Statistical properties of a photon gas in random media

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This paper is devoted to a derivation of the probability distribution of photon escape from a semi-infinite random medium, depending on the number of its interactions with macroscopic particles inside the medium. The consideration is limited to the case of highly developed multiple light scattering. The distribution function found facilitates the solution of both direct and inverse problems in light scattering media optics.

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I. INTRODUCTION

Statistical characteristics of a photon gas in random media is an important subject which has a number of applications [1–3]. We address the following question. We have a constant infinitely broad monochromatic photon flux incident in the direction specified by the unit vector $\vec{m}_0$ (ϑ_0,φ_0) on the surface of a semi-infinite plane-parallel random isotropic medium. Here ϑ_0 is the zenith angle and φ_0 is the azimuth angle of the incident light beam. It is assumed also that the energy of the photons is far from the absorption bands of substances contained in the random medium. We consider the angular distribution $N_0(\vec{m},\vec{m}_0)$ of photons emerging in the direction specified by the unit vector $\vec{m}(\vartheta,\varphi)$ from an arbitrary point S on the surface of a layer. Here ϑ is the zenith angle and φ is the azimuth angle of the emerging light beam. Clearly, due to the symmetry of the problem this distribution does not depend on the choice of S on the surface of a random isotropic medium. Also we have for normal illumination $\vec{m}_0\cdot\vec{x}=0$, $\vec{x}\in L$, where L is the plane containing the medium surface and $\vec{x}$ is an arbitrary unit vector in the plane L.

The function $N_0(\vec{m},\vec{m}_0)$ can be obtained by solving Ambartsumian's nonlinear integral equation [4,5]. We are interested, however, in the representation of $N_0(\vec{m},\vec{m}_0)$ as a sum of contributions $p(\vec{m},\vec{m}_0,n)$ due to photons scattered n times in the medium [1–3]:

$$N_0(\vec{m},\vec{m}_0)=\sum_{n=1}^{\infty} p(\vec{m},\vec{m}_0,n). \tag{1}$$

Photons scattered different numbers of times, of course, will have different path lengths in the scattering medium. The value of $p(\vec{m},\vec{m}_0,n)$, which is often called the photon weight, can be considered also in the framework of the Feynman path integral approach [6,7]. This approach, being very general, can be applied to any type of scattering medium. We will use here, however, the essential features of the medium under consideration, namely, its infinite extension in the space below the plane L and the absence of photon absorption. These assumptions allow us to avoid path integral calculations. Clearly, the average number of scattering events in such an artificial medium is infinite as well. The primary goal of this paper, therefore, is to derive the analytical representation for the weights $p(\vec{m},\vec{m}_0,n)$.

II. THE PROBABILITY DISTRIBUTION FUNCTION

Let us introduce the probabilities

$$f(n)=\frac{p(\vec{m},\vec{m}_0,n)}{\sum_{n=1}^{\infty} p(\vec{m},\vec{m}_0,n)} \tag{2}$$

with the normalization condition

$$\sum_{n=1}^{\infty} f(n)=1. \tag{3}$$

The value of $f(n)$ can be interpreted as the probability for photons injected in the medium in the direction $\vec{m}_0$ and scattered n times to emerge in the direction specified by the vector $\vec{m}$. The condition (3) states that the total probability of photon escape is equal to 1. This is due to the assumed absence of absorption.

To derive the function $f(n)$ we will use the random walk theory [8]. This theory states that the probability of a particle appearing at a given place, time, and direction after a large number n of interactions is given by

$$f(n)=\sqrt{\alpha/\pi}n^{-3/2}\exp(-\alpha/n), \tag{4}$$

where the constant α depends on the physical process under study. The only problem left is, therefore, to find the constant α for our particular case. Clearly, it does not depend on the position of the point S on the surface of the medium. It also does not depend on time because we consider the steady case. So the only dependence left is due to local optical characteristics of the random medium and the vectors $\vec{m},\vec{m}_0$.

To derive the parameter α, we consider now the case of an absorbing medium with the same scattering law in a single scattering event as for the nonabsorbing semi-infinite random medium in question.

The probability of photon survival ω_0 in a single scattering event differs from 1 for absorbing media. Then we have instead of Eq. (1) [9,10]

$$N(\vec{m},\vec{m_0})=\sum_{n=1}^{\infty} p(\vec{m},\vec{m}_0,n)\omega_0^n. \qquad (5)$$

Clearly, it follows that $N\equiv N_0$ at $\omega_0=1$. We can also use the distribution function $f(n)$. Then we have for $\xi\equiv N/N_0$

$$\xi=\sum_{n=1}^{\infty} f(n)\omega_0^n, \qquad (6)$$

where for the sake of simplicity we omitted the indications of the dependencies of ξ and f on $\vec{m}$, $\vec{m}_0$. We will consider now a random turbid medium with $\omega_0\approx 1$.

First of all we note that the expansion (6) is only slowly convergent for $\omega_0\approx 1$. So we will use an expansion in the parameter $\beta=1-\omega_0$, which is the probability of photon absorption, instead of the expansion in ω_0 in Eq. (6). Then we have from Eq. (6)

$$\xi=\sum_{n=1}^{\infty} f(n)(1-\beta)^n \qquad (7)$$

or

$$\xi=\sum_{n=1}^{\infty} f(n)\left(1-\beta n+\frac{\beta^2 n(n-1)}{2}-\frac{\beta^3 n(n-1)(n-2)}{6}+\cdots\right), \qquad (8)$$

where we used the expansion

$$(1-\beta)^n=\sum_{j=0}^{n} (-1)^j\binom{n}{j}\beta^j \qquad (9)$$

with $(n/j)\equiv n!/j!(n-j)!$.

It follows from Eq. (8) that approximately

$$\xi\approx 1-\beta\overline{n}+\frac{\beta^2\overline{n^2}}{2}-\frac{\beta^3\overline{n^3}}{6}+\cdots\approx\overline{\exp(-\beta n)}, \qquad (10)$$

where

$$\overline{n^k}=\sum_{n=1}^{\infty} f(n)n^k, \quad k=1,\ldots,\infty,$$

$$\overline{\exp(-\beta n)}=\sum_{n=1}^{\infty} f(n)\exp(-\beta n). \qquad (11)$$

Note that we have assumed that $n(n-1)\approx n^2$, $n(n-1)(n-2)\approx n^3$,... in the derivation of Eq. (10). This is possible due to the large number of scattering events n. For the same reason we have

$$\overline{\exp(-\beta n)}=\int_0^{\infty} f(n)\exp(-\beta n)dn, \qquad (12)$$

where [see Eq. (3)]

$$\int_0^{\infty} f(n)dn=1. \qquad (13)$$

The integral (12) can be evaluated analytically. The answer is

$$\overline{\exp(-\beta n)}=\exp(-2\sqrt{\alpha\beta}), \qquad (14)$$

where we used Eq. (4). Thus we have, taking into account Eq. (10),

$$N(\vec{m},\vec{m}_0)=N_0(\vec{m},\vec{m}_0)\exp(-2\sqrt{\alpha\beta}) \qquad (15)$$

or, as $\beta\to 0$,

$$N(\vec{m},\vec{m}_0)=N_0(\vec{m},\vec{m}_0)(1-2\sqrt{\alpha\beta}). \qquad (16)$$

Now we have an opportunity to find the value of α, comparing Eq. (16) with the exact solution of the radiative transfer equation at small β [3]:

$$N(\vec{m},\vec{m}_0)=N_0(\vec{m},\vec{m}_0)[1-yu(\vec{m},\vec{m}_0)], \qquad (17)$$

where $y=4\sqrt{\beta/3(1-g)}$ and $g=\frac{1}{2}\int_0^{\pi}p(\theta)\sin\theta\cos\theta d\theta$ is the asymmetry parameter [3], $p(\theta)$ is the probability of a photon scattering in the direction specified by the scattering angle θ, and

$$u(\vec{m},\vec{m}_0)\equiv[K_0(\mu)K_0(\mu_0)]/[R_0(\mu,\mu_0,\psi)], \qquad (18)$$

where [1]

$$K_0(\mu)=\frac{3}{2}\int_0^1 R_\infty^0(\mu,\mu_0)(\mu+\mu_0)d\mu_0 \qquad (19)$$

is the escape function and

$$R_\infty^0(\mu,\mu_0)=\frac{1}{2\pi}\int_0^{2\pi} R_\infty^0(\mu,\mu_0,\psi)d\psi \qquad (20)$$

is the azimuthally averaged reflection function $R_\infty^0(\mu,\mu_0,\psi)$ [3]. Here $\mu=\cos\vartheta$, $\mu_0=\cos\vartheta_0$, $\psi=\varphi-\varphi_0$. Clearly we have $\mu_0=\sqrt{1-(\vec{m}_0\cdot\vec{x})^2}$, $\mu=\sqrt{1-(\vec{m}\cdot\vec{x})^2}$.

It follows from Eqs. (16) and (17) that

$$\alpha=(4u^2)/[3(1-g)], \qquad (21)$$

which is the result we tried to establish from the very beginning. Finally, we have from Eqs. (4) and (21)

$$f(n)=\frac{2u\exp[-4u^2/3n(1-g)]}{n^{3/2}\sqrt{3\pi(1-g)}}, \qquad (22)$$

where u is given by Eq. (21). We note here the importance of the viewing function u, which combines all angular dependencies.

Our Eq. (22) transforms to the similar equations derived in [9,10] if one uses the expansion of the exponent in a power series in the value of n^{-1}. Only the first term of such an expansion was explicitly derived in [9,10].

It follows from Eq. (22) that $f'(n)=0$ at $n=n_{\max}=2\alpha/3$ and

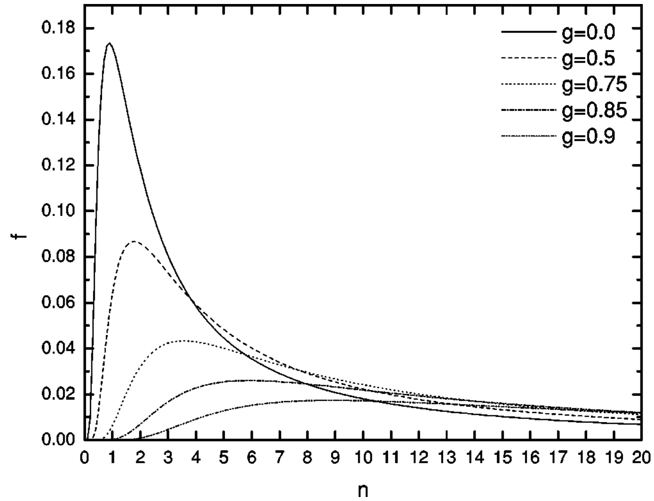

FIG. 1. The dependence $f(n)$ for various values of g at $u=1$.

$$f_{\max} \equiv f(n_{\max}) = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{3}{2}\right)^{3/2}\exp\left(-\frac{3}{2}\right), \tag{23}$$

where α is given by Eq. (21). We also have [see Eq. (21)]

$$n_{\max} = (8u^2)/[9(1-g)]. \tag{24}$$

We see that $n_{\max}$ increases with g and $f_{\max}$ is linearly proportional to $1-g$.

Finally, it follows from Eqs. (2) and (22)

$$p(\mu,\mu_0,n) = \frac{2K_0(\mu)K_0(\mu_0)\exp[-4u^2(\mu,\mu_0,\psi)/3n(1-g)]}{n^{3/2}\sqrt{3\pi(1-g)}}, \tag{25}$$

where we used the normalization condition $N_0(\vec{m},\vec{m}_0) = R_0(\mu,\mu_0,\psi)$. The function $K_0(\mu)$ can be approximated by [11,12] the expression $K_0(\mu) = \frac{3}{7}(1+2\mu)$ for arbitrary random media with discrete particles. The accuracy of this formula is better than 2% at $\mu > 0.2$ [11,12]. Simple approximations for $R_0(\mu,\mu_0,\psi)$ are derived in [1,5,11,12].

We present results of calculations with Eq. (22) at $u=1$ in Fig. 1. It follows that the maximum of the probability f increases linearly with increasing $1-g$, which is in correspondence with Eq. (23). Clearly, we obtain that $f\equiv 0$ at $g \equiv 1$. This means that photons do not have a chance to escape from the medium. They only propagate along straight lines (no scattering).

The maximum shifts to larger numbers of scatterings n for larger values of g, which is in correspondence with Eq. (24). The influence of the parameter $s=u^{-2}$ on the curves $f(n)$ is similar to that of the parameter $1-g$. The value of $n_{\max}$ only slowly changes with g at $g<0.7$. However, it increases rapidly at $g>0.9$.

III. APPLICATIONS

The main result, given by Eq. (22), can be generalized to other situations and measurement setups. For instance, if one is interested in functions $\overline{f}(n)$ for total photon numbers, integrated over the escape angle, it is possible to obtain instead of Eq. (22), following the same line of reasoning,

$$\overline{f}(n) = \frac{2K_0(\mu_0)\exp[-4K_0^2(\mu_0)/3n(1-g)]}{n^{3/2}\sqrt{3\pi(1-g)}}. \tag{26}$$

Using similar arguments we obtain for the case of integration on both the incident and escaped photon directions:

$$\tilde{f}(n) = \frac{2\exp[-4/3n(1-g)]}{n^{3/2}\sqrt{3\pi(1-g)}}. \tag{27}$$

This corresponds to the case of diffuse illumination and diffuse reflectance measurements. Clearly, we have for the diffuse reflection coefficient $r(\vartheta_0)$

$$r(\vartheta_0) = \sum_{n=0}^{\infty} \overline{f}(n)\omega_0^n, \tag{28}$$

which gives us after transfer to the continuous basis

$$r(\vartheta_0) = \int_0^{\infty} \overline{f}(n)\omega_0^n dn. \tag{29}$$

This formula can also be presented in the following form:

$$r(\vartheta_0) = \int_0^{\infty} \overline{f}(n)\exp(pn)dn, \tag{30}$$

where $p=\ln(1/\omega_0)\approx 1-\omega_0=\beta$ as $\omega_0\to 1$. It follows from Eqs. (30) and (26) that

$$r(\vartheta_0) = \exp[-yK_0(\mu_0)], \tag{31}$$

where $y=4\sqrt{[\ln(1/\omega_0)]/3(1-g)}$. Also, we have for the total reflectivity

$$r = \sum_{n=0}^{\infty} \tilde{f}(n)\omega_0^n \tag{32}$$

or, following the same steps as in the derivation of Eq. (31), $r=\exp(-y)$. All cases considered here correspond to weak absorption and, therefore, we can use the limiting value of y for ω_0 close to 1: $y=4\sqrt{(1-\omega_0)/3(1-g)}$. The expressions for $r(\vartheta_0)$ and r derived here have been known for a long time [11]. They were obtained, however, using a different approach. Our derivations allow us to make clearer their physical basis.

Also, using Eqs. (22) and (6) we conclude that

$$\xi \equiv \frac{R_\infty}{R_\infty^0} = \exp(-uy). \tag{33}$$

This formula is also the well known result of radiative transfer theory [11,12]. We see that the derived function (22) appears to be a key point for the derivation of many important relations in scattering media optics in a simple and straightforward manner.

It can also be used to establish the temporal statistics of photons emerging from a random light scattering layer. For instance, accounting for the fact that $n = vt/L$, where t is time needed for a photon travel the distance nL with the group speed v, we may obtain from Eq. (22) the distribution of photons according to arrival times t or paths nL. Here $L = \sigma^{-1}$, where σ is the extinction coefficient of the random medium. The value of L is called the photon free path length. Taking this into account, we obtain from Eq. (22)

$$f(t) = 2u\left(\frac{L}{v}\right)^{3/2}\frac{t^{-3/2}}{\sqrt{3\pi(1-g)}}\exp\left(-\frac{4Lu^2}{3(1-g)vt}\right). \quad (34)$$

Clearly the functions $\overline{f(n)}$ and $\tilde{f}(n)$ transform to

$$\overline{f}(t) = 2K_0(\mu_0)\left(\frac{L}{v}\right)^{3/2}\frac{t^{-3/2}}{\sqrt{3\pi(1-g)}}\exp\left(-\frac{4LK_0^2(\mu_0)}{3(1-g)vt}\right) \quad (35)$$

and

$$\tilde{f}(t) = 2\left(\frac{L}{v}\right)^{3/2}\frac{t^{-3/2}}{\sqrt{3\pi(1-g)}}\exp\left(-\frac{4L}{3(1-g)vt}\right). \quad (36)$$

For instance, we have from Eq. (35) at normal incidence

$$\overline{f(t)} = \sqrt{[B/\pi(1-g)]}\left(\frac{L}{vt}\right)^{3/2}\exp\left(-\frac{BL}{(1-g)vt}\right), \quad (37)$$

where

$$B = \tfrac{4}{3}K_0^2(1). \quad (38)$$

The value of B is approximately equal to 2.2. Equation (37) has been derived earlier [7,13]. However, the value of B in [13] is equal to 0.75. We believe that B, given in Eq. (38), is closer to the exact result. Note that the value of B is equal to 2.19 in [7], which is close to the result given by Eq. (38).

The distribution on path lengths is obtained by substitution of t in Eqs. (34)–(36) by s/v, where $s = nL$ is the total distance traveled by a photon after n scatterings.

The probability distribution function $f(n)$ can be used also to find the statistical moments

$$\overline{n^k} = \int_0^\infty n^k f(n)\omega_0^n dn. \quad (39)$$

It can be done analytically; namely, we have

$$\overline{n^k} = (-1)^{k-1}\sqrt{\alpha}\frac{\partial^{k-1}}{\partial p^{k-1}}\left(\frac{\exp(-2\sqrt{\alpha p})}{\sqrt{p}}\right), \quad (40)$$

where α is given by Eq. (21) and $p = \ln(1/\omega_0)$. In particular, we have

$$\overline{n} = \sqrt{\alpha/p}\exp(-2\sqrt{\alpha p}). \quad (41)$$

Thus, one can obtain for the average number of scatterings involved in forming the observed absorption line [14]

$$\langle N\rangle = \left(\int_0^\infty nf(n)\omega_0^n dn\right) \Big/ \left(\int_0^\infty f(n)\omega_0^n dn\right) \quad (42)$$

the following simple relation:

$$\langle N\rangle = (2u)/\gamma, \quad (43)$$

where $\gamma = \sqrt{3(1-g)(1-\omega_0)}$ is the diffusion exponent of radiative transfer theory [3]. The average distance traveled by a photon having $\langle N\rangle$ scatterings is given by $l = \langle N\rangle L$. This distance varies with the observation geometry due to the presence of the viewing function u in Eq. (43). Thus, we conclude that the strength of the absorption line in the scattering atmosphere will also depend on the viewing geometry.

IV. CONCLUSION

In conclusion, we derived here the probability distribution function (22). It describes the photon migration from the direction $\vec{m}_0$ to the direction $\vec{m}$ after n interactions with scatterers. This function might be of importance for a wide range of applications. Some of them are outlined above. Interestingly enough, the geometry of the observation enters Eq. (22) as a single number u. This gives significance to the function u, given by Eq. (18), for radiative transfer theory.

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