Harmonic modulation instability and spatiotemporal chaos

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It is shown from the conserved Zakharov equations that many solitary patterns are formed from the modulational instability of unstable harmonic modes that are excited by a perturbative wave number. Pattern selection in our case is discussed. It is found that the evolution of solitary patterns may appear in three states: spatiotemporal coherence, chaos in time but the partial coherence in space, and spatiotemporal chaos. The spatially partial coherent state is essentially due to ion-acoustic wave emission, while spatiotemporal chaos characterized by its incoherent patterns in both space and time is caused by collision and fusion among patterns in stochastic motion. So energy carried by patterns in the system is redistributed.

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Pattern formation in spatially extended nonequilibrium systems has become a major frontier area in science [1]. The existence of spatiotemporal chaos (STC) characterized by its extensive, incoherent (irregular) pattern dynamics in both space and time significantly enriches the study of pattern formation. STC has been discussed numerically and experimentally in many physical systems such as dissipative hydrodynamics [1–3], optics [4], liquid crystals [5], etc. However, the mechanism leading to STC still remains unclear. Recently, transition to STC-like phenomena with coexisting temporal chaos (TC) and spatially partial coherence (SPC) from solitary patterns excited by a perturbative wave number (the master mode) and unstable harmonic modes in a plasma system through the modulational instability of a spatially homogeneous background field has been observed in numerical simulations [6,7]. In this paper, we consider the evolution of patterns in space and time using the well-known Zakharov equations. It is found that the evolution of solitary patterns may undergo the following states: spatiotemporal coherence, coexistence of TC and SPC, as well as STC. Collision and fusion among solitary patterns in the stochastic motion are the main cause of STC.

The conserved one-dimensional Zakharov equations (ZEs) in dimensionless form [8]

$$i\partial_t E + \partial_x^2 E = nE,$$

$$\partial_t^2 n - \partial_x^2 n = \partial_x^2 |E|^2 \tag{1}$$

are used as a model for studying the evolution of patterns. Here $E(x,t)$ is a slowly varying envelope of the Langmuir wave electric field and $n(x,t)$ is the ion density perturbation. The ZEs and their derivatives have been widely used to describe laser-plasma interaction, strong turbulence in plasmas [8], as well as many other physical phenomena. The study of the transition to STC based on the ZEs may therefore contribute to the understanding of the onset of plasma turbulence as well as the evolution of patterns in various branches of science.

Linear stability analysis of the perturbation $\exp[i(kx-\omega t)]$ of a spatially homogeneous field E_0 for Eqs. (1) gives the growth rate of the modulational instability as [6]

$$\gamma = E_0 k[\sqrt{(1-2E_0^2k^2)^2+8E_0^2} - (1+2E_0^2k^2)]^{1/2}, \tag{2}$$

for $0<k<1$. Here the unstable wave number k has been renormalized by $\sqrt{2}E_0$. Equations (1) constitute a near-integrable Hamiltonian system and have at least two types of fixed points: one (0,0) is a center and the other $(E_0,0)$ a saddle.

To investigate the global behavior, we add a small spatial inhomogeneity at $t=0$ on the spatially homogeneous saddle state $\Psi_0 = (\mathrm{Re}\,E, \mathrm{Im}\,E, n, \partial_t n)_{t=0} = (E_0,0,0,0)$ as follows:

$$\Psi_{t=0} = \Psi_0 + (c_1 f_1 + c_2 f_2) E_0 \cos(kx)/500, \tag{3}$$

where f_1 and f_2 are the components of the eigenvector in the saddle subspace for the linearized Eqs. (1) with the constant coefficients c_1, c_2, and the factor 1/500 is to emphasize that the perturbation is very small. The initial condition ensures that the manifolds in the phase space will locally lie in a saddle subspace.

Spatial modulation from the unstable wave number results in a local accumulation of the wave field E and a depletion of the ion density there. Since $n<0$, the latter leads to an increase of the index of refraction $\eta = (1-\omega_{pe}^2/\omega_0^2)^{1/2}$, where ω_0 and $\omega_{pe} = [4\pi(n_0+n)e^2/m_e]^{1/2}$ are the wave and the electron plasma frequencies, respectively, n_0 is the average plasma density. Thus the wave energy is further enhanced in the cavity. When the unstable wave number $k \ll 1$, corresponding to a small growth rate γ, the waves are trapped by the ion density cavity, and a solution of Eqs. (1) is $n = -|E|^2/(1-u^2) \sim \mathrm{sech}^2(\alpha\xi)$, where $\xi = x - ut$ and the sound speed satisfies $u \ll 1$ [8]. That is, the localized structure propagates below the ion sound speed. The ZEs are derived under the quasineutrality assumption. When space charge effects are included the solution still has a soliton structure (the dressed soliton) but with additional small spatial oscillations for larger ξ. When the frequency shift vanishes and $u \ll 1$, the dressed soliton degenerates to the simple soliton here [9].

As k decreases, the stationary state gradually disintegrates. Ion-acoustic wave emission $[I(x,t)]$ becomes important. The solution of Eqs. (1) may be approximately expressed as $n(x,t) \simeq -|E(x,t)|^2 + I(x,t)$ [7], and the system is more unstable. This results in a distortion of the solitary pattern structure.

As $k<1/2$, the master mode k can in principle result in the excitation of $N-1$ unstable harmonic modes, where $N=[k^{-1}]$. Many solitary patterns may be formed from the modulation lengths $l_m = L/m$, where $m=1,2,3,\ldots,M$, $l_1 = L = 2\pi k^{-1}$ is for the master mode, and the others are for the unstable harmonic modes. The envelope E can be written as

$$E(x,t)=\sum_{m=1}^{M} E_m(t)\exp(imkx)+\sum_{m=M+1}^{\infty} E_m(t)\exp(imkx), \tag{4}$$

where the first term on the right-hand side of Eq. (4) comes from the master mode and unstable harmonic modes, and $M<N-1$ is from pattern selection. The second term on the right-hand side of Eq. (4) is from nonlinear interaction.

In the discussion below, for convenience the normalized time-averaged spatial two-point correlation function $\Gamma(x-x')=\Gamma(r)$ [1] and the temporal correlation function (or the Lyapunov exponent) will be used to demonstrate the spatiotemporal features of pattern evolution in the system. As discussed in Refs. [1] and [10,11], if $\Gamma(r)$ behaves like $\sim\exp(-r/\xi)$ for $r\to\infty$ and $\xi>L$ the system is spatially coherent (regular), or $\Gamma(r)=1$, where ξ is the correlation length and L is the system size. In the opposite limit $\xi\ll L$, or $\Gamma(r)\to 0$, the system is spatially incoherent (irregular). However, as we shall see below, in many situations there exists an intermediate state that is SPC, or $0<\Gamma(r)<1$.

We now proceed with the numerical solution of Eqs. (1) and (3) to consider the pattern dynamics and STC. The split-step method for E with a preestimate for the density perturbation n is used to integrate Eq. (1) with periodic boundary conditions. The largest length L from the master mode ($kL=2\pi$) is chosen. The number of grids is 2048 for L and $E_0=2.0$ is taken. In the computation, the conserved quanties are preserved to order 10^{-8}. Furthermore, unless otherwise stated, we set $c_1=c_2=1.0$ (far from homoclinic orbit [10]) without loss of generality.

It has been shown [6] that for $1/2\leqslant k\leqslant 1$, for each set of c_1 and c_2 there exists a critical wave number k_c. Since $k_c<k<1$, the motion of the solitary patterns is temporal recurrent (periodic) or pseudorecurrent (quasiperiodic), and that of the solitary pattern is STC. Since $1/2\leqslant k\leqslant k_c$, the motion of the center of the solitarylike pattern, whose initial peak is located at $x=\pm mL$, $m=0,1,2,\ldots$, exhibits stochastic behavior [10]. The amplitude of the pattern oscillates and its width varies temporally. The system is in TC, as shown by the positive Lyapunov exponent, and the spatial behavior is of SPC due to ion-acoustic wave emission. It has been shown [6] that resonant overlapping may be the cause for TC.

New phenomena are observed first for $k<1/2$ ($N>1$). First, a few harmonic modes are excited. For example, for $k=0.23$ the solitary pattern from the master mode, or master pattern, is first formed, Then patterns from the harmonic modes begin to appear. Two points are of interest: first, during the pattern formation, ions are repelled from the regions

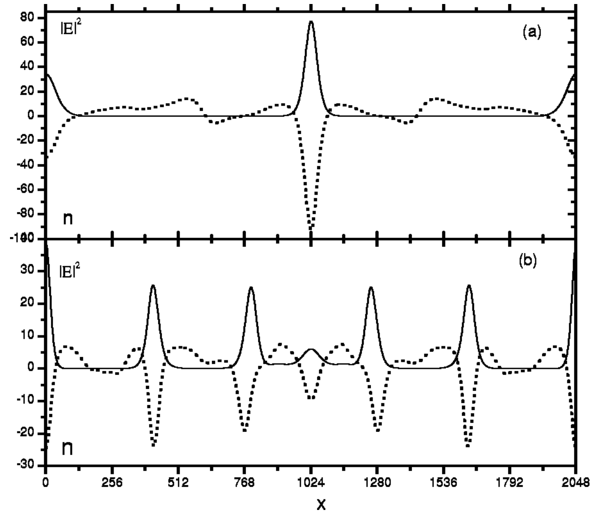

FIG. 1. $|E(x,t)|^2$ (solid line) and $n(x,t)$ (dotted line) vs x at $t=10$. The positive ion perturbation density n is distributed among the patterns formed. This shows pattern selection in the process of the pattern formation. (a) $k=0.23$, (b) $k=0.11$.

of the master mode and some harmonic patterns by the wave ponderomotive force and form spatial distributions with $n>0$ (dotted line of Fig. 1). These density humps, consisting of ions ponderomotively expelled by the master and harmonic wave fields, move stochastically on either side of the patterns. The waves cannot be trapped because of a corresponding decrease of the index of refraction. Second, for the case studied, no initial perturbation at $X/L=1/4$ and 3/4 appears. Finally, only one harmonic pattern is later formed at $X/L=1/2$. Thus, pattern selection leads to less than $N-1$ observed harmonic patterns. As shown in Fig. 2, actually only one master pattern and one harmonic pattern eventually move stochastically in the form of pattern trains. As a result, similar to the regime $1/2\leqslant k\leqslant k_c$, the system is in TC and SPC.

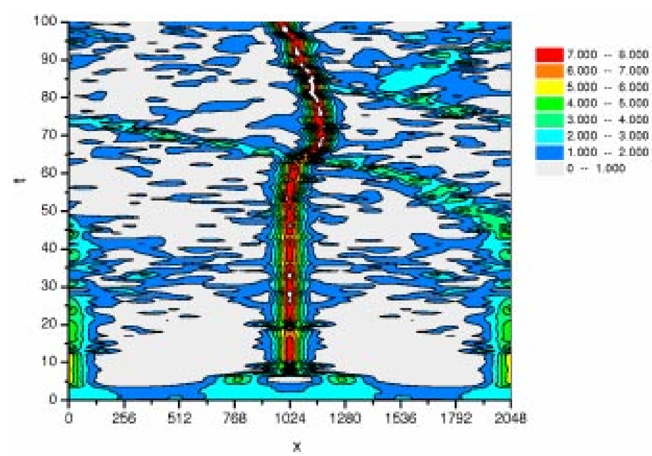

FIG. 2. (Color online) Contours of $|E(x,t)|=\text{const}$ for $k=0.23$. The solitary master pattern, initially peaked at $x=\pm mL(m=0,1,2,\ldots)$, is formed by the master mode. The pattern selection leads to only one harmonic pattern, initially peaked at $x=L/2$. The system is in the coexistent TC and SPC state.

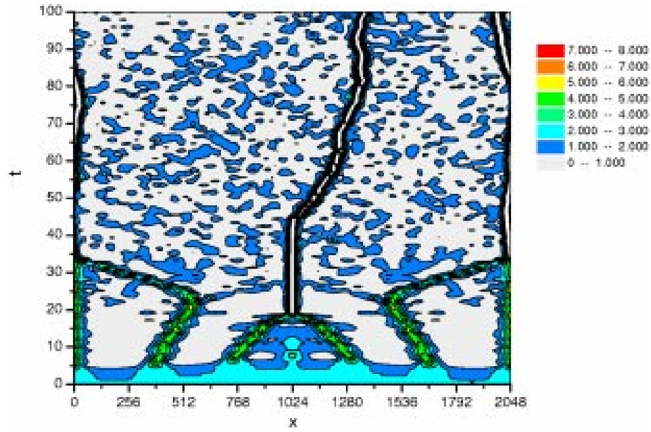

FIG. 3. (Color online) Contours of $|E(x,t)|=\text{const}$ for $k=0.11$. Five solitary patterns, initially peaked at $x\sim\pm(m+m'/5)L$, $m=0,1,2,\ldots,$ $m'=1$, 2, 3, 4 are formed. Two collisions, at $t\simeq 20$ and 33, respectively, occurred. The system is still in the coexistence of TC and STC.

As k reduces to 0.11, pattern selection leads to only five solitary patterns [Fig. 1(b)], formed initially from the master and harmonic modes (Fig. 3). At $t\simeq 20$, two solitary patterns, initially peaked at $x\simeq 768$ and 1280, respectively, collide and fuse into a new pattern with a strengthened amplitude and narrower width. At the same time, strong ion-acoustic wave emission $I(x,t)>|E(x,t)|^2$ is observed everywhere and the pattern structures are distorted. At $t\simeq 33$, the other patterns, initially peaked at 384 and 1664, collide with the master pattern and fuse into another new one. After two collisions, the four-harmonic patterns produce one new pattern that coexists with the distorted master pattern. However, as seen in Fig. 4, case (1), the spatial correlation function is still in SPC, i.e., $0<\Gamma(r)<1$. This implies that a few collisions do not seem to be enough to cause STC. The coherence in the system is partially retained, so that TC and SPC still coexist in the system.

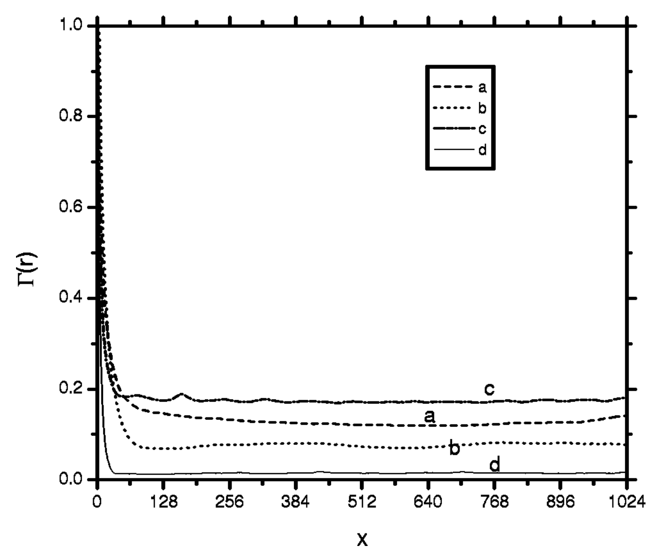

FIG. 4. The spatial correlation function $\Gamma(r)$ vs r: (1) for $k=0.11$, and (a) $t=0-100$, (b) $t>100$; (2) for $k=0.047$, and (c) $t=0-100$ (the system is in SPC), (d) $t>100$ (the system is in STC), with the correlation length $\xi\simeq 10\ll 2048$.

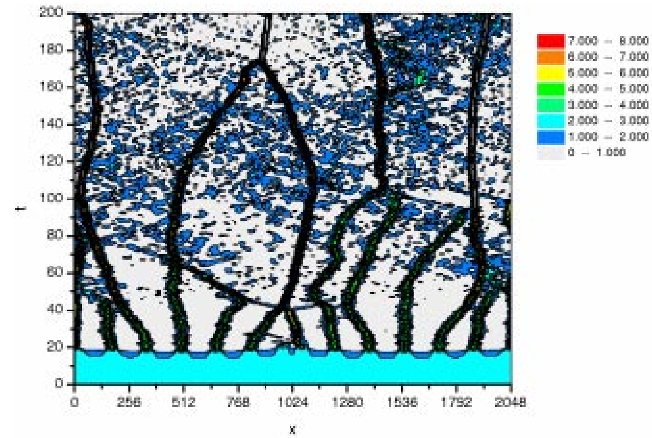

FIG. 5. (Color online) Contours of $|E(x,t)|=\text{const}$ for $k=0.047$. Twelve solitary patterns, initial peaks at $x\sim\pm(m+m'/12)L$, $m=0,1,2,\ldots,$ $m'=1,2,\ldots,11$ are formed. After some time, several incoherent patterns are fused from the original twelve solitary patterns by collisions. Ion-acoustic wave emission also occurs everywhere and the STC state emerges.

We now consider the case when many unstable modes are excited and saturated to form initially twelve solitary pattern trains by pattern selection (Fig. 5). The first collision occurs at $t\simeq 40$. The four patterns, initially at $x\simeq 170$, 340, 820, and 1100, respectively, are fused into two new patterns, accompanied by strong ion-acoustic wave emission. At $t\simeq 90-110$, collisions again take place among some of the pattern trains, and new incoherent patterns accompanied by strong ion-acoustic wave emission appear. At longer times, the new incoherent patterns collide with each other repeatedly. The patterns are also much distorted and the original twelve solitary pattern trains are finally fused into a few incoherent patterns. For $t>100$, the spatial correlation function $\Gamma(r)\to 0$ and the system tends to STC. The correlation length $\xi\simeq 10\ll L=2048$ is shown in Fig. 4, case (2). A certain amount of the system energy is carried by the incoherent patterns as well as many stable high-harmonic modes with shorter wavelengths $(k>1)$ excited through nonlinear interaction. The system energy is thus spatially redistributed in the process of pattern collision and fusion. Therefore, if initially there exist many unstable modulation lengths to form patterns, collision and fusion of many pattern trains can lead to the STC state. There should then exist a critical point k_{cs} where STC transition from TC can occur. In our model, k_{cs} is less than 0.11 but larger than 0.08.

In summary, solitary patterns may be formed by the harmonic modes excited by a spatially modulational master mode. If only the master pattern $(1/2<k<1)$ exists, the system may be in either the STC state or the TC and SPC state, depending on the modulation wave number [6]. However, as $k<1$, pattern selection restricts the number of the harmonic patterns formed to be less than that of the unstable harmonic modes. If a few harmonic patterns coexist with the master pattern, the system will still be in TC and SPC. However, if many harmonic patterns occur in the early phase, the system may experience the SPC state. With the lapse of time collision and fusion among patterns occur frequently and several new incoherent patterns are formed. The system enters the STC state. Energy is transferred to the stable high-harmonic modes with shorter wavelengths, and transport occurs in the system. In addition, as seen in Fig. 4, the more the patterns formed, the faster is the dynamic transition to STC. This implies that the excitation of a sufficient number of spatially harmonic modulation lengths is the main condition to cause STC.

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