Relativistic $E\times B$ acceleration

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The relativistic motion of charged particles is analyzed theoretically in electric and magnetic fields that are constant, uniform, and mutually perpendicular. In the relativistic regime where the magnitude of the electric field **E** is equal to or greater than that of the magnetic field **B**, i.e., $|\mathbf{E}|\geqslant|\mathbf{B}|$, the particle is effectively accelerated and gains energy indefinitely. This is quite different from the $\mathbf{E}\times\mathbf{B}$ drift motion in the nonrelativistic regime.

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I. INTRODUCTION

Understanding the motion of charged particles in electric and magnetic fields is basic to plasma research [1]. In particular, the $\mathbf{E}\times\mathbf{B}$ drift motion is well known as a popular drift motion. However, it is difficult to analyze the drift motion in the relativistic regime because of the strong nonlinearity of the Lorentz factor.

Landau and Lifshitz [2] have calculated the drift motion in the relativistic regime where $|\mathbf{E}|=|\mathbf{B}|$. Some of their results, presented in parametric form, are focused on a special case.

Jackson [3] has predicted that if $|\mathbf{E}|<|\mathbf{B}|$, the $\mathbf{E}\times\mathbf{B}$ drift motion in the nonrelativistic regime is significant; on the other hand, when the condition $|\mathbf{E}|\geqslant|\mathbf{B}|$ is satisfied, the particle acceleration becomes dominant rather than the drift motion. In a special case where $|\mathbf{E}|=|\mathbf{B}|$, this effective acceleration has already been used as a velocity spectrometer [4]. However, this is only a part of the $\mathbf{E}\times\mathbf{B}$ acceleration. Specific calculations and explanations of the acceleration mechanism have not been performed.

Therefore, we have derived here exact solutions from the relativistic equation of motion and investigate the relativistic $\mathbf{E}\times\mathbf{B}$ acceleration in detail.

II. BASIC EQUATIONS

The relativistic equation of motions of a particle with mass m and charge q in electric and magnetic fields is given by

$$m\frac{d\gamma\mathbf{v}}{dt}=q\mathbf{E}+\frac{q}{c}\mathbf{v}\times\mathbf{B},\tag{1}$$

where $\gamma\equiv1/\sqrt{1-(\mathbf{v}/c)^2}$ is the Lorentz factor and c is the velocity of light. We choose the uniform fields

$$\mathbf{E}=(E_0,0,0),\qquad \mathbf{B}=(0,0,-B_0),\tag{2}$$

and normalize physical quantities in the form: $\beta\equiv v/c$, $\tau\equiv\Omega t$, $\tilde{E}\equiv E_0/B_0$, where $\Omega\equiv qB_0/mc$ is the cyclotron frequency. Thus, the equation can be rewritten as

$$d(\gamma\beta_x)/d\tau=\tilde{E}-\beta_y,\tag{3}$$

$$d(\gamma\beta_y)/d\tau=\beta_x,\tag{4}$$

$$d(\gamma\beta_z)/d\tau=0,\tag{5}$$

and the other new equation corresponding to the energy equation is derived as

$$d\gamma/d\tau=\tilde{E}\beta_x.\tag{6}$$

III. PARTICLE TRAJECTORIES

Introducing the following relations:

$$d\xi/d\tau=\beta_x,\qquad d\eta/d\tau=\beta_y,\qquad d\zeta/d\tau=\beta_z,$$

$$\xi-\xi_0\equiv X,\qquad \eta-\eta_0\equiv Y,\qquad \zeta-\zeta_0\equiv Z,$$

it is possible to integrate Eqs. (3)–(6) as follows:

$$\gamma\beta_x=\gamma_0\beta_{x0}-(\eta-\eta_0)+\tilde{E}\tau\equiv G,\tag{7}$$

$$\gamma\beta_y=\gamma_0\beta_{y0}+(\xi-\xi_0)\equiv d+X,\tag{8}$$

$$\gamma\beta_z=\gamma_0\beta_{z0}\equiv k,\tag{9}$$

$$\gamma=\gamma_0+\tilde{E}(\xi-\xi_0)\equiv\gamma_0+\tilde{E}X.\tag{10}$$

Substituting Eqs. (7)–(10) into the modified Lorentz factor as $\gamma^2-1=(\gamma\beta_x)^2+(\gamma\beta_y)^2+(\gamma\beta_z)^2$, we obtain

$$G^2=(\tilde{E}^2-1)X^2+2\gamma_0(\tilde{E}-\beta_{y0})X+\gamma_0^2\beta_{x0}^2,\tag{11}$$

and redefine in a simpler form:

$$G=\sqrt{aX^2+bX+c}\equiv f_1(X),$$

where $G\equiv\gamma_0\beta_{x0}-Y+\tilde{E}\tau$ as shown in Eq. (7) and

$$a\equiv\tilde{E}^2-1,\qquad b\equiv2\gamma_0(\tilde{E}-\beta_{y0}),\qquad c\equiv\gamma_0^2\beta_{x0}^2.$$

From Eqs. (7)–(9), we can derive the differential equations

$$\frac{\gamma\beta_y}{\gamma\beta_x}=\frac{d\eta}{d\xi}=\frac{dY}{dX}=\frac{X+d}{G}\equiv\frac{X+d}{f_1(X)},\tag{12}$$

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$$\frac{\gamma\beta_z}{\gamma\beta_x}=\frac{d\zeta}{d\xi}=\frac{dZ}{dX}=\frac{k}{G}\equiv\frac{k}{f_1(X)}, \tag{13}$$

and obtain the following from Eqs. (7) and (10):

$$\frac{\gamma}{\gamma\beta_x}=\frac{d\tau}{dX}=\frac{\tilde{E}X+\gamma_0}{G}\equiv\frac{\tilde{E}X+\gamma_0}{f_1(X)}. \tag{14}$$

These equations have exact solutions as presented in the Appendix.

When the condition $a\neq 0$ is satisfied, then the exact solutions are described in the forms

$$Y=\frac{1}{a}[f_1(X)-f_0]+\left(d-\frac{b}{2a}\right)[I(X)-I_0], \tag{15}$$

$$Z=k[I(X)-I_0], \tag{16}$$

$$\tau=\frac{\tilde{E}}{a}[f_1(X)-f_0]+\left(\gamma_0-\frac{b\tilde{E}}{2a}\right)[I(X)-I_0], \tag{17}$$

where f_0 and I_0 are the initial values at $X=0$. Combining these three equations leads to another new relation

$$\beta_{z0}\tau=\tilde{E}\beta_{z0}Y+(1-\tilde{E}\beta_{y0})Z. \tag{18}$$

First, let us consider the case where $a<0$, i.e., $E_0<B_0$. In this case the particle drifts in the $\mathbf{E}\times\mathbf{B}$ direction with gyration, and its trajectory is described by Eq. (11) as

$$(Y-\gamma_0\beta_{x0}-\tilde{E}\tau)^2-a\left(X+\frac{b}{2a}\right)^2=c-\frac{b^2}{4a}. \tag{19}$$

This implies that the trajectory is elliptical in the XY plane because of $a<0$; in addition, its guiding center moves along the Y direction with a constant velocity, namely the drift velocity $V_g=cE_0/B_0$. In the limit of $\tilde{E}=0$, the trajectory becomes a circle or cyclotronlike motion as described below:

$$(Y-\gamma_0\beta_{x0})^2+(X+\gamma_0\beta_{y0})^2=\gamma_0^2(\beta_{x0}^2+\beta_{y0}^2), \tag{20}$$

and the trajectory in the Z direction obeys the relation $Z=\beta_{z0}\tau$ derived from Eq. (18).

If $a>0$ or $E_0>B_0$, then the electric force becomes stronger than the Lorentz force. Therefore, the particle can never gyrate and moves linearly along the trajectories

$$Y\approx\frac{X}{\sqrt{a}}=\frac{X}{\sqrt{\tilde{E}^2-1}}, \tag{21}$$

$$Z\approx\frac{k}{\sqrt{a}}\ln|4aX|=\frac{\gamma_0\beta_{z0}}{\sqrt{\tilde{E}^2-1}}\ln|4(\tilde{E}^2-1)X|. \tag{22}$$

When $a=0$ or $E_0=B_0$ is satisfied, the trajectory of the particle follows a slow arc described by the other exact solutions:

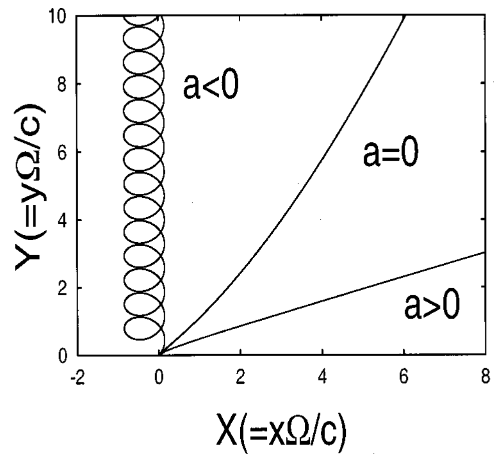

FIG. 1. Particle trajectories projected on the XY plane. $a<0$, trajectory of the $E\times B$ drift described by Eq. (19) where $\tilde{E}=0.1$; $a=0$, slow curved trajectory described by Eq. (23) or Eq. (25) where $\tilde{E}=1.0$; $a>0$, linear trajectory described by Eq. (15) or Eq. (21) where $\tilde{E}=3.0$. Initial values are given as $(X_0,Y_0)=(0{,}0)$ and $(\beta_{x0},\beta_{y0},\beta_{z0})=(0.3{,}0.4{,}0.1)$.

$$Y=\frac{2X}{3b}f_2(X)+\frac{6bd-4c}{3b^2}[f_2(X)-f_0], \tag{23}$$

$$Z=\frac{2k}{b}[f_2(X)-f_0], \tag{24}$$

where $f_2(X)=\sqrt{bX+c}$. As time goes on X grows larger, thus approximate forms of the trajectories can be shown by

$$Y\approx\frac{2X^{3/2}}{3\sqrt{b}}=\frac{\sqrt{2}X^{3/2}}{3\sqrt{\gamma_0(1-\beta_{y0})}}, \tag{25}$$

$$Z\approx\frac{2kX^{1/2}}{\sqrt{b}}=\beta_{z0}\left(\frac{2\gamma_0X}{1-\beta_{y0}}\right)^{1/2}. \tag{26}$$

Some typical trajectories are shown in Fig. 1.

IV. ENERGY GAIN

To obtain the net energy gain of the particle, we can rewrite Eq. (6) as follows:

$$\gamma\frac{d\gamma}{d\tau}=\frac{1}{2}\frac{d\gamma^2}{d\tau}=\tilde{E}\gamma\beta_{,x}, \tag{27}$$

and further by the use of Eqs. (3), (8), and (10), the above equation can be modified in the form

$$\frac{1}{2}\frac{d^2\gamma^2}{d\tau^2}=a+\frac{\gamma_0(1-\tilde{E}\beta_{y0})}{\gamma}\equiv a+\frac{e}{\gamma}. \tag{28}$$

Replacing the variable γ^2 by Γ and performing the energy integral, we can obtain

$$\frac{1}{4}\left(\frac{d\Gamma}{d\tau}\right)^2 = a\Gamma + 2e\sqrt{\Gamma} + h, \tag{29}$$

where h is the initial value given by

$$h \equiv -a\Gamma_0 - 2e\sqrt{\Gamma_0} + \frac{1}{4}\left(\frac{d\Gamma}{d\tau}\right)^2_{\tau=0}$$
$$= -a\gamma_0^2 - 2e\gamma_0 + (\tilde{E}\gamma_0\beta_{x0})^2. \tag{30}$$

The last term on the right hand side is derived from Eq. (27). Furthermore, Eq. (29) can be rewritten as

$$\frac{d\tau}{d\gamma} = \frac{\gamma}{f_3(\gamma)}, \tag{31}$$

where $f_3(\gamma) = \sqrt{a\gamma^2 + 2e\gamma + h}$ and its exact solution is also given in the form

$$\tau = \frac{f_3(\gamma) - f_3(\gamma_0)}{a} - \frac{e[I(\gamma) - I(\gamma_0)]}{a}. \tag{32}$$

When $a>0$, the first two terms on the right hand side of the above equation are important, while when $a<0$, the last two terms are dominant.

If the condition $a=0$ is satisfied, then

$$\tau = \frac{\gamma f_4(\gamma) - \gamma_0 f_4(\gamma_0)}{3e} - \frac{h[f_4(\gamma) - f_4(\gamma_0)]}{3e^2}, \tag{33}$$

where $f_4(\gamma) = \sqrt{2e\gamma + h}$.

As shown in Eq. (6), if ξ increases linearly, then γ must also increase because of $\Delta\gamma = \tilde{E}\Delta\xi$. When $\gamma \gg 1$ and $a>0$ is assumed, we can obtain

$$\gamma \approx (\tilde{E}^2 - 1)^{1/2}\tau. \tag{34}$$

On the other hand, if $a=0$ is satisfied, the equation can be derived as

$$\gamma \approx [9\gamma_0(1-\beta_{y0})/2]^{1/3}\tau^{2/3}. \tag{35}$$

In both cases, net (energy) gains of the particles increase indefinitely as time elapses.

Time evolutions of some typical (energy) gains are depicted in Fig. 2.

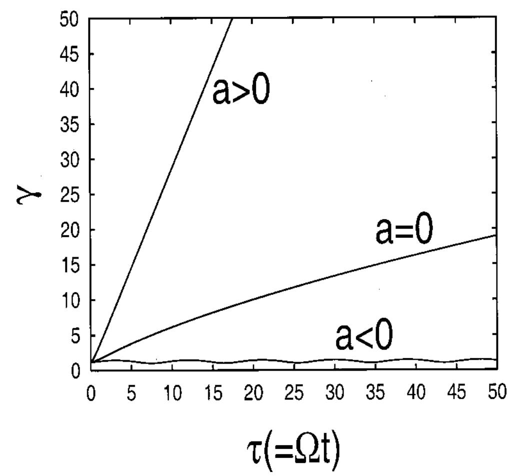

FIG. 2. Time evolution of particle (energy) gains. $a<0$, periodic gain due to the $E\times B$ drift described by Eq. (32) where $\tilde{E}=0.5$; $a=0$, slow curved increment described by Eq. (35) where $\tilde{E}=1.0$; $a>0$, linear gain described by Eq. (34) where $\tilde{E}=3.0$. Initial values are the same as those given in Fig. 1.

V. DISCUSSIONS

With an appropriate Lorentz transformation, the equation of motion can be rewritten simply [3,5]. Let us consider the case where the condition $E_0<B_0$ is satisfied. The particle that stays in the system K' moving with the drift velocity $V_g=(E_0/B_0)c$ relative to the original frame K experiences the electric and magnetic fields:

$$E' = 0, \qquad B' = \frac{B_0}{\gamma_g} = \sqrt{B_0^2 - E_0^2}, \tag{36}$$

where $\gamma_g \equiv 1/\sqrt{1-(V_g/c)^2}$. This implies that the particle gyrates in the uniform magnetic field in the system K'.

If $E_0>B_0$ is satisfied, the velocity V_g will be greater than the velocity of light. Then, we must introduce the other velocity described by $\mathbf{V_f}=c\mathbf{E}_0\times\mathbf{B}_0/E_0^2$. According to the transformation, the fields acting on the particle in the system K'' moving with the velocity V_f are given by

$$E'' = \frac{E_0}{\gamma_f} = \sqrt{E_0^2 - B_0^2}, \qquad B'' = 0, \tag{37}$$

where $\gamma_f \equiv 1/\sqrt{1-(V_f/c)^2}$. The particle only experiences the purely electrostatic field and is accelerated indefinitely with a hyperbolic trajectory in the system K''.

The motion of the particle in the moving frames can be derived more easily than for the original frame. Nevertheless, as calculated in previous sections, it is in the original frame where we can observe the trajectories and the energy gains. Accordingly, the inverse Lorentz transformation from the system K' or K'' to the original frame K is needed and leads to the same result as that in the original frame.

An alternative an acceleration mechanism [magnetic trapping acceleration (MTA)] [5,6] has been presented to account for ultrahigh energy cosmic rays, in which the energy gain of the particle becomes indefinite. The $\mathbf{E}\times\mathbf{B}$ acceleration also has as a feature of indefinite acceleration. If the condition $E_0 \geqslant B_0$ would be satisfied anywhere in the universe, this mechanism might be a candidate for high energy particle generations.

VI. CONCLUSION

The relativistic motions are determined exactly in mutually perpendicular electric and magnetic fields. When the condition $a\geqslant 0$ or $E_0 \geqslant B_0$ is satisfied, the particle can never gyrate anymore and is accelerated indefinitely.

This implies that the drift velocity V_g has a physical meaning only if $V_g/c=E_0/B_0<1$ is satisfied. This is quite different from the drift motion in the nonrelativistic regime. In the limit of $\tilde{E}^2 \ll 1$ and $\gamma \approx 1$, the trajectories in the relativistic motions coincide with the nonrelativistic ones.

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APPENDIX

Mathematical formulas of indefinite integrals are presented below:

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx=\frac{p}{a}\sqrt{ax^2+bx+c}+\left(q-\frac{bp}{2a}\right)I(x), \quad \text{(A1)}$$

where the function $I(x)$ must be classfied into the following two functions. For $a>0$,

$$I(x)=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right|; \quad \text{(A2)}$$

and for $a<0$ and $b^2-4ac>0$,

$$I(x)=-\frac{1}{\sqrt{|a|}}\arcsin\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right). \quad \text{(A3)}$$

If $a=0$, then the above formula is reduced in the following:

$$\int \frac{px+q}{\sqrt{bx+c}}dx=\frac{2p(bx-2c)+6bq}{3b^2}\sqrt{bx+c}. \quad \text{(A4)}$$

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