Field-theoretic renormalization group for a nonlinear diffusion equation

N. V. Antonov[1] and Juha Honkonen[2]
[1]*Department of Theoretical Physics, St. Petersburg University, Uljanovskaja 1, St. Petersburg, Petrodvorez 198504, Russia*
[2]*Theory Division, Department of Physical Sciences, FIN-00014 University of Helsinki, Finland*
(Received 8 July 2002; published 10 October 2002)

This paper is an attempt to relate two vast areas of the applicability of the renormalization group (RG): field-theoretic models and partial differential equations. It is shown that the Green function of a nonlinear diffusion equation can be viewed as a correlation function in a field-theoretic model with an ultralocal term, concentrated at a space-time point. This field theory is shown to be multiplicatively renormalizable, so that the RG equations can be derived in a standard fashion, and the RG functions (the β function and anomalous dimensions) can be calculated within a controlled approximation. A direct calculation carried out in the two-loop approximation for the nonlinearity of the form ϕ^{α}, where $\alpha>1$ is not necessarily integer, confirms the validity and self-consistency of the approach. The explicit self-similar solution is obtained for the infrared asymptotic region, with exactly known exponents; its range of validity and relationship to previous treatments are briefly discussed.

DOI: 10.1103/PhysRevE.66.046105 PACS number(s): 05.10.Cc, 11.10.Gh, 02.30.Jr

I. INTRODUCTION

The renormalization group (RG) has proved to be the most efficient tool for studying self-similar scaling behavior. This tool first appeared within the context of quantum field theory [1], and was then successfully applied to a variety of problems as disparate as phase transitions, polymer dilutes, random walks, hydrodynamical turbulence, growth processes, and so on. See, e.g., the monographs [2,3], the proceedings [4], and references therein.

The most powerful and well-developed formulation of the RG is the field-theoretic one, see Refs. [1–3]. It is this version of the RG that is simplest and most convenient in practical calculations, especially in higher orders. It is also important that it has a reliable basis in the form of quantum-field renormalization theory, including the renormalization of composite operators and operator product expansion. For this reason, the first step in the RG analysis of a given problem is to reformulate it as a field-theoretic model. This means that the quantities under study should be represented as functional averages with the weight $\exp S(\phi)$, where ϕ is a classical random field (or set of fields) and $S(\phi)$ is a certain action functional. For parabolic differential equations with an additive random source, such a formulation is provided by the well-known Martin-Siggia-Rose (MSR) formalism, see Refs. [5,6]. In problems involving fluctuation effects in chemical reactions, the somewhat more complicated approach of Doi [7] (see also Refs. [8,9]) has also been widely used [10–12]. No general recipe, however, seems to exist to cast a nonlinear problem to a field-theoretic form.

Such a reformulation, however, is by no means superfluous: once the field-theoretic formulation has been found, it becomes possible to apply standard tools (power counting of the one-irreducible correlation functions, etc.) to verify the renormalizability of the model, i.e., the applicability of the RG technique, to derive corresponding RG equations, and to calculate its coefficients (β functions and anomalous dimensions) within controlled approximations. An instructive example is provided by the model of the so-called true self-avoiding random walks [13–15]. After its field-theoretic formulation had been found [14], it became clear that the model in its original formulation was not renormalizable, and the direct application of the RG to it would lead to completely erroneous results. The renormalizable version of the model can be obtained by adding infinitely many terms to the original action, see Ref. [15].

It has long been known, however, that symmetries of the RG-type also appear in various physical problems described by ordinary or partial differential equations and integro-differential equations, whose solutions exhibit self-similar scaling behavior [16]. Since then, the list of such problems has been essentially increasing; see, e.g., Refs. [17–24] and references therein. As a rule, the field-theoretic formulation for these models does not exist (or, at least, is not known), and the derivation of the corresponding RG equations is a nontrivial task. Quoting the authors of Ref. [24], "the procedure of revealing RG transformations . . . in any partial case . . . up to now is not a regular one. In practice, it needs some imagination and atypical manipulation 'invented' for every particular case." In Ref. [24], a general approach was proposed to construct RG symmetries for certain classes of partial differential equations, but its relationship to the field-theoretic RG techniques is not clear.

The present paper is an attempt to "bridge the gap" between these two vast areas of the applicability of the RG: field-theoretic models and partial differential equations. To be specific, we shall consider nonlinear diffusion equation of the form

$$\partial_t \phi = \nu_0 \partial^2 \phi + V(\phi), \tag{1.1}$$

where $\phi(x) \equiv \phi(t,\mathbf{x})$ is a scalar field, ν_0 is the diffusion coefficient, ∂^2 is the Laplace operator, and $V(\phi)$ is some nonlinearity dependent on the field ϕ and its spatial derivatives. Within the RG context, various special examples of Eq. (1.1) were studied earlier in Refs. [18–22]. In practical calculations, we shall confine ourselves to the nonlinearity of the form $V(\phi)=-\lambda_0\phi^\alpha$, where $\alpha>1$ is not necessarily an integer.

We shall show that the problem (1.1) can be cast into a field-theoretic model and apply the standard RG formalism to it to establish the scaling behavior and to calculate corresponding anomalous dimensions. Then we shall discuss the range of applicability of the results obtained and their relationship to the previous RG treatments of the model.

II. FIELD-THEORETIC FORMULATION AND RENORMALIZATION OF THE PROBLEM

We begin the analysis of the Cauchy problem (1.1) with a localized initial condition that corresponds to the equation

$$\partial_t G=\nu_0\partial^2 G+V(G)+\delta(x-x_0) \tag{2.1}$$

for the Green function $G(x|e_0)$. It will be shown later that the large-scale asymptotic behavior of this problem survives for all integrable initial conditions [i.e., such that $\int d\mathbf{x}\phi(0,\mathbf{x})$ converges]. In Eq. (2.1) we denote $\delta(x-x_0)\equiv\delta(t-t_0)\delta^{(d)}(\mathbf{x}-\mathbf{x}_0)$, where d is the dimensionality of the $\mathbf{x}$ space, and $e_0=\{x_0,\nu_0,\lambda_0\}$ is the full set of parameters.

The functional derivation of the MSR formalism [6] can be adopted to represent the solution of Eq. (2.1) as a functional integral over the doubled set of fields, ϕ and ϕ':

$$G(x|e_0)=\int\mathcal{D}\phi'\int\mathcal{D}\phi\,\phi(x)\exp[S(\phi',\phi)+\phi'(x_0)]. \tag{2.2}$$

Here the normalization constant is included into the differential $\mathcal{D}\phi'\mathcal{D}\phi$, the action functional has the form

$$S(\phi',\phi)=\int dx\,\phi'(x)\{-\partial_t\phi(x)+\nu_0\partial^2\phi(x)+V(\phi(x))\}, \tag{2.3}$$

with $dx=dt\,d\mathbf{x}$. The last term in Eq. (2.1) can be treated as an addition to the "interaction" $V(\phi)$ and gives rise to the last term in the exponential of Eq. (2.2). The term quadratic in ϕ', typical to the MSR actions, is absent in Eq. (2.3) owing to the absence of the random force in Eq. (1.1).

Representation (2.2) shows that the Green function (2.1) can be viewed as the correlation function $\langle\phi(x)\exp\phi'(x_0)\rangle$ in the field-theoretic model with the action (2.3). It is not convenient, however, to deal with the exponential composite operator $\exp\phi'$. A more useful interpretation is the following: the integral (2.2) describes the correlation function $\langle\phi(x)\rangle$ for the extended action $S'=S+\phi'(x_0)$ with an "ultralocal" interaction term concentrated on a single space-time point x_0.

The renormalization of field-theoretic models with ultralocal terms, concentrated on surfaces, was studied in Ref. [25] in detail. The analysis of Ref. [25], which we also naturally take to apply to our case, has shown that the standard renormalization theory is applicable to such models, with some obvious modification (see below).

TABLE I. Canonical dimensions of the fields and parameters in the model (2.3).

F	ϕ	ϕ', g	ν, ν_0	μ	g_0
d_F^k	d	0	-2	1	$2+d(1-\alpha)\equiv\varepsilon$
d_F^ω	0	0	1	0	0
d_F	d	0	0	1	ε

The analysis of ultraviolet (UV) divergences is based on the analysis of canonical dimensions, see Refs. [1–3]. Dynamical models of the type (2.3), in contrast to static models, have two scales—the length scale L and the time scale T. Therefore, the canonical dimension of any quantity F (a field or a parameter in the action functional) is described by two numbers—the momentum dimension d_F^k and the frequency dimension d_F^ω—determined so that $[F]\sim[L]^{-d_F^k}[T]^{-d_F^\omega}$. The dimensions are found from the obvious normalization conditions $d_k^k=-d_{\mathbf{x}}^k=1$, $d_k^\omega=d_{\mathbf{x}}^\omega=0$, $d_\omega^k=d_t^k=0$, $d_\omega^\omega=-d_t^\omega=1$, and from the requirement that each term of the action functional be dimensionless (with respect to the momentum and frequency dimensions separately). Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F=d_F^k+2d_F^\omega$ (in the free theory, $\partial_t\propto\partial^2$), which plays in the theory of renormalization of dynamical models the same role as the conventional (momentum) dimension does in static problems; see Refs. [2,3].

Now let us turn to the special case of the model (2.1) with the extended action of the form

$$S'(\phi',\phi)=\int dx\,\phi'(x)\{-\partial_t\phi(x)+\nu_0\partial^2\phi(x)-g_0\nu_0\phi^\alpha(x)\}+\phi'(x_0), \tag{2.4}$$

where we have introduced the new parameter $g_0\equiv\lambda_0/\nu_0$, which plays the part of the coupling constant (a formal small parameter of the ordinary perturbation theory). Canonical dimensions for the model (2.4) are given in Table I, including the dimensions of renormalized parameters, which will appear later on. From Table I, it follows that the model is logarithmic (the coupling constant g_0 is dimensionless) for $2+d(1-\alpha)=0$. In what follows, we fix the exponent α in Eq. (2.4) and consider the model in variable space dimension $d=(2-\varepsilon)/(\alpha-1)$. Then the UV divergences take on the form of the poles in $\varepsilon\equiv2+d(1-\alpha)$ in the correlation functions. The interaction is therefore irrelevant (in the sense of Wilson) for $\varepsilon<0$, marginal (logarithmic) for $\varepsilon=0$, and relevant for $\varepsilon>0$; compare the analysis in Ref. [19]. This means that for $\varepsilon\geqslant0$, the ordinary perturbation expansion (i.e., series in g_0) fails to give correct infrared (IR) behavior and has to be summed up. The desired summation can be accomplished using the renormalization group.

It is a result of the renormalization theory that for the analysis of UV divergences of all correlation functions of the fields ϕ and ϕ' it is sufficient to consider one-particle-irreducible (1PI) correlation functions, whose graphical representation contains only graphs that remain connected after the removal of one (arbitrary) line (i.e., a free-field correlation or response function) of the graph.

The total canonical dimension of an arbitrary 1PI correlation function

$$\Gamma(x_1,\ldots,\ x_N;\,y_1,\ldots,y_{N'};\ x_0) = \frac{\delta^{N+N'}\Gamma(\phi,\phi')}{\delta\phi(x_1)\cdots\delta\phi(x_N)\,\delta\phi'(y_1)\cdots\delta\phi'(y_{N'})}, \quad (2.5)$$

in the time-coordinate representation is given by the relation

$$d_\Gamma = N(d+2-d_\phi)+N'(d+2-d_{\phi'}), \quad (2.6)$$

where N and N' are the numbers of corresponding fields. In Eq. (2.5) $\Gamma(\phi,\phi')$ is the (dimensionless) generating functional of 1PI Green functions. It should be noted, however, that due to the presence of the ultralocal term in the action, the functional $\Gamma(\phi,\phi')$ is *not* the Legendre transform of the functional $W(J,J') = \ln\mathcal{G}(J,J')$, where $\mathcal{G}(J,J') = \int\mathcal{D}\phi'\int\mathcal{D}\phi\exp[S'(\phi',\phi)+J\phi+J'\phi']$ is the generating functional of Green functions of the model. Moreover, contrary to the usual field theories, the functional $\ln\mathcal{G}(J,J')$ does not include all connected graphs of $\mathcal{G}(J,J')$. By definition of the generating functional, the 1PI Green function with N external ϕ legs and N' external ϕ' legs may be obtained by N functional differentiations of $\Gamma(\phi,\phi')$ with respect to the field ϕ and N' differentiations with respect to ϕ'. The canonical dimensions of the functional derivatives are related to the dimensions of the corresponding fields as $d^k[\delta/\delta\phi] = d - d^k_\phi$, $d^\omega[\delta/\delta\phi] = 1 - d^\omega_\phi$, and similarly for the auxiliary field ϕ'. Then the total canonical dimension of the function (2.5) in the frequency-momentum representation (obtained by the Fourier transformation with respect to all $N+N'$ independent differences of the time and coordinate arguments) is obtained from Eq. (2.6) by subtracting the term $(N+N')(d+2)$ and has the form

$$d_\Gamma = -d_\phi N - d_{\phi'}N' = -dN, \quad (2.7)$$

where the data from Table I are used in the last equality.

The quantity (2.7) is the formal index of the UV divergence for the function Γ. Like for usual (local) models, superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which $\delta \equiv d_\Gamma|_{\varepsilon=0}$ is a non-negative integer, see Refs. [1–3].

From Eq. (2.7) we conclude that for any positive d, such divergences can exist only in the 1PI functions with $N=0$ and arbitrary value of N'. For all these functions $\delta=0$, that is, the divergences are logarithmic and the corresponding counterterms in the frequency-momentum representation are constants.

At first glance, we have established that the model (2.4) requires infinitely many counterterms, and hence it is not renormalizable. However, it turns out to be sufficient to renormalize the 1PI Green function $\Gamma(x;x_0)$ only to render the model finite, as we shall now show.

The first few Feynman diagrams of G are shown in Fig. 1 for $\alpha=2$; the symmetry coefficients are shown for general α

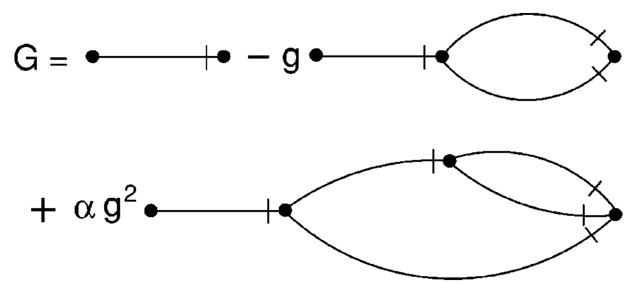

FIG. 1. First Feynman graphs for the Green function of the nonlinear diffusion equation (2.1) for $V(G) = -\lambda_0 G^2$.

(it would be embarrassing to depict the diagrams for fractional α, but the idea is the same). The lines with a slash denote the bare propagator

$$\Delta(t,\mathbf{r}) = \langle\phi\phi'\rangle_0 = \theta(t)\frac{\exp[-r^2/4\nu_0 t]}{(4\pi\nu_0 t)^{d/2}}. \quad (2.8)$$

The end with a slash corresponds to the field ϕ', and the end without a slash corresponds to ϕ. The initial (left) point in each diagram corresponds to x, and the final (right) point with a variable number of attached lines corresponds to x_0. The crucial point is that, as is easily seen from Fig. 1, all possible 1PI subdiagrams entering into the diagrams of G belong to the only 1PI function $\Gamma(x;x_0)$; no other 1PI functions are involved. The function G appears to be "closed with respect to the renormalization," i.e., we can eliminate their UV divergences by the only counterterm corresponding to its 1PI part $\Gamma(x;x_0)$.

Moreover, the renormalization of the only function $\Gamma(x;x_0)$ is in fact sufficient to completely renormalize all functions with $N'>1$. A typical diagram for $N'=3$ is shown in Fig. 2. It is clear that any such diagram reduces to a product of blocks that belong to the simplest function with $N'=1$ (we recall that there is no integration over x_0, the

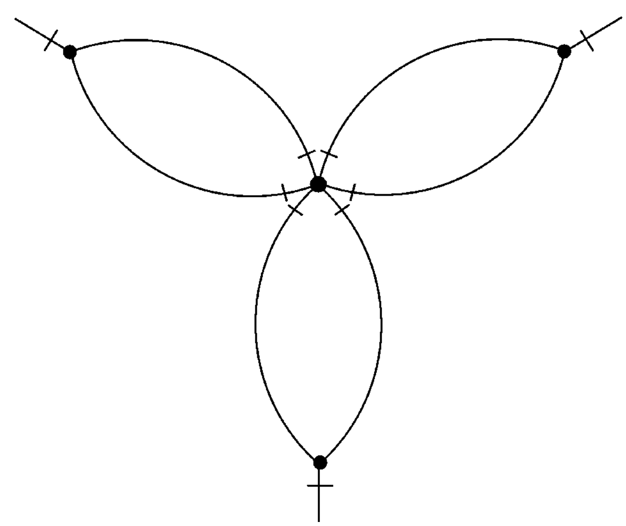

FIG. 2. A three-loop Feynman graph for the three-point correlation function of the nonlinear diffusion equation (2.1) for $V(G) = -\lambda_0 G^2$ illustrating the factorization property $\Gamma(x_1,x_2,x_3;x_0) = \Gamma(x_1;x_0)\Gamma(x_2;x_0)\Gamma(x_3;x_0)$.

only point that connects the blocks). Therefore, the diagram contains no superficial divergences; all its divergences are those of the subdiagrams and they are completely removed by the renormalization of the function with $N'=1$. This is equally true for any diagram of any function with $N'>1$.

In the generic case, all the loops are created by the presence of a single local vertex with any number of ϕ' legs, from which continuous chains of retarded diffusion propagators emanate. Due to the structure of the nonlinear term these chains do not branch, but they may merge (the single ϕ' field in the nonlinearity allows only one outgoing propagator from each ordinary vertex, whereas up to α incoming chains are allowed). A little reflection along the lines sketched above shows then that all divergent 1PI Green functions are factorized,

$$\Gamma(x_1,\ldots,x_N;x_0)=\Gamma(x_1;x_0)\cdots\Gamma(x_N;x_0).$$

Thus, we are left with the only counterterm to the function $\Gamma(x;x_0)$. It is constant (see above), which in the time-coordinate representation corresponds to the function $\delta(x-x_0)\equiv\delta(t-t_0)\delta^{(d)}(\mathbf{x}-\mathbf{x}_0)$. In the action functional, after the integration over the field argument, this gives $\phi'(x_0)$. Such term is present in the extended action (2.4), so that our model is renormalized multiplicatively, with the only renormalization constant, which we denote Z. The renormalized action has the form

$$S'_R(\phi',\phi)=\int dx\ \phi'(x)\{-\partial_t\phi(x)+\nu\partial^2\phi(x)-g\nu\mu^\varepsilon\phi^\alpha(x)\}+Z\phi'(x_0). \tag{2.9}$$

Here and below the g and ν are the renormalized analogs of the bare parameters; μ is the reference mass in the minimal subtraction (MS) scheme, which we use in practical calculations; and the constant Z depends on the dimensionless parameters g, α, and ε. The renormalized Green function G_R, which is finite for $\varepsilon\to 0$, is given by the representation (2.2) with the substitution $S'\to S'_R$.

If we now replace the local initial condition with an integrable one, $\phi(0,\mathbf{x})=a(\mathbf{x})$, then—after Fourier transforming—we obtain wave-vector integrals in which all the propagator lines starting from the initial condition contain a multiplicative factor $\tilde a(\mathbf{k})=\int d\mathbf{x}\,e^{-i\mathbf{k}\cdot\mathbf{x}}a(\mathbf{x})$. For the large-scale asymptotic analysis using RG it is sufficient to keep the leading small wave-number terms in all the lines, which amounts to the replacement $\tilde a(\mathbf{k})\to\tilde a(0)=\int d\mathbf{x}\,a(\mathbf{x})$, and we thus return to loop integrals of the problem with localized initial condition in which $\int d\mathbf{x}\,a(\mathbf{x})$ is the amplitude of the initial δ function.

To clarify the idea, consider the one-loop graph of Fig. 1, whose analytic expression with the initial condition $\phi(0,\mathbf{x})=a(\mathbf{x})$ is

$$\Gamma^{(1)}(t,\mathbf{x})=-\lambda_0\int d\mathbf{y}\int_0^\infty dt'\Delta(t-t',\mathbf{x}-\mathbf{y})\int d\mathbf{y}_1\times\Delta(t',\mathbf{y}-\mathbf{y}_1)a(\mathbf{y}_1)\int d\mathbf{y}_2\Delta(t',\mathbf{y}-\mathbf{y}_2)a(\mathbf{y}_2).$$

Here, $\Delta(t,\mathbf{x})$ is the diffusion propagator (2.8). Fourier transforming $\Gamma^{(1)}(t,\mathbf{x})$ with respect to $\mathbf{x}$, we arrive at the expression

$$\Gamma^{(1)}(t,\mathbf{k})=-\lambda_0\int_0^\infty dt'\Delta(t-t',\mathbf{k})\times\int\frac{d\mathbf{q}}{(2\pi)^d}\Delta(t',\mathbf{q})\tilde a(\mathbf{q})\Delta(t',\mathbf{k}-\mathbf{q})\tilde a(\mathbf{k}-\mathbf{q}), \tag{2.10}$$

where $\Delta(t,\mathbf{k})$ is the spatial Fourier transform of the diffusion kernel (2.8). From the point of view of RG, the IR relevant terms are given by the leading terms of the gradient expansion of the initial condition: $\tilde a(\mathbf{q})=\tilde a(0)+o(q)/q$. This allows one to replace (2.10) by

$$\Gamma^{(1)}(t,\mathbf{k})\sim-\lambda_0\int_0^\infty dt'\Delta(t-t',\mathbf{k})\int\frac{d\mathbf{q}}{(2\pi)^d}\Delta(t',\mathbf{q})\times\Delta(t',\mathbf{k}-\mathbf{q})\tilde a^2(0),$$

which corresponds to the localized initial condition with the amplitude $\tilde a(0)=\int d\mathbf{x}\,a(\mathbf{x})$.

III. RG EQUATIONS AND RG FUNCTIONS

It follows from Eqs. (2.3), (2.4), and (2.9) that the original and renormalized action functionals satisfy the relation $S'(Z\phi',Z^{-1}\phi,e_0)=S'_R(\phi',\phi,e,\mu)$, if the bare and renormalized parameters are related as follows:

$$\nu_0=\nu,\quad g_0=g\mu^\varepsilon Z^{\alpha-1}, \tag{3.1}$$

with the only renormalization constant Z from Eq. (2.9). This implies the relation $G(e_0)=Z^{-1}G_R(e,\mu)$ for the corresponding Green functions in Eq. (2.2); i.e., this quantity is multiplicatively renormalizable. We use $\tilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of this equation with it. This gives the basic RG equation

$$[\mathcal{D}_\mu+\beta(g)\partial_g-\gamma(g)]G_R(e,\mu)=0, \tag{3.2}$$

where $\mathcal{D}_\mu+\beta(g)\partial_g$ is nothing else than the operation $\tilde{\mathcal{D}}_\mu$ expressed in the renormalized variables. In Eq. (3.2), we have written $\mathcal{D}_x\equiv x\partial_x$ for any variable x, and the RG functions (the β function and the anomalous dimensions γ) are defined as

$$\gamma(g)\equiv\tilde{\mathcal{D}}_\mu\ln Z,\quad \beta_g\equiv\tilde{\mathcal{D}}_\mu g=g[-\varepsilon-(\alpha-1)\,\gamma(g)]. \tag{3.3}$$

The relation between β and γ results from the definitions and the relations (3.1).

We shall see below that, for small $\varepsilon>0$, an IR stable fixed point g_* of the RG equation (3.2) exists in the physical region $g>0$, i.e., $\beta(g_*)=0$, $\beta'(g_*)>0$. The functions G and G_R coincide up to a constant (i.e., independent of the time and space variables) factor Z and the choice of the parameters (bare e_0 or renormalized e, μ) and can equally be used in the analysis of the IR behavior. The general solution of the RG equations is discussed in detail, e.g., in Refs. [2,3]. It follows from this solution that, when an IR stable fixed point is present, the leading term of the IR behavior of the function $G_R \propto G$ satisfies Eq. (3.2) with the substitution $g \to g_*$,

$$[\mathcal{D}_\mu - \gamma^*] G_R(e,\mu) = 0. \tag{3.4}$$

In our case, the value of the anomalous dimension at the fixed point is found exactly owing to the relation between β and γ in Eq. (3.3),

$$\gamma^* \equiv \gamma(g_*) = -\varepsilon/(\alpha-1) = d - 2/(\alpha-1). \tag{3.5}$$

Dimensional considerations yield $G_R(t,r) = (\nu t)^{-d/2} \xi(1/t\mu^2\nu, r^2/t\nu)$, where ξ is some function of dimensionless variables. The dependence on g is not displayed explicitly, because the derivatives with respect to this parameter do not enter into Eq. (3.4). It follows from Eq. (3.4) that ξ satisfies—at the fixed point—the equation $[\mathcal{D}_s - \gamma^*/2]\xi(s,y) = 0$, its general solution is $\xi(s,y) = s^{\gamma^*/2}\chi(y)$, where χ is an arbitrary function of the second variable y. For the Green function (2.2) we then obtain

$$G(t,r) \sim G_R(t,r) \sim t^{-d/2+\gamma^*/2}\chi(r^2/t\nu) = t^{-1/(\alpha-1)}\chi(r^2/t\nu),$$

where the form on the "scaling function" $\chi(r^2/t\nu)$ is not determined by Eq. (3.4). The dependence on the parameters ν and μ can be easily restored from the dimensionality considerations (see Table I),

$$G(t,r) \sim (\nu_0 t)^{-1/(\alpha-1)}\chi(r^2/t\nu_0). \tag{3.6}$$

Although the value of γ^* in Eq. (3.5) and the solution (3.6) have been obtained without practical calculation of the constant Z and functions (3.3), such calculation is needed to check the existence, positivity, and IR stability of the fixed point. Within the ε expansion, these facts can be verified already in the simplest one-loop calculation.

In order to check the validity and self-consistency of the approach, we calculated the constant Z up to the two-loop approximation. The calculation is performed in the frequency-momentum (ω, k) representation and calls for the formulas derived in Ref. [26] for a model of critical dynamics.

Two key points are as follows: the convolution of two functions of the form $F(\alpha;a) \equiv (-\mathrm{i}\omega a + k^2)^{-\alpha}$ is a function of the same form,

$$F(\alpha;a)F(\beta;b) = K(\alpha,\beta;a,b)F(\alpha+\beta-d/2-1;a+b),$$

where a and b are both positive and the coefficient has the form

$$K(\alpha,\beta;a,b) = a^{d/2-\alpha} b^{d/2-\beta}(a+b)^{\alpha+\beta-d-1} \times \Gamma(\alpha+\beta-d/2-1)/\Gamma(\alpha)\Gamma(\beta)(4\pi)^{d/2},$$

while the product of two such functions can be represented as a single integral of a function of the same form with the aid of the generalized Feynman formula,

$$F(\alpha;a)F(\beta;b) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 ds\, s^{\alpha-1}(1-s)^{\beta-1} \times F(\alpha+\beta; as + b(1-s)).$$

For the sake of brevity, below we give only the final result,

$$Z = 1 + \frac{u}{\varepsilon} + \frac{\alpha u^2}{2\varepsilon^2} - \frac{u^2}{2\varepsilon}\tilde{I}_\alpha + O(u^3), \tag{3.7}$$

where we have introduced a new coupling constant

$$u \equiv \frac{g}{(4\pi)}\alpha^{-1/(\alpha-1)} \tag{3.8}$$

and have written $\tilde{I}_\alpha \equiv \alpha \ln \alpha + \alpha^{\alpha/(\alpha-1)} I_\alpha$ with the convergent single integral

$$I_\alpha \equiv \int_0^1 \frac{ds}{s}\{[s(\alpha-1)+1]^{(2-\alpha)/(\alpha-1)}[(s+1)(\alpha-1)+1]^{-1/(\alpha-1)} - \alpha^{-1/(\alpha-1)}\}, \tag{3.9}$$

in particular, $\tilde{I}_2 = 2\ln(4/3)$ and $\tilde{I}_3 = 6[\ln(3-\sqrt{5}) + \ln(3/2)]$.

Then for the corresponding β function we obtain $\beta_u \equiv \tilde{\mathcal{D}}_\mu u = -u[\varepsilon + \beta_u \partial_u \ln Z^{\alpha-1}]$, where we have used the last relation in Eq. (3.1) and the fact that $\tilde{\mathcal{D}}_\mu = \beta_u \partial_u$ for the functions dependent only on u. This yields

$$\beta_u(u) = \frac{-\varepsilon u}{1 + (\alpha-1)\mathcal{D}_u \ln Z}. \tag{3.10}$$

Substituting Eq. (3.7) into Eq. (3.10) gives

$$\beta_u(u) = -u[\varepsilon - u(\alpha-1) + u^2(\alpha-1)\tilde{I}_\alpha] + O(u^4). \tag{3.11}$$

Note that the poles in ε in the constant Z cancel out in the function (3.11); this is a manifestation of the general fact that the RG functions must be UV finite, i.e., finite as $\varepsilon \to 0$. The cancellation is possible by virtue of the correlation that exists between the u/ε and $(u/\varepsilon)^2$ terms in Eq. (3.7) and can be used as an additional check of the consistency of the approach. The simple (linear) dependence on ε is a feature specific to the MS scheme.

From Eq. (3.11) we find an explicit expression for the coordinate of the fixed point,

$$u_* = \frac{\varepsilon}{(\alpha-1)} + \tilde{I}_\alpha \frac{\varepsilon^2}{(\alpha-1)^2} + O(\varepsilon^3). \tag{3.12}$$

As already said above, for small positive ε and $\alpha > 1$ the fixed point is positive and IR stable: $\beta_u'(u_*) = \varepsilon + O(\varepsilon^2)$.

In the case $\alpha = 1$ the interaction $V(\phi) = -\lambda_0\phi$ reduces to a "mass term," Eq. (2.1) becomes linear with the solution

$$G_{\alpha=1}(t,\mathbf{r})=\theta(t)\frac{\exp[-\lambda_0 t-r^2/4\nu_0 t]}{(4\pi\nu_0 t)^{d/2}},$$

in which the purely time-dependent decay factor is exponential instead of the powerlike one in Eq. (3.6).

IV. DISCUSSION

We have applied the field-theoretic renormalization group to the nonstochastic differential equation (2.1) and established the scaling behavior in the IR asymptotic range, as a consequence of the existence of the IR stable fixed point in the physical range of parameters. The same asymptotic behavior is shown to be valid for integrable initial conditions which thus constitute the universality class of this fixed point.

The key points are the formulation of the problem as a field-theoretic model with an ultralocal term concentrated at a space-time point and the fact that this model appears multiplicatively renormalizable, in spite of the naive power counting that indicates nonrenormalizability.

The two-loop calculation confirms internal consistency of the approach.

The simple explicit form of the scaling dimensions follows from the fact that there is only one independent renormalization constant in the problem. In particular, this explains a simple value $z=2$ of the exponent in the argument $r^2/t^{2/z}$ of the scaling function (3.6) (in models of dynamical critical phenomena [2,3] and some models of nonlinear diffusion [27] this exponent differs from 2).

Recently, it has been conjectured [22] that the dynamic exponent $z\neq 2$ in the present problem. Our asymptotic solution (3.6), however, does not predict any deviation from the canonical value $z=2$, since there is no renormalization of the diffusion coefficient in the MS scheme we have used. In Ref. [22] with the use of a different renormalization procedure it was concluded that $z-2=O(\varepsilon^2)$. We think, however, that it is not consistent to prescribe physical quantities values of the order $O(\varepsilon^2)$ on the basis of the *one-loop* calculation carried out in Ref. [22], but a two-loop analysis is required for this accuracy.

The RG analysis allows one to derive the RG equation rigorously and to prove that the behavior (3.6) is indeed realized for $\varepsilon>0$, $g_0>0$ in the IR asymptotic range, specified by the relations $t\sim r^2$ and $r\ll\eta$, where $\eta\simeq g_0^{-1/\varepsilon}$ is the UV scale. The general solution of Eq. (3.2) interpolates between the ordinary perturbation theory for Eq. (2.1) and the self-similar asymptotic expression (3.6). The scaling function $\chi(y)$ can be calculated within the ε expansion; in the lowest order one easily obtains $\chi(y)=\exp[-(y/2)^2]+O(\varepsilon)$.

We hope that the ideas presented above might be useful in other models containing ultralocal contributions, which have several charges and hence richer IR behavior. Another direction of generalization would be the analysis of Green functions of vector quantities.

ACKNOWLEDGMENTS

We thank L. Ts. Adzhemyan, A. Kupiainen, M. Yu. Nalimov, and A. N. Vasil'ev for discussions. The work was supported by the Grant Center for Natural Sciences (Grant No. E00-3-24), the Nordic Grant for Network Cooperation with the Baltic Countries and Northwest Russia Grant No. FIN-18/2001, and the Academy of Finland (Grant No. 79781).

[1] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Wiley, New York, 1980).

[2] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989).

[3] A. N. Vasil'ev, *Quantum-Field Renormalization Group in the Theory of Critical Phenomena and Stochastic Dynamics* (St. Petersburg Institute of Nuclear Physics, St. Petersburg, 1998) [in Russian] [English translation: Gordon & Breach, in preparation].

[4] *Proceedings of the International Conference, Renormalization Group*, edited by D. V. Shirkov, D. I. Kazakov, and A. A. Vladimirov (World Scientific, Singapore, 1988); *Proceedings of the Second International Conference, Renormalization Group '91*, edited by D. V. Shirkov and V. B. Priezzhev (World Scientific, Singapore, 1991); *Proceedings of the Third International Conference, Renormalization Group '96*, edited by D. V. Shirkov, D. I. Kazakov, and V. B. Priezzhev (Joint Institute for Nuclear Research, Dubna, 1997).

[5] P.C. Martin, E.D. Siggia, and H.A. Rose, Phys. Rev. A **8**, 423 (1973).

[6] H.K. Janssen, Z. Phys. B **23**, 377 (1976); R. Bausch, H.K. Janssen, and H. Wagner, *ibid.* **24**, 113 (1976); C. De Dominicis, J. Phys. (Paris), Colloq. **37**, C1-247 (1976).

[7] M. Doi, J. Phys. A **9**, 1465 (1976); **9**, 1479 (1976).

[8] Ya.B. Zel'dovich and A.A. Ovchinnikov, Zh. Eksp. Teor. Fiz. **74**, 1588 (1978) [Sov. Phys. JETP **47**, 829 (1978)].

[9] P. Grassberger and M. Scheunert, Fortschr. Phys. **28**, 547 (1980).

[10] L. Peliti, J. Phys. A **19**, L365 (1986).

[11] B.P. Lee, J. Phys. A **27**, 2633 (1994).

[12] J. Cardy and U.C. Täuber, Phys. Rev. Lett. **77**, 4780 (1996); J. Stat. Phys. **90**, 1 (1998).

[13] D.J. Amit, G. Parisi, and L. Peliti, Phys. Rev. B **27**, 1635 (1983).

[14] S.P. Obukhov and L. Peliti, J. Phys. A **16**, L147 (1983); S.A. Bulgadaev and S.P. Obukhov, Phys. Lett. **98A**, 399 (1983); L. Peliti, Phys. Rep. **103**, 225 (1984).

[15] S.É. Derkachov, J. Honkonen, and A.N. Vasil'ev, J. Phys. A **23**, 2479 (1990).

[16] D.V. Shirkov, Dokl. Akad. Nauk SSSR **263**, 64 (1982) [Sov. Phys. Dokl. **27**, 197 (1982)]; Theor. Math. Phys. **60**, 778 (1984).

[17] D.V. Shirkov, Int. J. Mod. Phys. A **3**, 1321 (1988).

[18] N. Goldenfeld, O. Martin, and Y. Oono, J. Sci. Comput. **4**, 355 (1989); N. Goldenfeld, O. Martin, Y. Oono, and F. Liu, Phys. Rev. Lett. **64**, 1361 (1990).

[19] J. Bricmont and A. Kupiainen, Commun. Math. Phys. **150**, 193 (1992); Institut Mittag-Lefler, Report No. 5, 1994/95 (unpublished).
[20] J. Bricmont, A. Kupiainen, and G. Lin, Commun. Pure Appl. Math. **47**, 893 (1994).
[21] J. Bricmont, A. Kupiainen, and J. Xin, J. Diff. Eqns. **130**, 9 (1996).
[22] É.V. Teodorovich, Zh. Eksp. Teor. Fiz. **115**, 1497 (1999) [JETP **88**, 826 (1999)].
[23] L.Ts. Adzhemyan and N.V. Antonov, Theor. Math. Phys. **115**, 562 (1998).
[24] V.F. Kovalev, V.V. Pustovalov, and D.V. Shirkov, J. Math. Phys. **39**, 1170 (1998).
[25] K. Symanzik, Nucl. Phys. B **190**, 1 (1981).
[26] N.V. Antonov and A.N. Vasil'ev, Theor. Math. Phys. **60**, 671 (1984).
[27] A.N. Vasil'ev, M.M. Perekalin, and A.S. Stepanenko, Zh. Eksp. Teor. Fiz. **100**, 1781 (1991) [Sov. Phys.-JETP **73**, 985 (1991)].