Comment on "Kullback-Leibler and renormalized entropies: Applications to electroencephalograms of epilepsy patients"

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In a recent paper Quian Quiroga *et al.* [R. Quian Quiroga *et al.*, Phys. Rev. E **62**, 8380 (2000)] found renormalized entropy, formerly introduced as a complexity measure for the different regimes of a dynamical system, to be closely related to the standard Kullback-Leibler entropy. They assure this finding by reanalyzing electroencephalographic data of epilepsy patients, previously examined by exclusive use of renormalized entropy [K. Kopitzki *et al.*, Phys. Rev. E **58**, 4859 (1998)]. We argue that the general considerations undertaken by the authors and the experimental results do not justify this conclusion.

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Quian Quiroga *et al.* [1] investigate the relationship between Kullback-Leibler (KL) and renormalized entropy. The use of renormalized entropy as a complexity measure for the different regimes of a dynamical system, as described by the authors, was proposed by Saparin *et al.* [2]. Subsequently the method was applied to different physiological time series [2–5]. The procedure is based on Klimontovich's S theorem, which states that in the process of self-organization the entropy, renormalized to a given value of mean effective energy, decreases [6–8]. Given a reference distribution $q(X)$ of an observable X, representing the state of maximum disorder, the renormalized entropy $\Delta H(p,q)$ of a state $p(X)$ is given by

$$\Delta H[p,q]=H[p]-H[\tilde{q}] = -\int p(X)\ln p(X)dx+\int \tilde{q}(X)\ln \tilde{q}(X)dx. \tag{1}$$

Here the reference distribution $q(X)$ is renormalized into $\tilde{q}(X)$ according to the S theorem to ensure the equality of mean energies in $\tilde{q}(X)$ and $p(X)$. As shown by the authors of Ref. [1] for the discrete case $\Delta H[p,q]$ can be given in terms of a standard KL entropy $K(p|\tilde{q})$:

$$\Delta H[p,q]=-\int p(X)\ln\frac{p(X)}{\tilde{q}(X)}dx=-K(p|\tilde{q}). \tag{2}$$

Furthermore they prove that

$$|\Delta H[p,q]|\leqslant K(p|q) \tag{3}$$

holds true and infer from these findings that renormalized entropy is unlikely to be more useful than the standard KL entropy.

First, it should be mentioned that the relationship between renormalized and standard KL entropy [Eq. (2)] is immanent of Klimontovich's S theorem and already given in Refs. [2,6–8]. Being an addition to Boltzmann's H theorem, the S theorem uses the notion of free energy and KL entropy. The relationship derived by Quian Quiroga *et al.* [Eq. (12) in Ref. [1]] is the discrete form of Eq. (2), previously given by Saparin *et al.* in Ref. [2]. Independent of this the conclusion drawn in Ref. [1] seems questionable. Although the renormalized entropy can be given in terms of a KL entropy with respect to the renormalized reference $\tilde{q}(X)$, it can not be given in such terms with respect to the original reference $q(X)$. However, the renormalization of the reference distribution is the basic idea underlying the concept of renormalized entropy. It is due to the equality of the mean effective energies that the entropy difference of these two distributions in Eq. (1) can be given in terms of a Kullback-Leibler entropy.

Second, the relationship given in Eq. (3) does not establish a superiority of either renormalized or KL entropy [9]. The essential feature of a complexity measure is not to indicate every transition from one state to another but those which are linked to a change of complexity as defined within a certain framework. Here for example

$$|\Delta H[\tilde{q},q]|=0<K(\tilde{q}|q) \tag{4}$$

holds true for all $T\neq 0$, meaning that the corresponding states are assumed to have the same complexity [10].

To illustrate their findings Quian Quiroga *et al.* use both measures to reanalyze electroencephalographic data of epilepsy patients. These electrophysiologic time series were recorded at different intracranial locations relative to the brain area known to generate epileptic seizures. Both measures are calculated for the power spectra of consecutive segments of these time series resulting in time courses of renormalized and KL entropy assigned to different recording locations.

Although the time courses of renormalized entropy given in Ref. [1] do not coincide with those given in Ref. [5], the spatiotemporal behavior seems similar. The decrease of renormalized entropy within the transition from the preseizure to the seizure state is most pronounced for the recording location within the seizure-generating area [Fig. 2 in Ref. [1]]. In contrast the greatest increase of KL entropy is found in an adjacent brain area [Fig. 3 in Ref. [1]]. As mentioned in Ref. [5] this feature of renormalized entropy might be useful for clinical applications such as localization of the seizure-generating area. The aforementioned discrepancies are presumably caused by a systematic error in the calculation of renormalized entropy in Ref. [1]. This is suggested by the fact that the time courses of renormalized and KL entropy do not satisfy Eq. (3) within long time intervals as becomes most evident in Fig. 4. However, the figures do not indicate that KL entropy is closely related to renormalized entropy.

Summarizing the explanations given above, Eq. (3) seems to be the only genuine relationship between renormalized and KL entropy established by Quian Quiroga *et al.* [1]. But this equation does not provide any qualitative information about the relationship between these measures.

[1] R. Quian Quiroga, J. Arnhold, K. Lehnertz, and P. Grassberger, Phys. Rev. E **62**, 8380 (2000).

[2] P. Saparin, A. Witt, J. Kurths, and V. Anischenko, Chaos, Solitons Fractals **4**, 1907 (1994).

[3] J. Kurths, A. Voss, P. Saparin, A. Witt, and H.J. Kleiner, Chaos **5**, 88 (1995).

[4] A. Voss, J. Kurths, H.J. Kleiner, A. Witt, N. Wessel, P. Saparin, K.J. Osterziel, R. Schurath, and R. Dietz, Cardiovasc. Res. **31**, 419 (1996).

[5] K. Kopitzki, P.C. Warnke, and J. Timmer, Phys. Rev. E **58**, 4859 (1998).

[6] Yu.L. Klimontovich, Physica A **142**, 390 (1987).

[7] Yu.L. Klimontovich, *Turbulent Motion and the Structure of Chaos: A New Approach to the Statistical Theory of Open System* (Kluwer Academic Publishers, Dordrecht, 1991).

[8] Yu.L. Klimontovich, *Statistical Theory of Open Systems. Vol. 1: A Unified Approach to Kinetic Description of Processes in Active Systems* (Kluwer Academic Publishers, Dordrecht, 1995), Vol. 1.

[9] Equation (3) can be invalidated by basic mathematical operations on ΔH, which do not affect its characteristic features.

[10] Equation (4) also implies that $\Delta H[p,q]=0$ is not equivalent to $p\equiv q$. Thus this equivalence is not shown in Ref. [2] as stated by Quian Quiroga *et al.* [1].