PHYSICAL REVIEW E **66**, 041113 (2002)

Anomalous diffusion in nonlinear oscillators with multiplicative noise

Kirone Mallick*
Service de Physique Théorique, Centre d'Études de Saclay, 91191 Gif-sur-Yvette Cedex, France

Philippe Marcq†
Institut de Recherche sur les Phénomènes Hors Équilibre, Université de Provence, 49 rue Joliot-Curie, Boîte Postale 146, 13384 Marseille Cedex 13, France

(Received 18 April 2002; published 28 October 2002)

The time-asymptotic behavior of undamped, nonlinear oscillators with a random frequency is investigated analytically and numerically. We find that averaged quantities of physical interest such as the oscillator's mechanical energy, root-mean-square position, and velocity grow algebraically with time. The scaling exponents and associated generalized diffusion constants are calculated when the oscillator's potential energy grows as a power of its position: $\mathcal{U}(x) \sim x^{2n}$ for $|x| \to \infty$. Correlated noise yields anomalous diffusion exponents equal to half the value found for white noise.

DOI: 10.1103/PhysRevE.66.041113 PACS number(s): 05.40.−a, 05.10.Gg, 05.45.−a

I. INTRODUCTION

Randomness in the external conditions entails the parameters of a dynamical system to fluctuate. The extent of these fluctuations is independent of any thermodynamic characteristic of the system in contrast to intrinsic fluctuations the amplitude of which is proportional to the equilibrium temperature, in accordance with the fluctuation-dissipation theorem [1,2]. Usually, external randomness appears as a multiplicative noise in the dynamical equations. The interplay of noise and nonlinearity in a system far from equilibrium results in some unusual phenomena [3]. In fact, the presence of noise dramatically alters the properties of a nonlinear dynamical system both qualitatively and quantitatively (for a recent review, see [4]). For example, it was recently shown that in a spatially extended system, a multiplicative noise, white in space and time, generates an ordered symmetry-breaking state through a nonequilibrium phase transition, whereas no such transition exists in the absence of noise [5,6]. Noise can also induce spatial patterns [7,8] or improve the performance of a nonlinear device through stochastic resonance [9]. Furthermore, even if some important qualitative features of a deterministic system survive to external noise, their quantitative characteristics may change: a stable fixed point may become unstable [10], a bifurcation may be delayed (noise-induced stabilization) [11,12], and scale-invariant properties which manifest themselves as power laws may be altered with the appearance of nonclassical scaling exponents [13].

The discovery of Brownian motors that are able to rectify random fluctuations into a directed motion (noise-induced transport) has triggered renewed interest in the study of simple one-dimensional mechanical models of particles in a potential with random parameters [14]. It is well known that a linear oscillator subjected to parametric noise can be unstable even if damping is taken into account [10,15]. This noise-induced energetic instability has been observed in diverse experimental contexts such as electronic oscillators [16,17], nematic liquid crystals [18], and surface waves (Faraday instability) [19]. In engineering fields, this instability plays a crucial role in the study of the dynamic response of flexible structures to random environmental loading such as the wave-induced motion of offshore structures or the vibration of tall buildings in a turbulent wind [20]. The presence of nonlinear friction tends to limit the oscillation amplitude: the pendulum with a randomly vibrating suspension axis and undergoing nonlinear friction, known as the van der Pol oscillator, has been studied in the small-noise limit using perturbative expansions [16,21].

In the present work, we consider the motion of an undamped nonlinear oscillator trapped in a general confining potential and submitted to parametric random fluctuations. Because there is no dissipation, the energy of the system increases with time and we shall show that the position, the momentum, and the energy grow as power laws of time with scaling exponents that depend on the behavior of the confining potential at infinity [22]. A key feature of our method is to use the integrability properties of the associated deterministic nonlinear oscillator in order to derive exact stochastic equations in action-angle variables. We then use the averaging technique of classical mechanics [23], together with a reduction procedure [24,25], to calculate exactly the anomalous scaling exponents, irrespective of the amplitude of the noise. Some of our results were derived earlier, in the particular case of a cubic nonlinearity using an energy-envelope equation [26]. Our method enables us to derive the numerical prefactors appearing in the scaling laws (generalized diffusion constants), and our analytical predictions compare very satisfactorily with the numerical results. In the case of noise correlated in time, the anomalous diffusion exponents are modified: they can be obtained by dimensional analysis arguments, and the values thus found also agree with numerical results. Throughout this work, crossover phenomena between different scaling regimes are emphasized.

This article is organized as follows. In Sec. II, we recall that the energy of a linear oscillator with multiplicative noise

*Electronic address: mallick@spht.saclay.cea.fr
†Electronic address: marcq@irphe.univ-mrs.fr

grows exponentially with time and that this growth may be characterized by a Lyapunov exponent. In Sec. III, we consider a particle in an arbitrary confining potential that grows as a polynomial at large distances. Our technique allows us to study precisely the long-time behavior of the system. As a particular case, we analyze the classical Duffing oscillator in the presence of parametric, Gaussian white noise. In Sec. IV, we discuss the case of colored noise, where the presence of a new time scale (the correlation time) leads to a crossover from the white noise regime to another scaling regime. Our conclusions are presented in Sec. V. In Appendix A, the nonlinear oscillator in the presence of both additive and multiplicative noise is briefly studied: we show that at long times the effect of additive noise is irrelevant. Appendix B is devoted to numerical methods.

II. LINEAR OSCILLATOR WITH PARAMETRIC NOISE

In this section we review known results for an undamped linear oscillator submitted to parametric noise—a generic and widely studied model—in order to understand the role of external multiplicative noise [3,10,16]. The dynamical equation for such a system is

$$\frac{d^2}{dt^2}x(t)+(\omega^2+\xi(t))x(t)=0, \tag{1}$$

where $x(t)$ represents the position of the oscillator at time t and ω is its frequency. The random noise $\xi(t)$ is a Gaussian white noise of zero mean value and of amplitude $\mathcal{D}$:

$$\langle \xi(t)\rangle=0,$$

$$\langle \xi(t)\xi(t')\rangle=\mathcal{D}\delta(t-t'). \tag{2}$$

The physical interpretation of Eq. (1) is that the frequency of the oscillator is not constant in time but fluctuates around its mean value ω because of randomness in the external conditions (external noise). When these fluctuations are deterministic and periodic in time, Eq. (1) is a Mathieu equation, which has been extensively studied [23]. Here, we are interested in the case where these fluctuations are random with no deterministic part. The origin, $x=0$ and $dx/dt=0$, is an unstable stationary solution of Eq. (1). As shown in Ref. [10], this instability can be studied from the dynamical evolution of the probability distribution function (PDF) $P(x,v,t)$ of x and v (with $v=\dot{x}=dx/dt$). This PDF obeys the Fokker-Planck equation [1,2] associated with Eq. (1),

$$\frac{\partial P}{\partial t}=-v\frac{\partial P}{\partial x}+\omega^2 x\frac{\partial P}{\partial v}+\frac{\mathcal{D}}{2}\frac{\partial^2}{\partial v^2}(x^2P), \tag{3}$$

where Eq. (1) is understood according to Stratonovich rules.

This Fokker-Planck equation leads to a closed system of ordinary differential equations that couple the $n+1$ moments of order n, i.e., moments of the type $\langle x^{n-k}v^k\rangle$, where n and k are positive integers and $0\leqslant k\leqslant n$:

$$\frac{d}{dt}\langle x^{n-k}v^k\rangle=(n-k)\langle x^{n-k-1}v^{k+1}\rangle-\omega^2k\langle x^{n-k+1}v^{k-1}\rangle+\frac{\mathcal{D}}{2}k(k-1)\langle x^{n-k+2}v^{k-2}\rangle. \tag{4}$$

The divergence of the moments with time results from the existence of at least one positive eigenvalue of the linear system (4). In particular, the mean value of the mechanical energy E of the system (i.e., the sum of its kinetic and potential energies) grows exponentially with time,

$$\langle E\rangle=\tfrac{1}{2}\langle v^2\rangle+\tfrac{1}{2}\omega^2\langle x^2\rangle\propto e^{\mu t}, \tag{5}$$

where the growth rate μ is the positive real root of the equation,

$$\mu^3+4\omega^2\mu=2\mathcal{D}. \tag{6}$$

It has also been proved that the quenched average of the energy E grows linearly with time, hence the Lyapunov exponent Λ, defined as

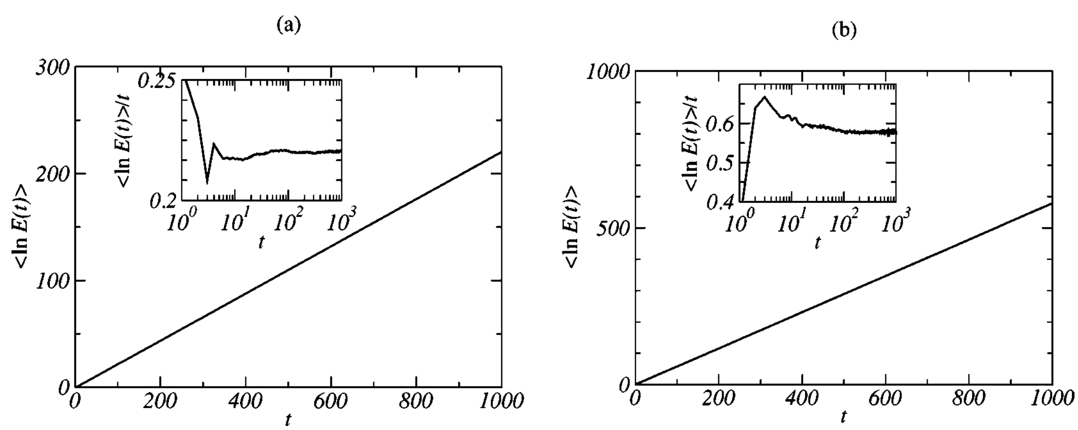

FIG. 1. Linear oscillator with multiplicative noise: Eq. (1) is integrated numerically for $\mathcal{D}=1$ with a time step $\delta t=10^{-3}$. Ensemble averages are computed over 10^4 realizations. We plot the average $\langle \ln E(t)\rangle$ and the ratio $\langle \ln E(t)\rangle/t$ (inset) vs time t. (a) $\omega=1$, $\Lambda(\omega=1)=\lim_{t\to\infty}\langle \ln E(t)\rangle/t=0.219(3)$; (b) degenerate case $\omega=0$, $\Lambda(\omega=0)\simeq 0.580(5)$. Both estimates of the Lyapunov exponent are in good agreement with the theoretical prediction [27,28].

$$\Lambda = \lim_{t\to\infty}\frac{1}{t}\langle \log E\rangle, \tag{7}$$

is finite and strictly positive [27,28]. The positivity of the Lyapunov exponent implies the instability of all moments at long times. Note that the growth rate μ, defined in Eq. (5), is larger than the Lyapunov exponent because of the convexity inequality, $\log\langle E\rangle \geqslant \langle \log E\rangle$.

In Fig. 1(a), we present the numerical solution of Eq. (1) averaged over a large number of realizations of the noise, where the pulsation is $\omega=1$. The algorithm used to solve this stochastic differential equation with multiplicative noise is inspired by [29] and explained in Appendix B. A numerical estimate of the Lyapunov exponent, given in Fig. 1(a), agrees very well with the analytic expression of Refs. [27,28]. The usual statistical equipartition of the total energy between kinetic and potential contributions is satisfied: $\langle E\rangle = \omega^2\langle x^2\rangle = \langle v^2\rangle$.

In Fig. 1(b), we show the same quantities for the degenerate linear oscillator obtained by taking ω equal to 0. This degenerate case exhibits the same behavior as the generic case and the Lyapunov exponent can be calculated by taking the $\omega\to 0$ limit in the formulas of Refs. [27,28]. We conclude that the instability triggered by the noise is the dominant effect and that the presence of the linear restoring force $-\omega^2 x$ is irrelevant.

Hence, in order to avoid an exponential increase of the energy and the amplitude of the oscillator, it is necessary to go beyond the linear approximation and consider the effect of nonlinear restoring forces [30].

III. GENERAL NONLINEAR OSCILLATOR WITH PARAMETRIC NOISE

We now consider the case of a particle trapped in a confining potential $\mathcal{U}(x)$ and subject to an external noise. As before, the potential is supposed to be harmonic for small oscillation amplitudes: when $|x|\to 0$,

$$\mathcal{U} \sim \tfrac{1}{2}\omega^2 x^2. \tag{8}$$

For the potential to be confining, we must have $\mathcal{U}\to +\infty$ when $|x|\to\infty$. We restrict our analysis to the case where $\mathcal{U}$ is a polynomial function of x, even in x, in order to respect the $x\to -x$ symmetry. Hence, when $|x|\to\infty$,

$$\mathcal{U} \sim \frac{1}{2n}\lambda x^{2n} \quad \text{with} \quad n\geqslant 2. \tag{9}$$

The dynamics of this mechanical system is given by

$$\frac{d^2}{dt^2}x(t) + (\omega^2+\xi(t))x(t) + \lambda x(t)^{2n-1} = 0, \tag{10}$$

where $\xi(t)$ is the Gaussian white noise of Eq. (2).

We shall prove that the nonlinear term is relevant and prevents the average amplitude from growing exponentially. Instead of an exponential behavior, the average energy of the oscillator, as well as the variances of its position and velocity, exhibit a power law behavior with time. We shall calculate exactly the associated scaling exponents.

A. Degenerate nonlinear oscillator with parametric noise

As before, we expect the amplitude of the oscillator to grow without bounds at large times. The linear part of the restoring force $-\omega^2 x$ is negligible in comparison to the nonlinear term when the amplitude of the oscillator is large. In order to study the long-time behavior of the oscillator, we therefore simplify Eq. (10) to that of a degenerate nonlinear oscillator:

$$\frac{d^2}{dt^2}x(t) + \xi(t)x(t) + x(t)^{2n-1} = 0. \tag{11}$$

The coefficient of the nonlinear term is set equal to unity by rescaling the variable $x(t)$ to $x(t)\lambda^{1/(2n-2)}$.

First we study the deterministic part of Eq. (11) and shall add the noise term afterwards [20]. In one dimension, the deterministic nonlinear oscillator is integrable because the energy E, defined as

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n}, \tag{12}$$

is a conserved quantity. The exact solution of the mechanical system, $\ddot{x} + x^{2n-1} = 0$, for a fixed value of E is given by

$$x = E^{1/2n}\mathcal{S}_n[(2nE)^{(n-1)/2n}t], \tag{13}$$

$$\dot{x} = (2n)^{(n-1)/2n}E^{1/2}\mathcal{S}_n'[(2nE)^{(n-1)/2n}t]. \tag{14}$$

The function $\mathcal{S}_n$ is defined as the inverse function of an hyperelliptic integral:

$$\mathcal{S}_n(X) = Y \leftrightarrow X = \sqrt{n}\int_0^{Y/(2n)^{1/2n}} \frac{du}{\sqrt{1-u^{2n}}} = \frac{\sqrt{n}}{(2n)^{1/2n}}\int_0^{Y}\frac{du}{\sqrt{1-\frac{u^{2n}}{2n}}}. \tag{15}$$

From this definition, we find a relation between $\mathcal{S}_n$ and its derivative $\mathcal{S}_n'$,

$$\mathcal{S}_n'(X) = \frac{(2n)^{1/2n}}{\sqrt{n}}\left(1 - \frac{[\mathcal{S}_n(X)]^{2n}}{2n}\right)^{1/2}. \tag{16}$$

From Eqs. (13) and (14), we define the action-angle variables of the nonlinear oscillator [23,33]. The action variable I corresponds to the area under a constant energy curve in phase space:

$$I=4\int_0^{(2nE)^{1/2n}}\sqrt{2E-\frac{x^{2n}}{n}}dx$$
$$=4(2^{n+1}n)^{1/2n}E^{(n+1)/2n}\int_0^1\sqrt{1-u^{2n}}du\propto E^{(n+1)/2n}. \tag{17}$$

The angle variable ϕ, canonically conjugate to the action I, is equal to $\omega(E)t$ (but for an unimportant additive constant), where $\omega(E)$ is the frequency corresponding to the energy E:

$$\omega(E)=(2nE)^{(n-1)/2n}. \tag{18}$$

Hence, for the free system (without noise) the second-order dynamical equation $\ddot{x}+x^{2n-1}=0$ is equivalent to the following two first-order equations, the first one representing energy conservation:

$$\dot{I}=\dot{E}=0,$$
$$\dot{\phi}=\omega(E)=(2nE)^{(n-1)/2n}. \tag{19}$$

The presence of external noise spoils the integrability of the dynamical system (11) and causes E to grow with time by continuously injecting energy into the system. From Eqs. (13) and (14), the phase is now identified as

$$\phi=\mathcal{S}_n^{-1}\left(\frac{x}{E^{1/2n}}\right)=\sqrt{n}\int_0^{x/[(2nE)^{1/2n}]}\frac{du}{\sqrt{1-u^{2n}}}. \tag{20}$$

The angle variable ϕ is well defined modulo the period $4K_n$ of the function $\mathcal{S}_n$, where

$$K_n=\sqrt{n}\int_0^1\frac{du}{\sqrt{1-u^{2n}}}. \tag{21}$$

In terms of the energy-angle coordinates (E,ϕ), the original variables $(x,\dot{x})$ read as follows:

$$x=E^{1/2n}\mathcal{S}_n(\phi), \tag{22}$$

$$\dot{x}=(2n)^{(n-1)/2n}E^{1/2}\mathcal{S}_n'(\phi). \tag{23}$$

We now take into account the external noise and rewrite the system (19) in (E,ϕ) coordinates. The stochastic evolution equation for the energy is given by

$$\dot{E}=x\dot{x}\xi(t)=(2n)^{(n-1)/2n}E^{(n+1)/2n}\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)\xi(t). \tag{24}$$

Using Eqs. (23) and (24), we obtain the stochastic evolution of the phase variable:

$$\dot{\phi}=(2nE)^{(n-1)/2n}-\frac{1}{(2n)^{1/n}}\frac{\mathcal{S}_n(\phi)^2}{(2nE)^{(n-1)/2n}}\xi(t). \tag{25}$$

With the help of the auxiliary variable Ω, defined as

$$\Omega=(2n)^{(n+1)/2n}E^{(n-1)/2n}, \tag{26}$$

we derive a compact form for the two stochastic evolution Eqs. (24) and (25):

$$\dot{\Omega}=(n-1)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi)\xi(t), \tag{27}$$

$$\dot{\phi}=\frac{\Omega}{(2n)^{1/n}}-\frac{\mathcal{S}_n(\phi)^2}{\Omega}\xi(t). \tag{28}$$

We emphasize that the coupled equations (27) and (28) are mathematically equivalent to the initial system and have been derived without any approximation. Moreover, the nature of the parametric perturbation has played no role in the derivation: the function $\xi(t)$ can be a deterministic or a stochastic function with arbitrary statistical properties.

We now perform a precise analysis of the long-time behavior of the nonlinear oscillator driven by a multiplicative Gaussian white noise. From an heuristic point of view, we observe from Eq. (27) that Ω undergoes a diffusion process and should scale typically as $t^{1/2}$. We also notice from Eq. (28) that, as Ω grows, the phase ϕ varies more and more rapidly with time. Hence, the phase ϕ is a fast variable and it is natural to average the dynamics over its rapid variations. This averaging process leads to some remarkable and general identities between different physical quantitites. Thus, we obtain the average of $\dot{x}^2$ from Eq. (23):

$$\langle\dot{x}^2\rangle=(2n)^{(n-1)/n}\overline{\mathcal{S}_n'(\phi)^2}\langle E\rangle=2\frac{\int_0^1du\sqrt{1-u^{2n}}}{\int_0^1\frac{du}{\sqrt{1-u^{2n}}}}\langle E\rangle. \tag{29}$$

The last equality is derived by writing $u=\mathcal{S}_n(\phi)$, and using Eqs. (16) and (21). Moreover, the following identity is true (as can be shown by integrating $\int_0^1 1\sqrt{1-u^{2n}}du$ by parts):

$$\int_0^1du\sqrt{1-u^{2n}}=n\int_0^1du\frac{u^{2n}}{\sqrt{1-u^{2n}}}$$
$$=-n\int_0^1du\sqrt{1-u^{2n}}+n\int_0^1\frac{du}{\sqrt{1-u^{2n}}}. \tag{30}$$

Substituting this identity into Eq. (29) leads to

$$\langle E\rangle=\frac{n+1}{2n}\langle\dot{x}^2\rangle. \tag{31}$$

From the definition (12) of the energy, we derive another statistical equality:

$$\langle\dot{x}^2\rangle=\langle x^{2n}\rangle. \tag{32}$$

We emphasize that these generalized equipartition relations are "universal" in the sense that they are independent of the form of the noise we consider. In particular, identities (31) and (32) are valid for multiplicative as well as for additive noise (see Appendix A). The only hypothesis is that the probability distribution function $P_t(\Omega,\phi)$ becomes uniform in ϕ over the interval $[0,4K_n]$ when $t\to\infty$. We observe from the numerical simulations presented in Fig. 2(c) that this condition is very well satisfied.

The same averaging procedure allows us to derive a closed equation for the stochastic evolution of the slow variable Ω. We start by writing the Fokker-Planck equation governing the evolution of the PDF $P_t(\Omega,\phi)$ associated with the system (27),(28). Since the noise term appears as a multiplicative factor, one must be cautious about the convention used to define stochastic calculus. Here, as well as in the following, we shall use Stratonovich rules because they are obtained naturally when white noise is considered as a limit of colored noise with vanishing correlation time [1,2]. The Fokker-Planck equation corresponding to Eqs. (27) and (28) reads

$$\partial_t P = -\frac{\Omega}{(2n)^{1/n}}\partial_\phi P + \frac{\mathcal{D}}{2}\left[\partial_\phi\left(\frac{g(\phi)}{\Omega}\partial_\phi\frac{g(\phi)}{\Omega}P\right) - \partial_\phi\left(\frac{g(\phi)}{\Omega}\partial_\Omega f(\phi)P\right) - \partial_\Omega\left(f(\phi)\partial_\phi\frac{g(\phi)}{\Omega}P\right) + \partial_\Omega[f(\phi)\partial_\Omega f(\phi)P]\right], \tag{33}$$

where we have defined

$$f(\phi)=(n-1)\mathcal{S}_n(\phi)\mathcal{S}_n'(\phi) \quad \text{and} \quad g(\phi)=\mathcal{S}_n(\phi)^2. \tag{34}$$

The Fokker-Planck equation (33) written in the variables (Ω,ϕ) is exact because we study the case of a Gaussian white noise. In order to pursue our calculations, we assume that $P_t(\Omega,\phi)$ becomes independent of ϕ when $t\to\infty$, i.e., that the probability measure for ϕ is uniform over the interval $[0,4K_n]$. We now average the Fokker-Planck equation (33) over the angular variable. We shall use the fact that the average of the derivative of any function is zero:

$$\overline{\partial_\phi(\;)=0}. \tag{35}$$

This implies that

$$\overline{\partial_\phi[g(\phi)\partial_\phi g(\phi)]}=0 \quad \text{and} \quad \overline{f(\phi)=0} \quad \text{because} \quad f(\phi)=\frac{n-1}{2}\partial_\phi g(\phi). \tag{36}$$

Using these properties and, in particular, the last identity in (36), we derive the phase-averaged Fokker-Planck equation:

$$\partial_t\tilde{P} = \frac{\mathcal{M}_n\mathcal{D}}{2}\left(\partial_\Omega^2\tilde{P} - \frac{2}{n-1}\partial_\Omega\frac{\tilde{P}}{\Omega}\right), \tag{37}$$

where $\tilde{P}_t(\Omega)$ is now a function of Ω and t only, and where $\mathcal{M}_n$ is given by

$$\mathcal{M}_n = \frac{(n-1)^2}{n}(2n)^{2/n}\frac{\int_0^1 du\, u^2\sqrt{1-u^{2n}}}{\int_0^1 \frac{du}{\sqrt{1-u^{2n}}}} = \frac{(n-1)^2}{n+1}(2n)^{2/n}\frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)}\frac{\Gamma\left(\frac{3n+1}{2n}\right)}{\Gamma\left(\frac{3n+3}{2n}\right)}, \tag{38}$$

$\Gamma(\;)$ being the Euler gamma function [31]. The effective Langevin dynamics for the variable Ω is thus

$$\dot{\Omega} = \frac{\mathcal{M}_n\mathcal{D}}{n-1}\frac{1}{\Omega} + \Xi_n(t), \tag{39}$$

where the effective Gaussian white noise $\Xi_n(t)$ satisfies the relation

$$\langle\Xi_n(t)\Xi_n(t')\rangle = \mathcal{M}_n\mathcal{D}\delta(t-t'). \tag{40}$$

The averaged distribution function $\tilde{P}_t(\Omega)$ can be calculated because Eq. (37) is exactly solvable due to its invariance under rescalings $t\to\lambda^2 t$, $\Omega\to\lambda\Omega$, λ being an arbitrary real number (this invariance is the same as that of the heat equation). Equation (37) is solved by using the self-similar ansatz

$$\tilde{P}_t(\Omega) = \frac{1}{\sqrt{t}}\Pi\left(\frac{\Omega}{\sqrt{t}}\right).$$

The PDF of Ω is found to be

$$\tilde{P}_t(\Omega) = \frac{1}{\Gamma\left(\frac{n+1}{2(n-1)}\right)}\frac{\Omega^{2/(n-1)}}{(2\mathcal{M}_n\mathcal{D}t)^{(n+1)/2(n-1)}} \times \exp\left\{-\frac{\Omega^2}{2\mathcal{M}_n\mathcal{D}t}\right\}, \tag{41}$$

from which we obtain the PDF of the energy

$$\tilde{P}_t(E) = \frac{1}{\Gamma\left(\frac{n+1}{2(n-1)}\right)}\frac{n-1}{nE} \times \left(\frac{(2n)^{(n+1)/n}E^{(n-1)/n}}{2\mathcal{M}_n\mathcal{D}t}\right)^{(n+1)/[2(n-1)]} \times \exp\left\{-\frac{(2n)^{(n+1)/n}E^{(n-1)/n}}{2\mathcal{M}_n\mathcal{D}t}\right\}. \tag{42}$$

The long-time behavior of the amplitude, velocity, and energy of the general nonlinear oscillator can now be derived. Using the equipartition identity (31) and Eq. (42), we obtain

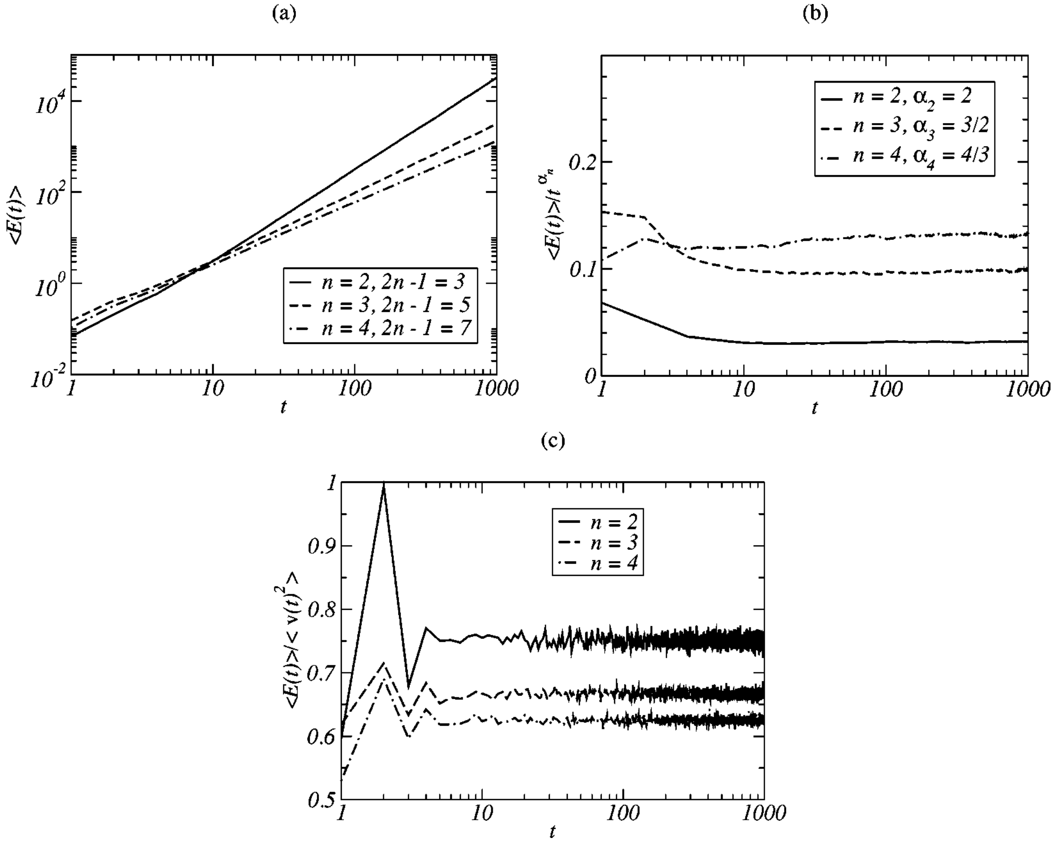

FIG. 2. General nonlinear oscillator: Eq. (11) is integrated numerically for $\mathcal{D}=1$ with a time step δt, and averaged over 10^4 realizations for $n=2$ $(2n-1=3)$, $\delta t=5\times10^{-4}$; $n=3$ $(2n-1=5)$, $\delta t=5\times10^{-4}$; $n=4$ $(2n-1=7)$, $\delta t=10^{-4}$. (a) Average energy $\langle E(t)\rangle$ vs time t. (b) The limit $\lim_{t\to\infty}\langle E(t)\rangle/t^{\alpha_n}$, $\alpha_n=n/(n-1)$ yields the following estimates of the diffusion constants $D_E^{(n)}$: $D_E^{(2)}=0.031(1)$, $D_E^{(3)}=0.097(3)$, $D_E^{(4)}=0.130(5)$. These are in excellent agreement with the predictions of Eqs. (48)–(50). (c) The measured equipartition ratio $\langle E(t)\rangle/\langle v(t)^2\rangle$ is close to the theoretical value $(n+1)/(2n)$ given in Eq. (31): $\frac{3}{4}$ for $n=2$; $\frac{2}{3}$ for $n=3$; $\frac{5}{8}$ for $n=4$.

$$\langle E\rangle=\frac{1}{(2n)^{(n+1)/(n-1)}}\frac{\Gamma\left(\frac{3n+1}{2n-2}\right)}{\Gamma\left(\frac{n+1}{2n-2}\right)}(2\mathcal{M}_n\mathcal{D}t)^{n/(n-1)}\propto t^{n/(n-1)}, \tag{43}$$

$$\langle\dot{x}^2\rangle=\frac{2n}{n+1}\langle E\rangle\propto t^{n/(n-1)}. \tag{44}$$

Using Eq. (38), we find that

$$\overline{\mathcal{S}_n^2(\phi)}=\frac{1}{K_n}\int_0^{K_n}\mathcal{S}_n^2(\phi)d\phi=(2n)^{1/n}\frac{\int_0^1\frac{u^2\,du}{\sqrt{1-u^{2n}}}}{\int_0^1\frac{du}{\sqrt{1-u^{2n}}}}=\frac{n+3}{(2n)^{1/n}(n-1)^2}\mathcal{M}_n, \tag{45}$$

where we made the change of variables $u=\mathcal{S}_n(\phi)$, and used the following identity (obtained by integrating by parts):

$$\int_0^1 u^2\sqrt{1-u^{2n}}\,du=\frac{n}{n+3}\int_0^1\frac{u^2\,du}{\sqrt{1-u^{2n}}}. \tag{46}$$

Finally, we deduce from Eqs. (13), (42), and (45)

$$\langle x^2\rangle=\overline{\mathcal{S}_n^2(\phi)}\langle E^{1/n}\rangle=\frac{2}{n-1}\frac{\mathcal{M}_n}{(2n)^{1/n}}\frac{\Gamma\left(\frac{3n+1}{2n-2}\right)}{\Gamma\left(\frac{n+1}{2n-2}\right)}\left(\frac{2\mathcal{M}_n\mathcal{D}t}{(2n)^{(n+1)/n}}\right)^{1/(n-1)}\propto t^{1/(n-1)}. \tag{47}$$

When $\mathcal{D}=1$, the analytical results for the nonlinear oscillators with cubic x^3 $(n=2)$, quintic x^5 $(n=3)$, and heptic nonlinearity x^7 $(n=4)$ are as follows:

$$\text{for } n=2,\quad \langle E\rangle=0.031t^2,\quad \langle\dot{x}^2\rangle=0.042t^2,\quad \langle x^2\rangle=0.125t; \tag{48}$$

$$\text{for } n=3,\quad \langle E\rangle=0.097t^{3/2},\quad \langle\dot{x}^2\rangle=0.145t^{3/2},$$

$$\langle x^2 \rangle = 0.290 t^{1/2}; \tag{49}$$

$$\text{for } n=4, \quad \langle E \rangle = 0.130 t^{4/3}, \quad \langle \dot{x}^2 \rangle = 0.208 t^{4/3},$$

$$\langle x^2 \rangle = 0.347 t^{1/3}. \tag{50}$$

The cubic oscillator ($n=3$) will be discussed in more detail in Sec. III B. Formulas (43), (44), and (47) were verified numerically. The scaling exponents and the prefactors given in Eqs. (48), (49), and (50) are in excellent agreement with the numerical values, as shown in Fig. 2.

In conclusion, we have derived the following scaling relations:

$$E \sim t^{n/(n-1)},$$

$$x \sim t^{1/[2(n-1)]},$$

$$\dot{x} \sim t^{n/[2(n-1)]}. \tag{51}$$

In particular, it should be noted that x undergoes an anomalous diffusion with time with exponent $1/(2n-2)$. If we make $n \rightarrow 1$ formally, this exponent diverges to infinity: this is consistent with the exponential growth of the linear oscillator (see Sec. II).

We end this section by considering the case of a general confining potential energy $\mathcal{U}$ neither necessarily polynomial in x nor even in x. The only requirement is that $\mathcal{U} \rightarrow +\infty$ when $|x| \rightarrow \infty$. We discuss the qualitative behavior of E, $\dot{x}$, and x at long times from elementary scaling considerations. Suppose first that $\mathcal{U} \sim |x|^r$ for large values of $|x|$, r being an arbitrary real number.

(i) If $r>2$, then balance between kinetic and potential energies leads to $E \sim \dot{x}^2 \sim x^r$; thus, the time evolution of the energy is given by $\dot{E} \sim x \dot{x} \xi \sim E^{[(1/r)-(1/2)]} \xi$. From the scaling relations $\dot{E} \sim E/t$ and $\xi \sim t^{-1/2}$, we conclude that

$$E \sim t^{r/(r-2)}, \quad x \sim t^{1/(r-2)}, \quad \dot{x} \sim t^{r/[2(r-2)]}.$$

This qualitative argument can be made rigorous by generalizing the results obtained above to noninteger values of r.

(ii) If $r \leqslant 2$, then the potential $\mathcal{U}$ is negligible with respect to the multiplicative noise term, and we are back to the case of the degenerate linear oscillator. Therefore, E, $\dot{x}$, and x grow exponentially with time.

If the potential grows exponentially, i.e., $\mathcal{U} \sim e^{x^\beta}$, β being a positive real number, then similar considerations lead to $E \sim t$ and $\dot{x} \sim t^{1/2}$ (disregarding logarithmic corrections). We then conjecture that the amplitude x diffuses in a logarithmically slow manner: $x \sim (\ln t)^{1/\beta}$.

B. The degenerate cubic oscillator

In this section, we study the particular case of a cubic nonlinearity,

$$\frac{d^2}{dt^2} x(t) + \xi(t) x(t) + x(t)^3 = 0. \tag{52}$$

The results obtained in Sec. III A adopt simpler expressions in terms of the classical Jacobi elliptic functions sn, cn, and dn [31,32].

The exact solution of the deterministic version of Eq. (52), $\ddot{x} + x^3 = 0$, is

$$x = E^{1/4} \frac{\mathrm{sn}\left((4E)^{1/4} t; \frac{1}{\sqrt{2}} \right)}{\mathrm{dn}\left((4E)^{1/4} t; \frac{1}{\sqrt{2}} \right)}, \tag{53}$$

$$\dot{x} = (2E)^{1/2} \frac{\mathrm{cn}\left((4E)^{1/4} t; \frac{1}{\sqrt{2}} \right)}{\mathrm{dn}^2\left((4E)^{1/4} t; \frac{1}{\sqrt{2}} \right)}, \tag{54}$$

for a fixed value of the energy E, defined as

$$E = \tfrac{1}{2} \dot{x}^2 + \tfrac{1}{4} x^4. \tag{55}$$

The quarter of the period of the elliptic functions sn, cn, and dn that appear in Eqs. (53) and (54) is given by

$$K_2 = K\left(\frac{1}{\sqrt{2}} \right) = \sqrt{2} \int_0^1 \frac{du}{\sqrt{1-u^4}} \simeq 1.854. \tag{56}$$

When the noise term is taken into account, the energy is not conserved. Inverting the relation (53), we obtain the definition of the phase variable:

$$\phi = \mathrm{sd}^{-1}\left(\frac{x}{E^{1/4}}, \frac{1}{\sqrt{2}} \right), \tag{57}$$

where we have introduced the function sd=sn/dn. The stochastic evolution of the variables E and ϕ becomes

$$\dot{E} = x \dot{x} \xi(t) = \sqrt{2} E^{3/4} \frac{\mathrm{sn}(\phi; 1/\sqrt{2})\, \mathrm{cn}(\phi; 1/\sqrt{2})}{\mathrm{dn}^3(\phi; 1/\sqrt{2})} \xi(t), \tag{58}$$

$$\dot{\phi} = (4E)^{1/4} - \frac{\mathrm{sd}^2(\phi; 1/\sqrt{2})}{2\sqrt{2} E^{1/4}} \xi(t). \tag{59}$$

Introducing the auxiliary variable $\Omega = 2\sqrt{2} E^{1/4}$, the Eqs. (58) and (59) can be written in the simpler form

$$\dot{\Omega} = \frac{\mathrm{sn}(\phi) \mathrm{cn}(\phi)}{\mathrm{dn}^3(\phi)} \xi(t), \tag{60}$$

$$\dot{\phi} = \frac{\Omega}{2} - \frac{\mathrm{sd}^2(\phi)}{\Omega} \xi(t), \tag{61}$$

where the elliptic modulus $1/\sqrt{2}$ common to all the elliptic functions has been omitted.

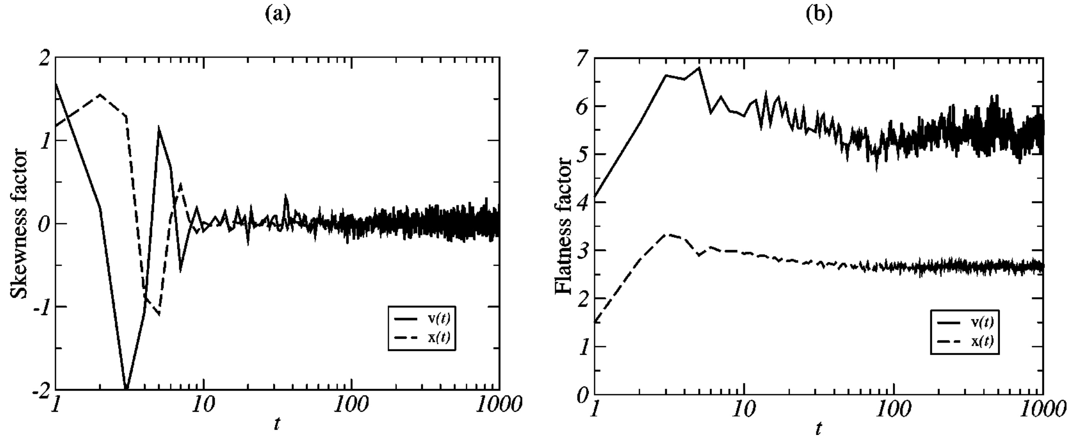

FIG. 3. Cubic oscillator. (a) Skewness factors of the position $x(t)$ (dashed line) and velocity $v(t)$ (solid line). (b) Flatness factors of the position $x(t)$ (dashed line) and velocity $v(t)$ (solid line). Numerical data is obtained from the same runs as in Fig. 2.

The averaging process, with respect to the fast variable, generates an effective Langevin dynamics for the slow variable $\Omega(t)$. Starting from the Fokker-Planck equation for the total PDF, $P_t(\Omega,\phi)$, and averaging over ϕ leads to the following evolution equation for the averaged probability distribution:

$$\partial_t \tilde{P} = \frac{\mathcal{D}\mathcal{M}_2}{2}\left(\partial_\Omega^2 \tilde{P} - 2\partial_\Omega \frac{\tilde{P}}{\Omega}\right), \tag{62}$$

where $\tilde{P}_t(\Omega)$ is now a function of Ω and t only, and where $\mathcal{M}_2$ is given by

$$\mathcal{M}_2 = \frac{1}{K}\int_0^K \frac{\mathrm{sn}^2(\phi)\mathrm{cn}^2(\phi)}{\mathrm{dn}^6(\phi)} d\phi = 2\frac{\int_0^1 u^2\sqrt{1-u^4}\,du}{\int_0^1 \frac{du}{\sqrt{1-u^4}}} = \frac{32\pi^2}{5\left(\Gamma\left(\frac{1}{4}\right)\right)^4} \simeq 0.3655\ldots. \tag{63}$$

The effective Langevin dynamics for the variable Ω is given by

$$\dot{\Omega} = \frac{\mathcal{D}\mathcal{M}_2}{\Omega} + \Xi(t), \tag{64}$$

where the effective noise $\Xi(t)$ satisfies the relation

$$\langle \Xi(t)\Xi(t')\rangle = \mathcal{M}_2 \mathcal{D}\delta(t-t'). \tag{65}$$

The Fokker-Planck equation (62) can be solved exactly. We obtain the following PDF for the energy:

$$\tilde{P}_t(E) = \frac{1}{\sqrt{\pi}E}\left(\frac{4E^{1/2}}{\mathcal{D}\mathcal{M}_2 t}\right)^{3/2} \exp\left\{-\frac{4E^{1/2}}{\mathcal{D}\mathcal{M}_2 t}\right\}. \tag{66}$$

The long-time behavior of the amplitude, velocity, and energy of the cubic oscillator can now be derived:

$$\langle E\rangle = \frac{15}{64}(\mathcal{D}\mathcal{M}_2 t)^2 = \frac{48\pi^4}{5\left[\Gamma\left(\frac{1}{4}\right)\right]^8}(\mathcal{D}t)^2 \simeq 0.0313(\mathcal{D}t)^2, \tag{67}$$

$$\langle \dot{x}^2\rangle = \frac{4}{3}\langle E\rangle = \frac{64\pi^4}{5\left[\Gamma\left(\frac{1}{4}\right)\right]^8}(\mathcal{D}t)^2 \simeq 0.0417(\mathcal{D}t)^2, \tag{68}$$

$$\langle x^2\rangle = \overline{\mathrm{sd}^2(\phi)}\langle E^{1/2}\rangle = \frac{15}{16}(\mathcal{M}_2)^2\mathcal{D}t = \frac{192\pi^4}{5\left[\Gamma\left(\frac{1}{4}\right)\right]^8}\mathcal{D}t \simeq 0.125\mathcal{D}t. \tag{69}$$

The prefactor in the mean value of the energy, Eq. (67), agrees with that of [26].

The distribution function allows us to calculate the PDF in the $(x,\dot{x})$ variables in the time-asymptotic regime. In particular, the skewness and flatness factors of the position and of the velocity can be calculated analytically. Since both variables x and $\dot{x}$ are parity symmetric, their skewness vanishes. The flatness is given by

$$\frac{\langle x^4\rangle}{\langle x^2\rangle^2} = \frac{4}{3[\overline{\mathrm{sd}^2(\phi)}]^2}\frac{\langle \Omega^4\rangle}{\langle \Omega^2\rangle^2} = \frac{16}{45\mathcal{M}^2} \simeq 2.66, \tag{70}$$

$$\frac{\langle v^4\rangle}{\langle v^2\rangle^2} = \frac{9}{4}\,\overline{\frac{\mathrm{cn}^4(\phi)}{\mathrm{dn}^8(\phi)}}\,\frac{\langle \Omega^8\rangle}{\langle \Omega^4\rangle^2} = \frac{27}{5} = 5.4. \tag{71}$$

These values are also in excellent agreement with the numerical computations shown in Fig. 3. We notice that the variables x and $\dot{x}$ are non-Gaussian because their flatness differs from the Gaussian value 3.

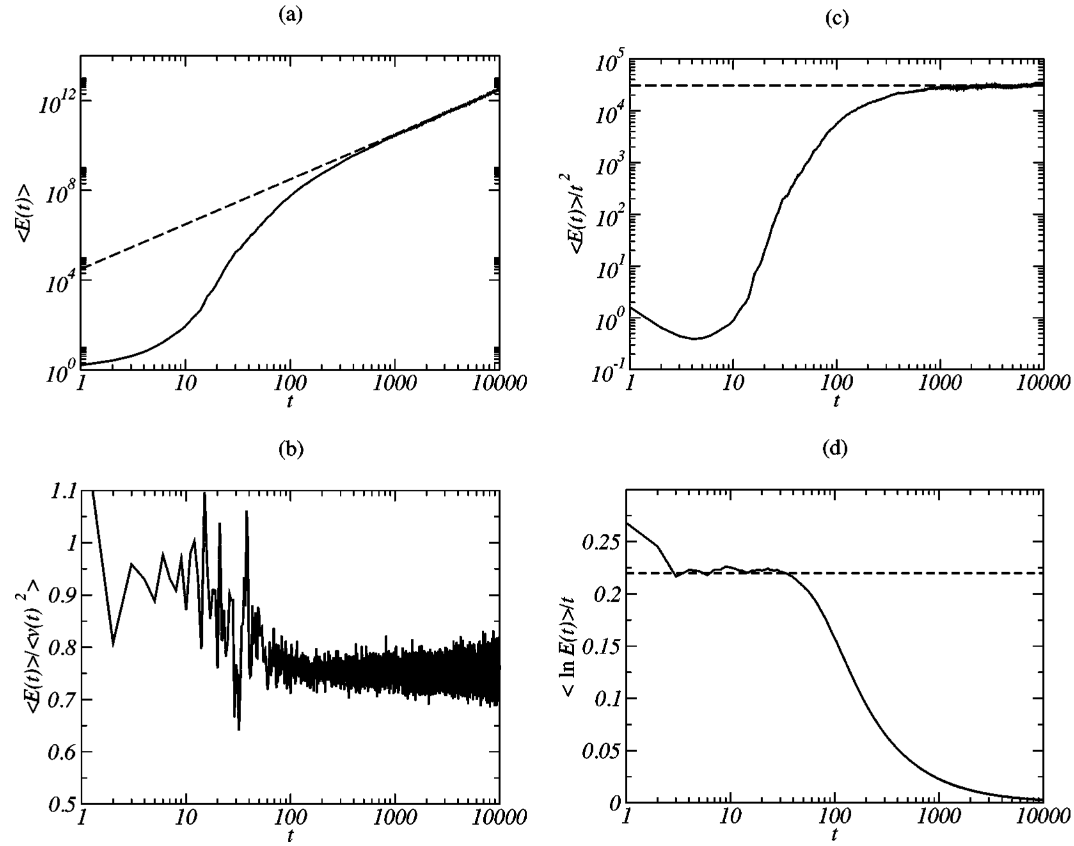

FIG. 4. Duffing oscillator: Eq. (72) is integrated numerically for $\omega=1$, $\lambda=10^{-6}$, and $\mathcal{D}=1$ with a time step $\delta t=10^{-4}$. Ensemble averages are computed over 10^3 realizations. (a) $\langle E(t)\rangle$ vs t; the dashed line corresponds to the expected power law $\langle E(t)\rangle \simeq D_E(\lambda)t^2$. (b) The numerical value of the equipartition ratio $\langle E(t)\rangle/\langle v(t)^2\rangle$ changes from a value close to 1 to $\frac{3}{4}$. (c) The numerical estimate of the generalized diffusion constant $D_E(\lambda)=\lim_{t\to\infty}\langle E(t)\rangle/t^2$ is $3.0(4)\times 10^4$; the dashed line corresponds to the expected diffusion constant $D_E(\lambda)=3.13\times 10^4$. (d) $\langle \ln E(t)\rangle/t$ vs time t; the dashed line corresponds to the Lyapounov exponent $\Lambda=0.22$ expected in the linear regime.

C. The Duffing oscillator with multiplicative noise

We now study the general case of a nonzero pulsation ω:

$$\frac{d^2}{dt^2}x(t)+[\omega^2+\xi(t)]x(t)+x(t)^3=0. \tag{72}$$

Here the coefficient of the nonlinear term has been rescaled to unity and the random noise is Gaussian and white, as defined in Eq. (2). The deterministic nonlinear mechanical system correponding to Eq. (72) is known as the Duffing oscillator.

The results of Secs. II and III B show two regimes: starting from a small initial condition, the amplitude of the oscillator grows exponentially with time until $x\sim\omega$, where the linear and nonlinear terms are of the same order and then the amplitude grows as the square-root of time according to Eq. (69). Because the deterministic system corresponding to Eq. (72) is integrable, this crossover from exponential to algebraic can be derived in a quantitative manner.

When ω is nonzero, Eqs. (53) and (54) become

$$x=\left(\frac{4E^2}{4E+\omega^4}\right)^{1/4}\frac{\mathrm{sn}[(4E+\omega^4)^{1/4}t;k]}{\mathrm{dn}[(4E+\omega^4)^{1/4}t;k]}, \tag{73}$$

$$\dot{x}=(2E)^{1/2}\frac{\mathrm{cn}[(4E+\omega^4)^{1/4}t;k]}{\mathrm{dn}^2[(4E+\omega^4)^{1/4}t;k]}, \tag{74}$$

where the elliptic modulus k varies with the energy and is given by

$$k^2=\frac{\sqrt{4E+\omega^4}-\omega^2}{2\sqrt{4E+\omega^4}}. \tag{75}$$

We notice that k tends to the limiting value $1/\sqrt{2}$ when the energy goes to infinity. Defining the angle variable as

$$\phi=\mathrm{sd}^{-1}\left(\frac{(4E+\omega^4)^{1/4}}{\sqrt{2E}}x;k\right), \tag{76}$$

we rewrite the dynamical equation in energy-angle coordinates. However, while deriving the dynamical equation for ϕ we must remember that the elliptic modulus k depends on the energy E and is, therefore, a function of time. After reintroducing the multiplicative noise term, the stochastic Duffing oscillator in the energy-angle variables becomes

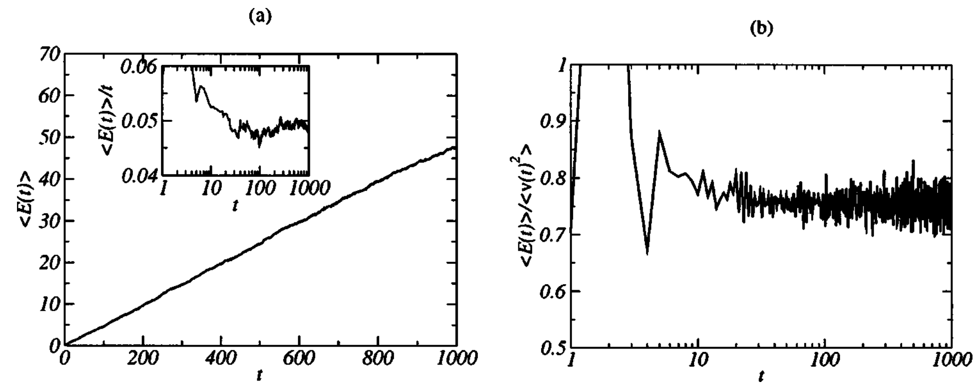

FIG. 5. Cubic oscillator with colored noise. Equations (80),(82) are integrated numerically for $n=2$, $2n-1=3$, $\mathcal{D}=1$, and $\tau=1$ with a time step $\delta t=10^{-5}$. Ensemble averages are computed over 10^3 realizations. (a) The average energy $\langle E(t)\rangle$ grows linearly with time. The inset gives an estimate of the diffusion constant $D_E \simeq 0.047(3)$. (b) The time-asymptotic value of the equipartition ratio $\langle E(t)\rangle/\langle v(t)^2\rangle$ is equal to $\frac{3}{4}$.

$$\dot{E}=\frac{2E}{(4E+\omega^4)^{1/4}}\frac{\mathrm{sn}(\phi;k)\mathrm{cn}(\phi;k)}{\mathrm{dn}^3(\phi;k)}\xi(t); \tag{77}$$

$$\dot{\phi}=(4E+\omega^4)^{1/4}-\frac{(2E+\omega^4)}{(4E+\omega^4)^{5/4}}\mathrm{sd}^2(\phi;k)\xi(t) - \frac{E\omega^2\xi(t)}{(4E+\omega^4)^{7/4}}\left(\mathrm{sd}^4(\phi;k)-\frac{\mathrm{sn}(\phi;k)\mathrm{cn}(\phi;k)}{\mathrm{dn}^3(\phi;k)} \times\int_0^{\phi}\mathrm{sd}^2(\theta;k)d\theta\right). \tag{78}$$

As compared to Eq. (59), two supplementary terms appear in Eq. (78). These terms are related to dk/dt and are proportional to ω^2.

Although the dynamical equations are more complicated than those of the purely cubic case, the analysis can be performed as above. We shall, however, simplify our discussion here by taking k equal to its asymptotic value $1/\sqrt{2}$. This approximation is justified as soon as the energy is large. We also replace the noise $\xi(t)$ by the effective noise $\Xi(t)$ defined in Eq. (65). This second approximation is only qualitatively correct, since it amounts to neglecting a deterministic force in the effective Langevin dynamics for E. We thus obtain

$$\dot{E}\simeq\frac{2E}{(4E+\omega^4)^{1/4}}\Xi(t). \tag{79}$$

We deduce from Eq. (79) that as long as $E\ll\omega^4$, the energy behaves as the exponential of a Brownian motion and, therefore, increases exponentially with time. However, when $E>\omega^4$, the nonlinear term becomes important. Equation (77) reduces to Eq. (58), and the energy grows as the square of time.

We expect that the crossover from exponential to algebraic growth will appear when $E\sim\omega^4$ or $x\sim\omega$. Using unscaled variables, the balance between linear and nonlinear terms in Eq. (72) is obtained when $x=x_c\sim\omega/\sqrt{\lambda}$. Figure 4 demonstrates that the two regimes are observed numerically when the nonlinear coefficient λ is very small compared to ω^2: we use the numerical values $\omega=1$ and $\lambda=10^{-6}$. We notice that in the short-time linear regime, the usual equipartition relation for a quadratic potential is verified ($\langle E\rangle \simeq\langle\dot{x}^2\rangle$), while the exponential growth of the energy is characterized by the Lyapunov exponent Λ, defined in Eq. (7). In the long-time regime, the equipartition ratio reaches its nonlinear value $\frac{3}{4}$, while the energy growth becomes algebraic with a generalized diffusion constant $D_E(\lambda)$, in good agreement with Eq. (67), up to the expected scaling factor $D_E(\lambda)=D_E/\lambda$.

IV. COLORED GAUSSIAN NOISE

We now consider the case where the Gaussian noise has a nonzero correlation time, and discuss how the previously found scalings are modified. The system we want to study satisfies the dynamical equation

$$\frac{d^2}{dt^2}x(t)+x(t)\eta(t)+x(t)^{2n-1}=0, \tag{80}$$

where η is a colored Gaussian noise of zero mean value. The statistical properties of η are determined by

$$\langle\eta(t)\rangle=0,$$

$$\langle\eta(t)\eta(t')\rangle=\frac{\mathcal{D}}{2\tau}\mathrm{e}^{-|t-t'|/\tau}, \tag{81}$$

where τ is the correlation time of the noise. The noise η can be obtained from white noise by solving the Ornstein-Uhlenbeck equation

$$\frac{d\eta(t)}{dt}=-\frac{1}{\tau}\eta(t)+\frac{1}{\tau}\xi(t), \tag{82}$$

where $\xi(t)$ is the Gaussian white noise defined in Eq. (2), and $t,t'\gg\tau$.

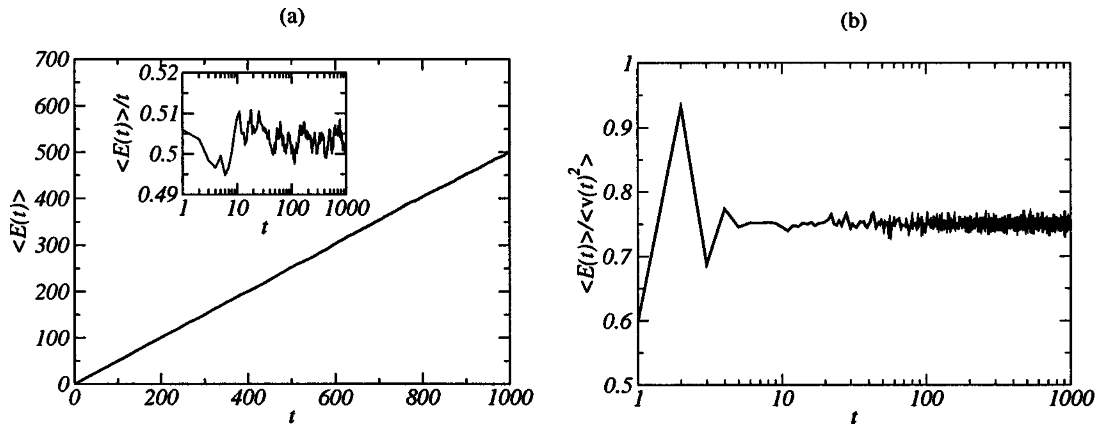

FIG. 6. Cubic oscillator with additive noise. Equation (A1) is integrated numerically for $n=2$, $\mathcal{D}=1$ with a time step $\delta t=5\times10^{-4}$. Ensemble averages are computed over 10^4 realizations. (a) The average energy $\langle E(t)\rangle$ grows linearly with time. The inset gives an estimate of the diffusion constant when noise is additive: $D_E\simeq0.505(5)$. (b) The time-asymptotic value of the equipartition $\langle E(t)\rangle/\langle v(t)^2\rangle$ is equal to $\frac{3}{4}$.

Introducing action-angle variables as in Sec. III, we obtain the same set of coupled Langevin equations, Eqs. (27),(28), where $\xi(t)$ is replaced by the colored noise $\eta(t)$. As emphasized previously, the generalized equipartition relations are independent of the nature of the noise: Eqs. (31) and (32) remain valid when the noise is correlated in time. This is confirmed by numerical simulations [see Fig. 5(b) for $n=2$].

The scalings found in Sec. III A are deduced by averaging the Fokker-Planck equation. Here, we must write the evolution equation for the joint PDF of x, $v=\dot{x}$ and η, $P_t(x,v,\eta)$:

$$\frac{\partial P}{\partial t}=-v\frac{\partial P}{\partial x}+(x^{2n-1}-x\eta)\frac{\partial P}{\partial v}+\frac{1}{\tau}\frac{\partial \eta P}{\partial \eta}+\frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 P}{\partial \eta^2}. \tag{83}$$

We perform a scaling analysis of this equation in the spirit of [34]. Balancing the diffusion term with the time derivative leads to $\eta\sim t^{1/2}$. Then we compare the terms of probability current $(vP)/x$ and $[(x^{2n-1}-x\eta)P]/v$. A consistent balance between these terms is possible only if $v^2\sim x^{2n}$ and $x^{2n-2}\sim\eta$. We thus find the following scaling relations:

$$E\sim t^{n/[2(n-1)]},$$

$$x\sim t^{1/[4(n-1)]},$$

$$\dot{x}\sim t^{n/[4(n-1)]}. \tag{84}$$

Thus, we predict that the scaling exponents for colored noise are *half* the exponents calculated for white noise (51). Numerical simulations indeed confirm that the average energy of a cubic oscillator $(n=2)$ with colored multiplicative noise grows linearly with time [see Fig. 5(a)].

The period T of a deterministic oscillator (without noise) decreases as the energy increases: $T\sim E^{-(n-1)/2n}$, from Eq. (25). When the equations are written in the energy-angle coordinates, two time scales T and τ appear. In the regime where $\tau\ll T$, the correlation time of the noise is much smaller than the typical variation time of the angle. Hence, the noise can be considered to be white, the averaging procedure can be applied as in Sec. III, and the scalings found in (51) are correct. When $T\sim\tau$, the noise becomes correlated over a period of the free system and cannot be treated as white anymore. Now, $T\sim\tau$ corresponds to $E\sim\tau^{-2n/(n-1)}$, which leads to crossover time t_c of the order $t_c^{n/(n-1)}\sim\tau^{-2n/(n-1)}$, i.e., $t_c\sim\tau^{-2}$. For times larger than t_c, the scalings (84) are observed.

V. CONCLUSION

A particle trapped in a confining potential with white multiplicative noise undergoes anomalous diffusion: if the confining potential grows as x^{2n} at infinity, the particle diffuses as $\langle x^2\rangle\sim D_x^{(n)}\,t^{\beta_n}$. We have calculated the anomalous diffusion exponent $\beta_n=1/(n-1)$, and the coefficient $D_x^{(n)}$. We have found similar laws for the diffusion of velocity and energy. Thanks to generalized equipartition identities, we have derived universal relations between the exponents and between the prefactors. Our calculations are based on the assumption that in the long-time limit the probability distribution function becomes uniform in the phase variable. By averaging out the phase variations, an effective projected dynamics for the action (or energy) can be defined. This technique enabled us to derive the asymptotic distribution law of the energy in the $t\to\infty$ limit, and to calculate its non-Gaussian features (skewness and flatness). Our analytical results agree with the numerical simulations within the numerical error bars. Thus, the averaging procedure produces very accurate results; it would be an interesting and challenging problem to characterize deviations from our results and to calculate subleading corrections.

In the case of colored multiplicative noise, we have deduced the anomalous diffusion exponents from an elementary scaling argument. Our result, supported by numerics, shows that the exponents are halved in the presence of time correlations. The efficiency of parametric amplification decreases if the noise is coherent over a period of the system and, therefore, the particle diffuses at a much slower rate. In this case, however, the averaging technique is harder to apply because the noise itself is averaged out to the leading order. A precise calculation in the case of colored Gaussian noise still remains to be done.

We have considered only Hamiltonian systems, i.e., systems where no friction is present. Nevertheless, if the damping is small, the results we have derived for the undamped oscillator remain valid until the crossover time (identical to the typical decay time of the energy) is reached. The general case of a nonlinear oscillator with (linear) friction leads to interesting results and is currently under study [35].

ACKNOWLEDGMENTS

It is a pleasure to thank Yves Pomeau for encouraging us to work in this field of nonlinear stochastic equations and for his advice. K.M. is grateful to Michel Bauer for many useful discussions.

APPENDIX A: NONLINEAR OSCILLATOR WITH EXTERNAL AND INTERNAL NOISE

In this Appendix we discuss the behavior of an oscillator subjected to both additive and multiplicative noises. Because we are considering nondissipative systems, there is no stationary probability distribution; the position, velocity, and energy of the system satisfy scaling laws.

We first consider the case where the noise is only additive. As before, the linear term can be neglected in the long-time limit and the dynamics is given by

$$\frac{d^2}{dt^2}x(t)+x(t)^{2n-1}=\xi(t), \tag{A1}$$

where $\xi(t)$ is the Gaussian white noise defined in Eq. (2). The oscillator's energy, defined in Eq. (12), now obeys the following equation:

$$\dot{E}=\dot{x}\xi(t). \tag{A2}$$

Although this equation can be analyzed as in Sec. III, we will only discuss our results qualitatively, referring the reader to [35] for a thorough analysis. Using the energy and angle variables, defined in Eqs. (22) and (23), we find that the angle variable ϕ is a fast variable: the equipartition relationships, Eqs. (31) and (32), remain valid. This is indeed confirmed by numerical simulations, as shown in Fig. 6(b) for a cubic oscillator.

Since $\dot{x}\sim E^{1/2}$ and $x\sim E^{1/2n}$, we obtain from Eq. (A2) the scaling laws for E, $\dot{x}$, and x:

$$E\sim t,$$
$$x\sim t^{1/2n},$$
$$\dot{x}\sim t^{1/2}. \tag{A3}$$

The scaling exponents for additive noise are different from, and smaller than, the exponents for multiplicative white noise [Eq. (51)]. For example, we observe that a particle subjected to an additive noise in a quartic potential ($n=2$) is subdiffusive with an anomalous exponent equal to $\frac{1}{2}$, whereas in the presence of multiplicative noise it behaves diffusively.

Finally, when both additive and multiplicative noises are present, the oscillator is governed by the equation

$$\frac{d^2}{dt^2}x(t)+x(t)\xi_{\rm mult}(t)+x(t)^{2n-1}=\xi_{\rm add}(t), \tag{A4}$$

where $\xi_{\rm mult}$ and $\xi_{\rm add}$ are independent white noises of amplitude $\mathcal{D}_{\rm mult}$ and $\mathcal{D}_{\rm add}$, respectively. If we study the energy variation due to noise, $\dot{E}\sim x\dot{x}\xi_{\rm mult}+\dot{x}\xi_{\rm add}\simeq E^{(n+1)/2n}\xi_{\rm mult}+E^{1/2}\xi_{\rm add}$, we observe that the first term has a dominant effect. From this simple argument, we conclude that the multiplicative noise is expected to dominate over the additive noise and, therefore, asymptotically, the scaling laws will be those derived for the multiplicative noise alone. However, a crossover between the two scalings (51) and (A3) should be observed by choosing $\mathcal{D}_{\rm mult}\ll\mathcal{D}_{\rm add}$. Comparing Eqs. (51) and (A3), we find that the effect of the multiplicative noise starts to dominate after a crossover time of the order of $(\mathcal{D}_{\rm mult}t_c)^{1/(2n-2)}\sim(\mathcal{D}_{\rm add}t_c)^{1/2n}$, i.e., $t_c\sim\mathcal{D}_{\rm add}^{n-1}/\mathcal{D}_{\rm mult}^{n}$.

APPENDIX B: NUMERICAL ALGORITHM

The algorithm used to integrate numerically the stochastic ordinary differential equations studied in this article is the one-step collocation method advocated in [29]. In this Appendix, we recall the general principles underlying this method, and give the algorithms we used to integrate Eqs. (11) and (80)–(82) for white and colored noise, respectively. All stochastic equations are understood according to Stratonovich rules.

Let $\{x_i\}_{i=1,\ldots,N}$ be N real variables of time t, and $\xi(t)$ a stochastic process assumed to be Gaussian and white. We wish to solve systems of N coupled Langevin equations of the form

$$\dot{x}_i=f_i(\{x_j(t)\})+g_i(\{x_j(t)\})\xi(t), \tag{B1}$$

where f_i and g_i are N (smooth) functions of the x_i's. Let δt be the integration time step. Upon formally integrating Eq. (B1) between 0 and δt, we obtain the following set of coupled equations implicit in $\{x_i(t)\}$:

$$x_i(\delta t)-x_i(0)=\int_0^{\delta t}f_i(\{x_j(s)\})ds+\int_0^{\delta t}g_i(\{x_j(s)\})\xi(s)ds. \tag{B2}$$

For small enough δt, the functions f_i and g_i may be Taylor expanded in the vicinity of $t=0$, e.g.,

$$f_i(\{x_j(s)\})=f_i(\{x_j(0)\})+\partial_k f_i(\{x_j(0)\})\delta x_k(s)+\tfrac{1}{2}[\partial_k\partial_l f_i(\{x_j(0)\})]\delta x_k(s)\,\delta x_l(s)+\cdots, \tag{B3}$$

where $\delta x_k(s)=x_k(s)-x_k(0)$. Replacing the functions f_i and g_i in Eq. (B2) by the expansion (B3), we obtain a new set of coupled integral equations implicit in $\{\delta x_i\}$. Upon solving these equations up to an arbitrary order in δt, we obtain $\delta x_k(\delta t)$. In practice, we implemented an algorithm exact to $O(\delta t^2)$.

1. White noise

We integrate the following set of first-order differential equations:

$$\dot{x}=v, \tag{B4}$$

$$\dot{v}=-\omega^2 x-\lambda x^{2n-1}+x\xi, \tag{B5}$$

where the Gaussian white noise $\xi(t)$ verifies Eq. (2). The exact evolution equations for $\delta x(\delta t)=x(\delta t)-x(0)$ and $\delta v(\delta t)=v(\delta t)-v(0)$ read

$$\delta x(\delta t)=\int_0^{\delta t} ds[v(0)+\delta v(s)], \tag{B6}$$

$$\delta v(\delta t)=\int_0^{\delta t} ds\{-\omega^2[x(0)+\delta x(s)]-\lambda[x(0)+\delta x(s)]^{2n-1} +[x(0)+\delta x(s)]\xi(s)\}. \tag{B7}$$

The auxiliary variables Z_1 and Z_2, defined by

$$Z_1(\delta t)=\int_0^{\delta t}\xi(s)ds, \tag{B8}$$

$$Z_2(\delta t)=\int_0^{\delta t} Z_1(s)ds, \tag{B9}$$

are Gaussian random variables with zero average and the following correlations: $\langle Z_1(\delta t)^2\rangle=\mathcal{D}\delta t$, $\langle Z_2(\delta t)^2\rangle=\mathcal{D}\delta t^3/3$, and $\langle Z_1(\delta t)Z_2(\delta t)\rangle=\mathcal{D}\delta t^2/2$. Up to order $(\delta t)^2$, we find

$$x(\delta t)=x(0)+v(0)\delta t+Z_2(\delta t) +\tfrac{1}{2}\delta t^2[-\omega^2 x(0)-\lambda x(0)^{2n-1}], \tag{B10}$$

$$v(\delta t)=v(0)+Z_1(\delta t)+\delta t[-\omega^2 x(0)-\lambda x(0)^{2n-1}] +\tfrac{1}{2}\delta t^2 v(0)[-\omega^2 x(0)-(2n-1)\lambda x(0)^{2n-2}]. \tag{B11}$$

In practice, we use two *independent* Gaussian random noises, Z_1 and Y_1, with zero mean and correlations $\langle Z_1(\delta t)^2\rangle=\mathcal{D}\delta t$, and $\langle Y_1(\delta t)^2\rangle=\mathcal{D}\delta t$. As shown in Ref. [29], the variable Z_2 may be approximated by the expression

$$Z_2(\delta t)=\delta t\left(\frac{1}{2}Z_1(\delta t)+\frac{1}{2\sqrt{3}}Y_1(\delta t)\right) \tag{B12}$$

when the algorithm is exact up to order δt^2.

2. Colored noise

When the noise $\eta(t)$ is correlated, we must solve a set of three coupled equations:

$$\dot{x}=v, \tag{B13}$$

$$\dot{v}=-\omega^2 x-\lambda x^{2n-1}+x\eta, \tag{B14}$$

$$\dot{\eta}=-\frac{1}{\tau}\eta+\frac{1}{\tau}\xi. \tag{B15}$$

The set of exact integral equations becomes

$$\delta x(\delta t)=\int_0^{\delta t} ds[v(0)+\delta v(s)], \tag{B16}$$

$$\delta v(\delta t)=\int_0^{\delta t} ds\{-\omega^2[x(0)+\delta x(s)]-\lambda[x(0)+\delta x(s)]^{2n-1} +[x(0)+\delta x(s)][\eta(0)+\delta\eta(s)]\}, \tag{B17}$$

$$\delta\eta(\delta t)=\int_0^{\delta t} ds\left\{-\frac{1}{\tau}[\eta(0)+\delta\eta(s)]+\frac{1}{\tau}\xi(t)\right\}. \tag{B18}$$

The algorithm to order δt^2 reads

$$x(\delta t)=x(0)+v(0)\delta t+\frac{1}{2}v(0)\delta t^2, \tag{B19}$$

$$v(\delta t)=v(0)+\delta t[-\omega^2 x(0)-\lambda x(0)^{2n-1}+x(0)\eta(0)] +\frac{1}{\tau}x(0)Z_2(\delta t)+\frac{1}{2}\delta t^2\bigg(v(0)[-\omega^2 x(0) -(2n-1)\lambda x(0)^{2n-2}+\eta(0)]-\frac{1}{\tau}x(0)\eta(0)\bigg), \tag{B20}$$

$$\eta(\delta t)=\eta(0)+\frac{1}{\tau}Z_1(\delta t)+-\frac{1}{\tau}\eta(0)\delta t-\frac{1}{\tau^2}Z_2(\delta t) +\frac{1}{2}\delta t^2\frac{\eta(0)}{\tau^2}. \tag{B21}$$

[1] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1992).

[2] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, New York, 1989).

[3] H. Horsthemke and R. Lefever, *Noise Induced Transitions* (Springer-Verlag, New York, 1984).

[4] P. S. Landa and P. V. E. McClintock, Phys. Rep. **323**, 1 (2000).

[5] C. van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).

[6] C. van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).
[7] J. M. R. Parrondo, C. van den Broeck, J. Buceta, and F. Javier de la Rubia, Physica A **224**, 153 (1996).
[8] M. San Miguel and R. Toral, in *Instabilities and Nonequilibrium Structures VI*, edited by E. Tirapegui and W. Zeller (Kluwer, Dordrecht, 1997).
[9] S. Barbay, G. Giacomelli, and F. Marin, Phys. Rev. E **61**, 157 (2000).
[10] R. Bourret, Physica (Amsterdam) **54**, 623 (1971); R. Bourret, U. Frisch, and A. Pouquet, *ibid.* **65**, 303 (1973).
[11] M. Lücke and F. Schank, Phys. Rev. Lett. **54**, 1465 (1985).
[12] J. Röder, H. Röder, and L. Kramer, Phys. Rev. E **55**, 7068 (1997).
[13] W. Genovese and M. A. Muñoz, Phys. Rev. E **60**, 69 (1999).
[14] P. Reimann, Phys. Rep. **361**, 57 (2002).
[15] K. Lindenberg and B. J. West, Physica A **128**, 25 (1984).
[16] R. L. Stratonovich, *Topics on the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. 1; (1967), Vol. 2.
[17] S. Kabashima, S. Kogure, T. Kawakubo, and T. Okada, J. Appl. Phys. **50**, 6296 (1979).
[18] T. Kawakubo, A. Yanagita, and S. Kabashima, J. Phys. Soc. Jpn. **50**, 1451 (1981).
[19] R. Berthet, S. Residori, B. Roman, and S. Fauve, Phys. Rev. Lett. **33**, 557 (2002).
[20] J. B. Roberts and M. Vasta, J. Appl. Mech. **67**, 763 (2000).
[21] P. S. Landa and A. A. Zaikin, Phys. Rev. E **54**, 3535 (1996).
[22] J. P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[23] L. D. Landau and I. Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1969).
[24] C. van den Broeck, M. Malek Mansour, and F. Baras, J. Stat. Phys. **33**, 557 (1982).
[25] F. Drolet and J. Viñals, Phys. Rev. E **57**, 5036 (1998).
[26] V. Seshadri, B. J. West, and K. Lindenberg, Physica A **107**, 219 (1981).
[27] D. Hansel and J. F. Luciani, J. Stat. Phys. **54**, 971 (1989).
[28] L. Tessieri and F. M. Izrailev, Phys. Rev. E **62**, 3090 (2000).
[29] R. Mannella, in *Noise in Nonlinear Dynamical Systems, Vol. 3: Experiments and Simulations*, edited by F. Moss and P. V. E. Mc Clintock (Cambridge University Press, Cambridge, 1989).
[30] C. Degli Esposti Boschi and L. Ferrari, Phys. Rev. E **63**, 026218 (2001).
[31] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1966).
[32] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer-Verlag, Berlin, 1954).
[33] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, Berlin, 1992).
[34] Y. Pomeau, J. Phys. I **3**, 365 (1993).
[35] K. Mallick and P. Marcq (unpublished).