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Speeding to a stop: The finite-time singularity of a spinning disk

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The final stages of a coin spinning on a flat surface have recently been proposed [H.K. Moffatt, Nature (London) **404**, 833 (2000)] as an example of a finite-time singularity, wherein the precession rate of the symmetry axis of the coin diverges as it comes to a stop. We report measurements by high-speed video imaging of the rolling motion of disks and rings on a variety of surfaces. We find that the precession rate, Ω, diverges as a power law in time: $\Omega(t)\propto(t-t_o)^{-1/n}$, where t_o is the instant the motion ceases. The exponent n varies between 2.7 and 3.2 under different experimental conditions. The value of n, as well as the systematic dependence of precession rate on coefficients of friction, establishes that the primary mechanism of energy dissipation is rolling friction rather than air drag, as previously suggested.

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In a system where the rate of energy dissipation is large enough, all motions can cease in a finite time. There are systems, however, in which a kinematic variable actually increases toward infinity even as the total kinetic energy tends toward zero. An example of such a finite-time singularity is the bouncing motion of an inelastic ball on a plane. In an idealized case, where the same fraction of kinetic energy is lost on each bounce, the frequency of bounces becomes infinite as the ball comes to a halt in a finite time. An interesting many-body version of this singularity is the so-called inelastic collapse [1] seen in simulations of gases of inelastic particles. As particles lose energy by collisions they organize into linear chains, and the interparticle collision rates within these chains diverge in a finite time. A less-known example of a finite-time singularity is the Painlevé paradox [2] which describes the motion of a rod sliding in a ring with Coulomb friction, which comes to rest with a divergent acceleration.

In a recent paper [3] entitled ''Euler's disk and its finite-time singularity'' [4], Moffatt proposes that another example of a finite-time singularity might be found in the familiar phenomenon of a coin spinning on a plane. A coin spun on its edge begins to tip [5] as it loses kinetic energy. In the late stages of motion, the rotation axis of the coin precesses, with the point of contact paradoxically rolling ever-faster as the coin loses energy. This speeding up is evident as the sound produced by the rolling motion becomes a whirring or chattering sound just before the coin comes to a stop. In a calculation that assumes that the dominant source of energy loss is the viscous drag in the sheared layer of air between coin and table, Moffatt [3] predicts that the precession rate of the coin's symmetry axis is a power law, $\Omega(t)\propto(t-t_o)^{-1/6}$, where t_o is the instant at which the coin comes to rest. To our knowledge, this power-law scaling has not been experimentally tested. Moreover, the proposed mechanism of dissipation has been a matter of controversy. A recent comment [6] on Ref. [3] reports experiments in which the time for a disk to settle was unchanged when the disk was spun in an evacuated chamber, leading them to claim that air drag was not the primary dissipation mechanism. Moffatt in his response points out that viscosity of air is quite insensitive to the pressure and therefore that these observations are inconclusive. Another recent calculation [7], incorporates the effects of the finite width of the viscous boundary layer, and produces an exponent of 2/9, rather than the 1/6 of Ref. [3]. Reference [7] also points out that the exponent of 2/9 is also that obtained in the ''chirping'' of two neutron stars as they collapse gravitationally. In this Rapid Communication we present measurements of the spinning motion of heavy steel disks and rings on a variety of surfaces and find that the precession rate of all these objects increase with a power law as suggested by Moffatt [3]: $\Omega(t)\propto(t-t_o)^{-1/n}$. However, n varies between 2.7 and 3.2 under different experimental conditions rather than being equal to 6. The exponent of the power law, as well as the systematic dependence of the precession rate on coefficients of friction, establishes that the primary mechanism of energy dissipation is rolling friction, rather than the viscous drag of the air.

The dynamical problem is shown in Fig. 1. In the situation sketched therein, the symmetry axis ($\hat{Z}'$) of a disk of mass M and radius a is tipped at an angle α to the direction

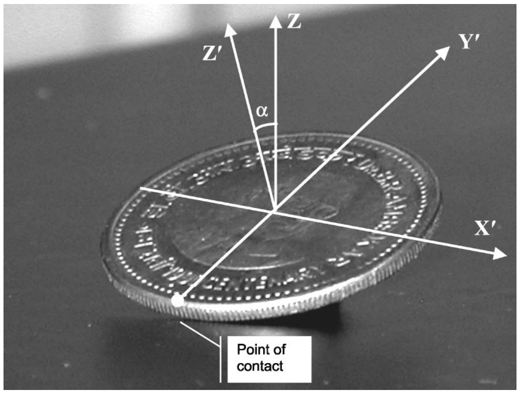

FIG. 1. Geometry of spinning disk. X',Y', and Z' are body-fixed axes, and α is the angle between the normal (Z') to the plane of the disk and the vertical direction Z. The instantaneous motion of the disk is rotation about the diameter Y' that contains the point of contact. As the disk rolls to a stop, α goes to zero, and the precession rate Ω of Z' about Z goes to infinity.

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$\hat{Z}$ normal to the horizontal plane. The angular velocity of precession of the symmetry axis, denoted by $\vec{\Omega}$, points along the Z axis. If the disk rolls without slipping, there is no angular momentum component about the symmetry axis. The instantaneous state of motion, therefore, is rotation about the diameter ($\vec{Y}'$) that contains the point of contact of the disk with the plane. The angular momentum is

$$\vec{L}=I(\vec{\Omega}\cdot\hat{Y})\hat{Y}=I\Omega\hat{Y}\sin\alpha,$$

where $I=\frac{1}{4}Ma^2$ is the moment of inertia of the disk about its diameter. The precession of the angular momentum is due to the gravitational torque,

$$[\vec{T}=(a\hat{Y}')\times(Mg\hat{Z})=-Mga\cos\alpha\hat{X}]. \tag{1}$$

Using the Euler equation [8] for the disk, the torque

$$\left[\vec{T}=\frac{d\vec{L}}{dt}=\vec{\Omega}\times\vec{L}=I\Omega^2\cos\alpha\sin\alpha\hat{X}\right]. \tag{2}$$

From Eqs. (1) and (2) we find that the precession rate Ω is completely determined by the angle of inclination α through the equation

$$[\Omega^2=Mga/(I\sin\alpha)]. \tag{3}$$

As pointed out in Ref. [5], Eq. (3) clearly suggests that the precession rate diverges as α goes to zero, i.e., as the coin settles on the surface. Reference [3] goes further in assuming a mechanism for the energy loss and determining the time dependence of the precession rate $\Omega(t)$ under an adiabatic assumption that the tip angle α varies slowly in time compared to the precessional motion.

The energy loss mechanism assumed in Ref. [3] is the shearing of the layer of air trapped between the disk and the surface. As the tip angle α gets smaller, the viscous dissipation rate Φ_{visc} increases, both because the precession rate goes up and because the layer is thinner and has to support a larger gradient in velocity. For small angles the dissipation rate is estimated [3] as $\Phi_{visc}\sim\alpha^{-2}$. The total energy of the coin is $E=Mga\sin\alpha+\frac{1}{2}I\Omega^2\sin^2\alpha=\frac{3}{2}Mga\sin\alpha$. Integrating $dE/dt=-\Phi_{visc}$ yields the principal result of Ref. [3]: $\Omega(t)\propto(t-t_o)^{-1/6}$.

We have made measurements of the motions of a disk, a ring, and coins by high-speed video imaging. The disks are spun on a variety of surfaces, by hand. In the early stages of the disk's motion there are degrees of freedom other than those described in Fig. 1 and the associated calculation. Initially, the point of contact of the disk can roll in a larger circle than the one of radius $a\cos\alpha$. This is apparent from a slow rotation of the body-fixed axes ($\vec{X}'$ and $\vec{Y}'$) over one rotation of the $\vec{Z}'$ axis about the Z axis. Depending on the way in which the disk was set into motion, the center of mass may also have some linear momentum. However, the disk rapidly settles into the motion described by Fig. 1, and all other motions appear to be damped out. In this regime we have not observed any significant deviations from Eq. (3). In Fig. 2 we present measurements of the angular precession rate $\Omega(t)$ versus $(t-t_o)$. The triangles represent data from five different measurements of a steel disk spun on a flat steel surface. As shown in the figure, the data are quite reproducible and in this regime of time do not reflect irreproducibilities in the initial motions discussed above. In the figure we also show data for the same steel disk spinning on a glass surface (plus signs) and a flat slate laboratory countertop (inverted triangles). We also show data for a steel ring of comparable mass and height spinning on a steel surface. In all cases, the data fall on straight lines on the log-log plot of Fig. 2, indicating that the data are consistent with a power-law description $\Omega(t)\propto(t-t_o)^{-1/n}$. For the best case of a steel disk on a steel surface, the power law spans a little over two decades in time (even with a video frame rate of 3000 frames/sec, we are unable to approach more closely the instant of the singularity). However, the exponent of the power law is not given by $n=6$ as in the prediction of Ref. [3], but varies between $n=2.7$ and 3.2 for the cases we have tried. Similar exponents resulted when coins were used in the experiment.

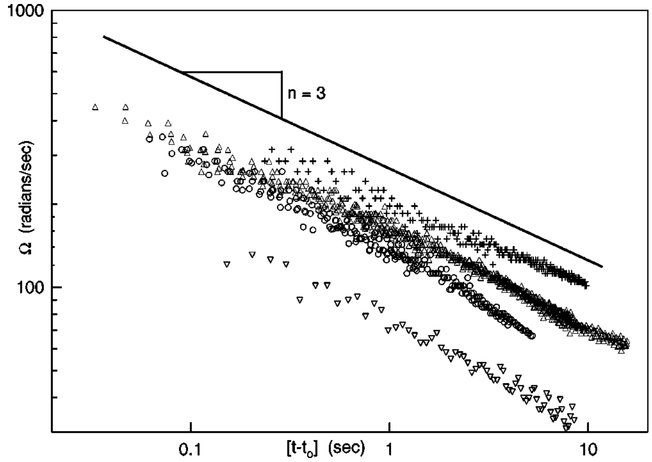

FIG. 2. Angular velocity Ω (in rad/sec) against $t-t_o$ (in sec) for a steel disk (radius, $R=3.7$ cm; mass, $M=440$ g m) spinning on glass (+), steel (△), and slate (▽) surfaces, and for a steel ring (○) of comparable mass and radius (inner radius, 3.75 cm; outer radius, 4.5 cm; and mass, $M=390$ g m) on a steel surface. We find the instant at which the coin finally comes to rest by finding the video frame at which the image remains unchanged in all pixels. This allows us determine t_o to a precision of about ten video frames ($\equiv\pm1.6$ msec). Variations of t_o within these error bars do not noticeably change the data shown above.

Three items of experimental evidence lead us to believe that the major source of dissipation in the slowing down of the disk is not air-drag, but friction at the point of contact between the disk and the surface. First, the prefactor to the power-law decreases systematically when the disk is rolled on surfaces where the rolling friction is greater (in the data discussed above, the glass, steel and slate surfaces are in increasing order of friction coefficient). Since the geometry of disk and plane do not change as the material of the surface is varied, the prefactor would have remained the same if air drag were the dominant source of dissipation. Second, the power-law exponent for a disk rolling on a surface is approximately the same as for a ring rolling on the same surface ($n \approx 2.8$), even though the layer of sheared air is limited to a small region along the perimeter of the ring. Finally, the measured power law exponent is consistent with the prediction of $n=3$ that would result from a simple model of rolling friction where the energy dissipated is proportional to path length traveled by the point of rolling contact. If the coefficient of static friction is μ, then the dissipation rate is given by $\Phi_{friction}=\mu Mg(a\cos\alpha)\Omega$. For small angles, integrating $dE/dt=-\Phi_{friction}$ yields $\Omega(t)\propto(t-t_o)^{-1/3}$.

We also tried to eliminate air drag as a possible source of dissipation by containing the spinning disk in an evacuated chamber, just as in Ref. [6]. However, to significantly alter the viscous effects of the air, the chamber has to be evacuated to the Knudsen regime, where the mean free path of molecules is comparable to the dimensions of the disk.

We now address the question of whether the observed power-law exponent is expected to hold asymptotically close to the singularity. We first note that we approach quite close to the point of singularity in the experiment: if the power-law behavior of $\Omega(t)\propto(t-t_o)^{-1/3}$ were to persist, then we are less than three rotations from the singularity, so that any other regime of behavior would be very short lived [a crossover to $(t-t_o)^{-1/6}$ scaling would only hasten the end]. In principle, it is to be expected that close to the singularity viscous drag will become the dominant dissipation mechanism since the dissipation rate from the air drag $\Phi_{visc}\propto\Omega^4$, while that from frictional drag $\Phi_{friction}\propto\Omega$. However, the angular frequency at which this crossover occurs is given by $\Omega_{crossover}=(16Mg^2\mu/\pi\eta)^{1/3}(1/a)$, where η is the viscosity of the air. We calculate that $\Omega_{crossover}\approx 3000$ rad/sec in our experiment, well beyond the range over which we are able to take data. Furthermore, as noted in Ref. [3], the viscous drag regime is cut off at a high angular frequency $\Omega_{cut\text{-}off}$ when the vertical acceleration of the center of mass exceeds gravity: $a\ddot{\alpha}=g$. Using Eqs. (4) and (5) of Ref. [3] it can be deduced that this condition implies a cutoff angular frequency of $\Omega_{cut\text{-}off}\simeq 2[\frac{81}{4}Mg^3/2\pi\eta a^4]^{1/5}$. This yields $\Omega_{cut\text{-}off}\approx 1000$ rad/sec for our experimental parameters. Since $\Omega_{cut\text{-}off}<\Omega_{crossover}$ it would not be possible to observe the viscous regime by being able to take measurements closer to t_o. In order to observe the $t^{-1/6}$ scaling over a reasonable range one needs to reverse the inequality of the previous sentence and arrange parameters such that $\Omega_{cut\text{-}off}\gg\Omega_{crossover}$. This is quite difficult to realize experimentally by varying a or the density of the disk because of the weak dependence on these parameters: $\Omega_{cut\text{-}off}/\Omega_{crossover}\sim(\eta^2/a\rho^2t^2)^{1/15}$ (assuming $M\sim a^2t\rho$, where t is the thickness of the disk, and ρ its density). It therefore appears that the frictional regime, rather than the viscous regime, will be more commonly observed for disks spinning in air (with the possible exception of very thin disks, where it will be a challenging task to ensure rigidity and circularity of the disk, as well as flatness of the surface).

Thus, in conclusion, we have shown experimentally that the spinning motion of a disk in its late stages does show a finite-time singularity as suggested in Ref. [3]. However, the dissipation mechanism that leads to the singular behavior is just rolling friction, rather than air drag, when objects like coins or the Euler disk toy [4] are used. The pleasing feature of this new example of a finite-time singularity identified in Ref. [3] is that the measured power-law dependence of this familiar phenomenon is relatively robust. This is in spite of the many potential complications in the mechanics of the contact between rolling object and coin, such as bouncing contact, slippage, and acoustic radiation. Thus the measurements we present here of a spinning coin perhaps represent a relatively clean experimental realization of a finite-time singularity in a discrete mechanical system.

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