	0.76	0.03	14.00
0.9	0.91	0.01	11.60

TABLE III. The least-squares fit of the parameters in the phenomenological equation (3.51). A and B to all the data are recorded for each value of the order of the fractional derivative.

α	A	B
0.6	0.077	0.278
0.7	0.20	0.209
0.8	0.358	0.147
0.9	0.509	0.102

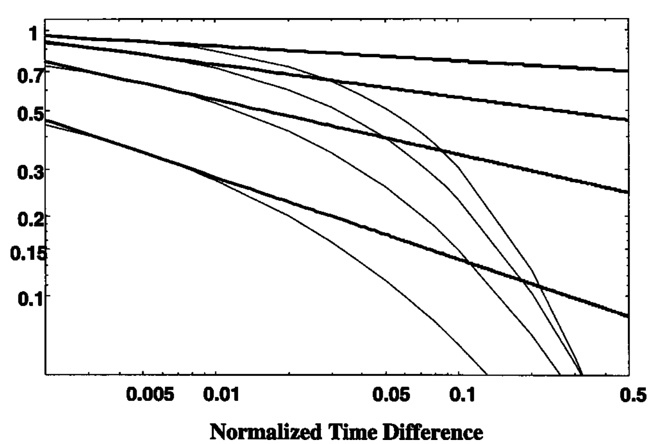

FIG. 4. The auto-correlation function (3.48) is plotted versus the dimensionless time interval τ/t on log-log graph paper and a least-squares fit to the time interval $(0.001t, 0.02t)$ with the phenomenological equation (3.50). Only the values $\alpha = 0.9$, 0.8, 0.07, and 0.6 are shown.

$$\mathcal{S}_\alpha(\omega) \propto \frac{1}{\omega^{0.19+0.84\alpha}} \tag{3.53}$$

as long as $\alpha < 1$.

We point out that we have assumed the equivalence of our theoretical autocorrelation function and that calculated by Gopkirishnan *et al.* [32] which uses the absolute values of the price fluctuations.

IV. CONCLUSIONS

We have presented here a model of long-range memory in market data. Empirical studies have identified market activity, as measured by the number of trades per unit time, as the likely candidate for the macroscopic stochastic process from which long-term memory originates and, mediated by market impact, is then transferred down to certain nonlinear functions of price fluctuations, including measures of volatility. This explains our choice of market activity as the stochastic variable to be modeled. We hypothesized that market activity constitutes a macroscopic stochastic process driven by a microscopic random noise representing the flow of information available to agents. No further hypothesis is present in the model, as no temporal correlations were assumed in the noise term.

We have shown that a fractional-differential operator can couple a short-term memory process to a long-term memory one and that the exponent found in the empirical power-law can be successfully reproduced by suitably adjusting the index of the fractional derivative. However, the questions what is the physical meaning of the model and how can its results be interpreted in economic terms remain.

In physical systems, the emergence of power laws is generally associated with an underlying scale invariance, so it is tempting to put forth the working hypothesis that the same principle may be at work in economic systems as well [34–36]. In particular, power-law behavior arises in physical processes in which a constant, or nearly constant energy input is stored in a system and then released in "fits and starts," that is, in an intermittent fashion, thus exhibiting an alternation of periods of low to moderate activity interspersed with intense and sudden "bursts" of activity with no preferential time scale involved. Examples of such processes are earthquakes [37], rainfalls [38], turbulent fluid flow [39], relaxation of stress in viscoelastic materials [30], microcrack propagation [40], and other processes with stick-slip dynamics. The "economic analog" of the above picture would be the nearly steady input of information which builds up in the market until the different pieces come together to form, in the agents' mind, a sufficiently coherent signal, thus rousing them to action.

The interpretation of the flow of information as the driving energy source in an economic context is not new. It was advanced a few years ago [41] to suggest possible similarities between the price-formation process and the energy cascade in turbulence [39]. However, our model has its point of departure with the observation of the lack of separation of the scales pertaining to the information flow as a microscopic process and those relative to an empirically observable macroscopic process, such as market activity. All analogies with other physical systems are here invoked *a posteriori* to interpret the model's solution. Moreover, the same equation can describe slightly different power laws observed in different assets, by simply adjusting the corresponding fractional index, without modifying the driving noise terms, which would be desirable, as presumably different stocks may be affected differently by the same pieces of information with common temporal correlations. We had previously used an identical model to describe long-term memory effects exhibited by the volatility [42], whose empirical autocorrelation function decays as a power law with a different exponent [43-45]. However, we believe the connection between information flow and trading activity to be more direct, thus rendering the "physical" interpretation of the model more transparent.

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