Solutions of a (2+1)-dimensional dispersive long wave equation

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A special type of multisoliton solution with a particular dispersion relation is obtained for Wu-Zhang equation [which describes (2+1)-dimensional dispersive long waves] by the standard Weiss-Tabor-Carnvale Painlevé truncation expansion. Using a nonstandard truncation of a modified Conte's invariant Painlevé expansion, two different types of soliton solutions without any dispersive relation is found. Two types of periodic wave solutions expressed by Jacobi elliptic functions are found by the truncations of a special extended Painlevé expansion. The soliton solutions are special cases of the corresponding periodic solutions.

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I. INTRODUCTION

In Ref. [1], three sets of model equations are derived for modeling nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth. Omitting the higher order terms, one of these equations, the Wu-Zhang (WZ) equation, can be written as

$$
\begin{aligned}
&u_t+uu_x+vu_y+w_x=0,\\
&v_t+uv_x+vv_y+w_y=0,\\
&w_t+(uw)_x+(vw)_y+\frac{1}{3}(u_{xxx}+u_{xyy}+v_{xxy}+v_{yyy})=0,
\end{aligned}
\tag{1}
$$

where $w-1$ is the elevation of the water wave, u is the surface velocity of water along the x direction, and v is the surface velocity of water along the y direction. By scaling transformation and symmetry reduction, Eq. (1) can be reduced to the $(1+1)$-dimensional dispersive long wave equation

$$
\begin{aligned}
&v_t+vv_y+w_y=0,\\
&w_t+(wv)_y+\frac{1}{3}v_{yyy}=0.
\end{aligned}
\tag{2}
$$

A good understanding of all solutions of Eq. (1) is very helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of solutions of Eq. (1) is of fundamental interest in fluid dynamics. In this paper, we will find the soliton solutions for Eq. (1) directly by using the standard and nonstandard truncations of the Weiss-Tabor-Carnevale (WTC) approach and the modified Conte's invariant Painlevé expansion for the WZ equation.

It is well known that the Painlevé analysis developed by Weiss, Tabor, and Carnevale [2] not only is one of the most powerful methods to prove the integrability of a model, but also can be used to find some exact solutions. Conte [3], Pickering [4], and Lou [5] had generalized the WTC approach in some ways to find more exact and explicit solutions of nonlinear models.

Furthermore, the WTC method and the modification approaches may also be applied to nonintegrable systems. Though the nonlinear systems may have not the Painlevé property, some useful results such as exact solutions and Bäcklund transformations can still be produced from the truncated Painlevé test [6–8]. In this paper, basing on the truncated WTC Painlevé expansion and the Conte's modification, we study the exact solitary wave solutions of the system (1). The periodic solutions are also studied by the standard and nonstandard truncations of a special extended Painlevé expansion given in Ref. [5].

In Ref. [9], it was pointed out that for Eq. (2), there are some types of single soliton solutions without any dispersion relation. It is natural to ask whether the situation is remained for the WZ equation. The results of this paper show us a positive answer.

In Sec. II of this paper, after finishing a brief discussion on the non-Painlevé integrability of the WZ equation, we use the standard and nonstandard truncations of the WTC Painlevé expansion and a special extended Painlevé expansion to obtain soliton solutions with and without dispersion relations. Section III is devoted to find periodic solutions of Eq. (1) by using another special type of extended Painlevé expansion. The last section is a simple summary and discussions.

II. TRUNCATED PAINLEVÉ EXPANSIONS AND EXACT SOLITON SOLUTIONS

A. Non-Painlevé integrability of the WZ equation

Before to find some exact solutions of the model (1), we briefly discuss its non-Painlevé integrability by means of the standard WTC approach. Usually, when we say a model is integrable we should pointed out that the model is integrable under what special meaning(s). For instance, we say a model is Painlevé integrable if the model possesses the Painlevé property and a model is Lax or inverse scattering transformation (IST) integrable if the model has a Lax pair and then can be solved by the IST approach. An integrable model under some special meanings may not be integrable under other meanings. For instance, some Lax integrable models may not be Painlevé integrable [10]. On the other hand, though many scientists believe that the Painlevlé property is a sufficient condition on the integrability [11] and the Lax pairs can be found from the Painlevé analysis [12], the Lax pairs of various Painlevé integrable models have not yet been found [13]. In other words, whether possessing the Painlevé property is a sufficient condition for Lax and/or IST integrable is still unclear. In this subsection, we only discuss the non-Painlevé integrability of the model.

As usual, we take the following Laurent expansion of the function $u\equiv u(x,y,t)$, $v\equiv v(x,y,t)$, and $w\equiv w(x,y,t)$ about a singular manifold $\equiv \phi(x,y,t)$:

$$u=\sum_{j=0}^{\infty} u_j\phi^{j+\alpha}, \qquad v=\sum_{j=0}^{\infty} v_j\phi^{j+\beta}, \qquad w=\sum_{j=0}^{\infty} w_j\phi^{j+\gamma}. \tag{3}$$

Substituting the leading terms $(j=0)$ of Eq. (3) into Eq. (1), by means of the standard leading order analysis, we can obtain

$$\alpha=-1, \qquad \beta=-1, \qquad \gamma=-2 \tag{4}$$

and

$$u_0=\frac{2}{3}\sqrt{3}a\phi_x, \quad v_0=\frac{2}{3}\sqrt{3}a\phi_y,$$

$$w_0=-\frac{2}{3}(\phi_y^2+\phi_x^2), \;\; a^2=1. \tag{5}$$

Now substituting the full expansion (3) into Eq. (1) yields the recursion relation

$$A\begin{pmatrix} u_j \\ v_j \\ w_j \end{pmatrix}=\begin{pmatrix} F_{1,j-1} \\ F_{2,j-1} \\ F_{3,j-1} \end{pmatrix}, \tag{6}$$

where the coefficient matrix reads

$$A\equiv\begin{pmatrix} \frac{2a}{\sqrt{3}}[\phi_y^2(j-1)+\phi_x^2(j-2)] & -\frac{2a}{\sqrt{3}}\phi_y\phi_x & \phi_x(j-2) \\ -\frac{2a}{\sqrt{3}}\phi_x\phi_y & \frac{2a}{\sqrt{3}}[\phi_x^2(j-1)+\phi_y^2(j-2)] & \phi_y(j-2) \\ \frac{j}{3}(j-3)^2\phi_x(\phi_x^2+\phi_y^2) & \frac{j}{3}(j-3)^2\phi_y(\phi_x^2+\phi_y^2) & \frac{2a}{\sqrt{3}}(j-3)(\phi_x^2+\phi_y^2) \end{pmatrix}, \tag{7}$$

and the functions $F_{i,j-1}$, $i=1, 2, 3$ are complicated functions and only dependent on $u_k, v_k, w_k, k=0,1,\ldots, j-1$ and the derivatives of ϕ. All the functions $\{u_j, v_j, w_j\}$ can be determined by the recursion relation (6) except for those special resonance j which cause the determinant of the coefficient matrix A to vanish,

$$\Delta\equiv\det A$$
$$=-\frac{2}{9}a\sqrt{3}(j+1)(j-1)(j-2)(j-3)(j-4)(\phi_x^2+\phi_y^2)^3. \tag{8}$$

If the model possesses the Painlevé property, four resonance conditions located at $j=1$, 2, 3, and 4 should be satisfied identically. That means five arbitrary functions (ϕ and one of u_j, v_j, and w_j for every $j=1$, 2, 3, and 4) should be entered into the general expansion (3). However, the detailed calculations show us that the resonance conditions at $j=3,4$ are not satisfied identically. So the Eq. (1) does not pass the WTC Painlevé test. That means WZ equation has no Painlevé property and then it is not Painlevé integrable.

B. Standard truncation of WTC and multiple soliton solutions with a special dispersion relation

Though the model is non-Painlevé integrable, we can still construct some exact solutions by means of some suitable Painlevé truncated expansions. According to the WTC method [2], we consider the standard truncation of Eq. (3) at first,

$$u=\frac{u_0}{\phi}+u_1, \qquad v=\frac{v_0}{\phi}+v_1, \qquad w=\frac{w_0}{\phi^2}+\frac{w_1}{\phi}+w_2, \tag{9}$$

where $\{u_1, v_1, w_2\}$ is a particular seed solution of the WZ equation. Simple inspection of this system shows that if u_1, v_1, and w_2 are purely constants, the WZ system (1) is trivially satisfied. Substituting Eq. (9) with the trivial constant seed solution into WZ equation (1) and vanishing all the coefficients with different powers of ϕ, one can reobtain Eq. (5) and

$$w_1=\frac{2}{9}\frac{(\phi_y^2+3\phi_x^2)}{\phi_x^2+\phi_y^2}\phi_{xx}+\frac{2}{9}\frac{\phi_x^2+3\phi_y^2}{\phi_x^2+\phi_y^2}\phi_{yy}+\frac{8}{9}\frac{\phi_x\phi_y\phi_{xy}}{\phi_x^2+\phi_y^2}, \tag{10}$$

while ϕ is an arbitrary solution of the equation system

$$\phi_t=-\frac{\sqrt{3}}{2}w_1-(u_1\phi_x+v_1\phi_y), \tag{11}$$

$$2\sqrt{3}[3w_2(\partial_x^2+\partial_y^2)+(\partial_x^2+\partial_y^2)^2]\phi+9a(\partial_t+u_1\partial_x+v_1\partial_y)w_1=0, \tag{12}$$

and

$$\begin{aligned}&12w_2(\phi_x^2+\phi_y^2)-12w_1(\phi_{xx}+\phi_{yy})-9w_1^2\\&+8\phi_x(2\phi_{xxx}+2\phi_{xyy}-3w_{1x})\\&+8\phi_y(2\phi_{xxy}+2\phi_{yyy}-3w_{1y})\\&+4(3\phi_{xx}^2+3\phi_{yy}^2+4\phi_{xy}^2+2\phi_{xx}\phi_{yy})=0.\end{aligned} \tag{13}$$

In order to give out one special type of explicit solution, we choose the ansatz

$$\phi_y=c\phi_x, \tag{14}$$

with c being an arbitrary constant. Obviously, under the ansatz (14), we have

$$u_0=\frac{2}{3}\sqrt{3}a\phi_x, \quad v_0=\frac{2}{3}\sqrt{3}ac\phi_x,$$

$$w_0=-\frac{2}{3}(1+c^2)\phi_x^2, \quad w_1=\frac{2}{3}(1+c^2)\phi_{xx}, \tag{15}$$

while Eqs. (11)–(13) are simplified to

$$\phi_t=-\frac{\sqrt{3}}{3}a(c^2+1)\phi_{xx}-(u_1+cv_1)\phi_x \tag{16}$$

and

$$w_2\phi_{xx}=0, \quad w_2\phi_x^2=0. \tag{17}$$

Substituting solutions of Eqs. (16), (17) with Eq. (14) into (9), we obtain the first type of solution for the WZ equation (17). Combining the exponential solutions of the equation system of Eqs. (16) and (14),

$$\phi=1+\sum_i^N \exp(k_i(x+cy)+\omega_i t+\xi_{i0}), \tag{18}$$

with a particular type of dispersion relation between constants k_i and ω_i

$$\omega_i=-\frac{\sqrt{3}}{3}a(c^2+1)k_i^2-d_1k_i \tag{19}$$

yields a special type of multiple soliton solutions of Eq. (1), where $d_1=(u_1+cv_1)$ and ξ_{i0} are arbitrary constants.

From above, we know that Eq. (1) possesses two branches in the standard Painlevé analysis ($a=\pm 1$). So from the Conte's expansion one can obtain some new exact solutions by nonstandard truncation [4]. However, if we use directly the nonstandard truncated expansion basing on the Conte's Painlevé expansion, we cannot find the nonsingular ring (or bell) type soliton solutions for all the fields u, v, and w. To overcome the singularity problem to obtain the ring (or bell) type soliton solutions, one may use the two-singular manifold approach [14] or the Pickering's modification [15] or the extended Painlevé analysis approach.

C. Extended Conte's truncation and the soliton solution without dispersion relation

In Ref. [5], it is pointed out that the Painlevé expansion form (3) can be modified as

$$u=\sum_{j=0}^{\infty} U_j\xi^{j-1}, \quad v=\sum_{j=0}^{\infty} V_j\xi^{j-1}, \quad w=\sum_{j=0}^{\infty} W_j\xi^{j-2}, \tag{20}$$

with ξ being determined by

$$\xi_x=\sum_{j=0}^{N} S_j\xi^j, \quad \xi_y=\sum_{j=0}^{N} C_j\xi^j, \quad \xi_t=\sum_{j=0}^{N} K_j\xi^j. \tag{21}$$

When we take $N=2$, the general expansion (20) with Eq. (21) is just the Pickering's modification [15]. Now using the nonstandard truncation of Eq. (20), one can find some types of new exact solutions. More specifically, we fix the expansion function as

$$\xi_{N=2}\equiv g\equiv\lambda-\chi, \quad \chi\equiv\left(\frac{\phi_x}{\phi}-\frac{\phi_{xx}}{2\phi_x}\right)^{-1}, \tag{22}$$

with λ being an arbitrary constant and ϕ being an arbitrary function of space-time variables. When we take $\lambda=0$, the modified expansion (20) with Eq. (22) will be reduced back to the usual Conte's expansion. As in the usual Conte's expansion, the coefficients U_j, V_j, and W_j are all invariant under the Möbius transformation. From the special selection (22), Eq. (21) becomes

$$g_x=-1+\frac{S}{2}(\lambda-g)^2,$$

$$g_y=C-C_x(\lambda-g)+\frac{1}{2}(C_{xx}-CS)(\lambda-g)^2, \tag{23}$$

$$g_t=K-K_x(\lambda-g)+\frac{1}{2}(K_{xx}-KS)(\lambda-g)^2,$$

where

$$S\equiv\frac{3}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2-\frac{\phi_{xxx}}{\phi_x}, \quad C\equiv-\frac{\phi_y}{\phi_x}, \quad K\equiv-\frac{\phi_t}{\phi_x}, \tag{24}$$

which are the Möbius transformation invariants. It is straightforward to prove that all the compatibility conditions $g_{xy}=g_{yx}$, $g_{xt}=g_{tx}$, $g_{ty}=g_{yt}$ are satisfied automatically because of Eq. (24).

When the expansion function of Eq. (20) is selected as g shown by Eq. (22), the corresponding nonstandard truncation form reads

$$u=\frac{U_0}{g}+U_1+U_2g,\quad v=\frac{V_0}{g}+V_1+V_2g,$$

$$w=\frac{W_0}{g^2}+\frac{W_1}{g}+W_2+W_3g+W_4g^2. \tag{25}$$

Substituting the nonstandard truncated expansion (25) with Eq. (23) into Eq. (1) and vanishing all the coefficients of different powers of g we can find a set of complicated overdetermined equations to fix the functions U_j, V_j, $(j=0,1,2)$, W_k, $(k=0,\ldots,4)$, S, C, and K. However, if we only want to find the single soliton solution, we can take them as constants simply. Omitting the detailed calculations to solve these overdetermined equations, we only list the final result

$$U_0=\frac{\sqrt{3}}{3}a(S\lambda^2-2),$$

$$U_1=\frac{\sqrt{3}}{3}(-SaC^2\lambda-Sa\lambda+\sqrt{3}V_1C+\sqrt{3}K),$$

$$U_2=\frac{\sqrt{3}}{3}aS,\;\; V_0=-\frac{\sqrt{3}}{3}aC(S\lambda^2-2),\;\; V_2=-\frac{\sqrt{3}}{3}aSC, \tag{26}$$

$$W_0=-\frac{1}{6}(C^2+1)(S\lambda^2-2)^2,\;\; W_1=\frac{\lambda}{3}S(C^2+1)(S\lambda^2-2),$$

$$W_2=0,\quad W_3=\frac{\lambda}{3}S^2(C^2+1),\;\; W_4=-\frac{S^2}{6}(C^2+1),\;\; a^2=1$$

and V_1 is an arbitrary constant. When S, C, and K are taken as constants, the general solution of Eq. (23) reads

$$g=\frac{\sqrt{S}}{S}\left\{\sqrt{S}\lambda-\sqrt{2}\tanh\left[\frac{\sqrt{2}}{2}\sqrt{S}(x-Cy-Kt+d_1)\right]\right\}, \tag{27}$$

where d_1 is an arbitrary constant. The fields u, v, and w in Eq. (25) with Eqs. (27) and (26) are bell- or ring-type soliton solutions. Note that six arbitrary constants $(\{S,K,C,\lambda,d_1,V_1\})$ are included. One of the most interesting facts may be that there is no dispersion relation among the constants S, C, and K. However, if one puts some types of special boundary conditions to the solitary wave solutions, then some types of dispersion relations have to be introduced. For instance, if we take the following boundary conditions $[\xi\equiv\sqrt{S}(x-Cy-Kt+d_1)]$:

$$u|_{\xi\to\infty}\to 0,\;\; v|_{\xi\to\infty}\to 0,\;\; w|_{\xi\to\infty}\to 0,$$

then the corresponding dispersion relation reads

$$3K^2-2S-4SC^2-2SC^4=0.$$

By the way, using the standard truncated form of the extended Painlevé expansion Eq. (20) with Eqs. (22), i.e., Eq. (25) with $U_2=V_2=W_3=W_4=0$, one can obtain another type of soliton solution that has the form

$$u=\frac{\sqrt{3}}{3}\frac{a}{g}(S\lambda^2-2)-\frac{\sqrt{3}}{3}(aC^2S\lambda-\sqrt{3}V_1C+aS\lambda-\sqrt{3}K),$$

$$v=-\frac{\sqrt{3}}{3}\frac{aC}{g}(S\lambda^2-2)+V_1, \tag{28}$$

$$w=-\frac{1+C^2}{3}\left[\frac{(S\lambda^2-2)^2}{2g^2}-\frac{S\lambda}{g}(S\lambda^2-2)+\sqrt{3}aV_{1y}-S+\frac{S^2}{2}\lambda^2\right],$$

where g is given by Eq. (27) and V_1 is an arbitrary solution of the Burgers equation

$$V_{1t}=\frac{\sqrt{3}}{3}a(1+C^2)V_{1yy}+\frac{\sqrt{3}}{3}[(a\lambda SC-\sqrt{3}V_1)(1+C^2)-\sqrt{3}CK)]V_{1y} \tag{29}$$

with

$$V_{1x}=CV_{1y}. \tag{30}$$

This type of solution is the generalization of that obtained by the standard truncation solution of the WTC expansion. If we take $S\lambda^2=2$, the solution (28) is equivalent to the first type of solution obtained by the standard truncation solution of the WTC expansion. When we take $V_1=0$, Eq. (28) with Eq. (27) is a new type of soliton solution without dispersion relation. The fields v and u in Eq. (28) are kink or antikink solitons while w is a bell type soliton for $V_1=0$.

It is also worthy to mention that the usual tanh expansion method can also be considered as the general expansion of Eq. (25) with

$$U_0=V_0=W_0=W_1=0. \tag{31}$$

Detailed calculations show us that the soliton solution obtained by the truncated expansion of Eq. (25) with Eq. (31) is equivalent to Eq. (28).

III. PERIODIC SOLUTIONS OF EQ. (1)

Usually, the single soliton solutions of integrable models are special limited cases of the elliptic function solutions. In our knowledge, the periodic solutions expressed by Jacobi or Weierstrass elliptic functions cannot be obtained by the WTC standard truncated expansion, Conte's standard and nonstandard truncated expansions, and the Pickering's modification and the two-singular manifold approach.

If we take $N\to\infty$ in Eq. (21) and select the functions S_j, C_j, and K_j such that the summations become some closed forms, then it is also possible to obtain some types of new exact solutions of the DLWE by using the standard and nonstandard truncations of the extended Painlevé expansion (20). For instance, a special type of summation form of Eq. (21) may have the forms

$$\xi_x=\sqrt{\sum_{j=0}^{M}\frac{2js_j}{\xi^j}},\quad \xi_y=\sqrt{\sum_{j=0}^{M}\frac{2jc_j}{\xi^j}},\quad \xi_t=\sqrt{\sum_{j=0}^{M}\frac{2jk_j}{\xi^j}}. \tag{32}$$

Using the expansion functions expressed by Eq. (32) in Eq. (20), we may use both the standard and nonstandard truncated expansions to find new exact solutions. To find the solutions with elliptic functions, we take s_j, c_j, and k_j as constants and $M=4$. The compatibility conditions of Eq. (32) for s_j, c_j, and k_j being constants reads

$$c_j=c^2 s_j,\qquad k_j=k^2 s_j. \tag{33}$$

When ξ is determined by Eq. (32) with $M=4$ and Eq. (33), the nonstandard truncation form can be taken as

$$u=\frac{U_0}{\xi}+U_1+U_2\xi,\quad v=\frac{V_0}{\xi}+V_1+V_2\xi,$$

$$w=\frac{W_0}{\xi^2}+\frac{W_1}{\xi}+W_2+W_3\xi+W_4\xi^2, \tag{34}$$

while the standard truncation has the same form as Eq. (34) but with $U_2=V_2=W_3=W_4=0$.

Substituting Eqs. (34),(32),(33) and $M=4$ into Eq. (1) and vanishing all the coefficients of different powers of ξ, we have

$$cV_2U_2+U_2^2+2W_4=0,\quad kU_2+W_3+cV_1U_2+U_1U_2=0,\quad cV_0U_2-cV_2U_0=0,$$

$$(U_1+k+cV_1)U_0+W_1=0,\qquad U_0^2+cV_0U_0+2W_0=0,\qquad U_2V_2+cV_2^2+2cW_4=0,$$

$$kV_2+cV_1V_2+U_1V_2+cW_3=0,\qquad kV_0+cW_1+U_1V_0+cV_1V_0=0,$$

$$3U_2W_4+U_2s_4+3cV_2W_4+V_2s_4c+c^2U_2s_4+c^3V_2s_4=0,\qquad cV_0^2+2cW_0+U_0V_0=0,$$

$$cV_2W_3+kW_4+U_1W_4+U_2W_3+cV_1W_4+\frac{s_3}{3}(c^2U_2+U_2+c^3V_2+V_2c)=0, \tag{35}$$

$$\frac{s_2}{3}(1+c^2)(U_2+V_2c)+W_2(U_2+cV_2)+W_4(U_0+cV_0)+W_3(U_1+cV_1+k)=0,$$

$$\frac{s_2}{3}(1+c^2)(U_0+cV_0)+W_0(U_2+cV_2)+W_2(cV_0+U_0)+W_1(cV_1+U_1+k)=0,$$

$$c^3V_0s_1+U_1W_0+kW_0+cV_0W_1+cV_1W_0+U_0W_1+cV_0s_1+U_0s_1+c^2U_0s_1=0,$$

$$2cV_0s_0+2c^3V_0s_0+3U_0W_0+3cV_0W_0+2c^2U_0s_0+2U_0s_0=0.$$

Solving out the equation system (35) yields three nontrivial cases. The first case

$$U_1=\frac{2s_1(1+c^2)}{3U_0}-cV_1-k,\quad U_2=\frac{U_0s_3}{3s_1},\quad V_0=cU_0,$$

$$V_2=\frac{cU_0s_3}{3s_1},\quad W_0=-\frac{U_0^2}{2}(1+c^2),\quad W_1=-\frac{2}{3}(1+c^2)s_1, \tag{36}$$

$$W_2=\frac{8c^2s_1^3+3U_0^4s_3c^2-6c^2s_2U_0^2s_1+3U_0^4s_3+8s_1^3-6s_2U_0^2s_1}{18U_0^2s_1},$$

$$W_4=-\frac{U_0^2s_3^2(1+c^2)}{18s_1^2},\quad W_3=-\frac{2}{9}s_3(1+c^2),$$

$$s_0=\frac{3}{4}U_0^2,\qquad s_4=\frac{U_0^2s_3^2}{6s_1^2}$$

corresponds to the nonstandard truncation. The second case

$$U_1=\frac{2s_1}{3U_0}(1+c^2)-cV_1-k,\quad V_0=cU_0, \tag{37}$$

$$W_0=-\frac{U_0^2}{2}(1+c^2),\quad W_1=-\frac{2}{3}s_1(c^2+1),\quad s_0=\frac{3}{4}U_0^2,$$

$$W_2=\frac{1+c^2}{9U_0^2}(4s_1^2-3s_2U_0^2),\quad U_2=V_2=W_3=W_4=0$$

is related to the standard truncation while the last case is equivalent to the second case. Now the remaining problem is to solve the equation system (32) and give out the explicit expressions of the expansion functions. In these cases, the expansion functions can be expressed by the usual Jacobi elliptic functions with help of the mapping deformation approach proposed in Refs. [16–18].

Similar to the method used in Ref. [17] or more concretely in Ref. [18], to solve the system (32), one may simplify the equations by means of the symmetry property of the expression. It is easy to check that Eq. (32) is form invariant under the Möbious transformation. In other words, if we make the transformation

$$\xi\rightarrow\frac{a+bg}{c_1+dg}\quad(bc_1-ad\neq 0),\tag{38}$$

then the function g satisfies the same equation as Eq. (32)

$$g_x=\sqrt{\sum_{j=0}^{4}\overline{s_j}g^j}=\frac{g_y}{c}=\frac{g_t}{k},\tag{39}$$

with

$$\overline{s_0}=\frac{s_4a^4+s_3a^3c_1+s_2a^2c_1^2+s_1ac_1^3+s_0c_1^4}{(ad-bc_1)^2},\tag{40}$$

$$\overline{s_1}=\frac{c_1^2(3ad+c_1b)s_1}{(ad-bc_1)^2}+\frac{2ac_1(ad+c_1b)s_2}{(ad-bc_1)^2}+\frac{a^2(ad+3c_1b)s_3}{(ad-bc_1)^2}+\frac{4a^3bs_4}{(ad-bc_1)^2}+\frac{4c_1^3ds_0}{(ad-bc_1)^2},$$

$$\overline{s_2}=\frac{c_1d(ad+c_1b)s_1}{(ad-bc_1)^2}+\frac{(a^2d^2+b^2c_1^2+4abc_1d)s_2}{(ad-bc_1)^2}+\frac{3ab(ad+c_1b)s_3}{(ad-bc_1)^2}+\frac{6a^2b^2s_4}{(ad-bc_1)^2}+\frac{6c_1^2d^2s_0}{(ad-bc_1)^2},$$

$$\overline{s_3}=\frac{d^2(ad+3c_1b)s_1}{(ad-bc_1)^2}+\frac{2bd(ad+c_1b)s_2}{(ad-bc_1)^2}+\frac{b^2(3ad+c_1b)s_3}{(ad-bc_1)^2}+\frac{4ab^3s_4}{(ad-bc_1)^2}+\frac{4c_1d^3s_0}{(ad-bc_1)^2},$$

$$\overline{s_4}=\frac{s_4b^4+s_3b^3d+s_2b^2d^2+s_1bd^3+s_0d^4}{(ad-bc_1)^2}.$$

From Eq. (40), we know that the expansion function ξ can be expressed explicitly with help of the standard Jacobi elliptic functions by the appropriate selections of the constants a, b, c_1, and d such that

$$\overline{s_1}=\overline{s_3}=0.\tag{41}$$

In Ref. [16], various solutions of Eqs. (39) with (41) were listed in a table. Then using the results of Ref. [16], say,

$$g_x=\sqrt{\frac{m^2}{p^2}g^4-(m^2+1)g^2+p^2},\quad g=p\,\mathrm{sn}\,x,\tag{42}$$

we may obtain many kinds of periodic solutions of the DLWE (1). In Eq. (42), sn$x$ is the usual Jacobi elliptic sine function and m is the modular of the function sn$x$. In this section, we only write down two special cases, which are the generalizations of the soliton solutions listed in the last section.

If we rewrite the arbitrary constants in the first case (36) as

$$U_0^2=\frac{n^2a_1^2}{3p^2}(m^2+p^4-p^2m^2-p^2),\quad s_1=\frac{s_3}{3}a_1^2,$$

$$s_2=\frac{n^2}{2p^2}(3m^2+p^2m^2+p^2+3p^4),\quad s_3=\frac{3n^2}{2a_1p^2}(p^4-m^2),\tag{43}$$

where m, n, a_1, and p are new arbitrary constants. Then the general solution of Eq. (32) with Eqs. (36) and (43) has the form

$$\xi=a_1\frac{1-p\ \mathrm{sn}\,z}{1+p\ \mathrm{sn}\,z},\quad z\equiv n(x+cy+kt).\tag{44}$$

The corresponding periodic solution of Eq. (1) reads

$$u=\frac{A_1+p^2A_2\,\mathrm{sn}^2z}{A_0(p^2\ \mathrm{sn}^2\ z-1)},$$

$$v=-\frac{B_1+p^2B_2\,\mathrm{sn}^2z}{a_1(p^2\,\mathrm{sn}^2z-1)},\tag{45}$$

$$w=\frac{C_0(C_1-2p^2C_2\,\mathrm{sn}^2z+p^4C_3\,\mathrm{sn}^4z)}{(p^2\,\mathrm{sn}^2z-1)^2},$$

where

$$A_0=-3U_0p^2,\quad B_1=2cU_0+V_1a_1,\quad B_2=2cU_0-V_1a_1,\tag{46}$$

$$A_1=a_1n^2[p^4(c^2+3)-m^2(c^2-1)-2p^2(m^2+1)]-3U_0p^2(cV_1+k),$$

$$A_2=n^2a_1[m^2(c^2+3)-p^4(c^2-1)-2p^2(m^2+1)]+3U_0p^2(cV_1+k),$$

$$C_0=\frac{n^4a_1^2(1+c^2)}{9p^4U_0^2},$$

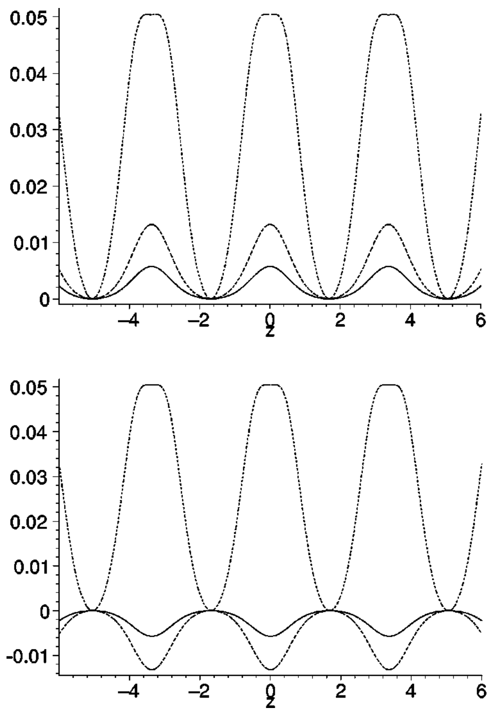

FIG. 1. Plot of the periodic solution (45) with Eq. (47) for the WZ equation for small m $(m=0.5)$. (a) The periodic solution related to the upper sign of Eq. (47). (b) The periodic solution related to the lower sign of Eq. (47). The solid lines, dotted lines, and dash-dotted lines denote the values of the fields v, u, and w, respectively.

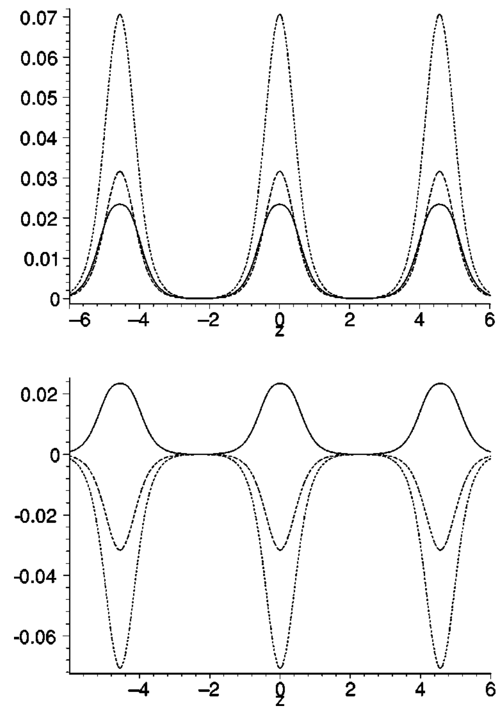

FIG. 2. Structures of the soliton lattice solution (45) with Eq. (50) and $m=0.9$ for the WZ equation. (a) The bell-bell soliton lattice expressed by Eqs. (45) and (48) with the upper sign. (b) The bell-ring soliton lattice related to Eqs. (45) and (48) with the lower sign. The solid lines, dotted lines, and dash-dotted lines denote the values of the fields w, v, and u, respectively.

$$C_1=m^4+2p^4m^2(2p^2-3)+p^6(4-3p^2),$$

$$C_2=m^4(4p^4-6p^2+3)-2p^2m^2(3p^4-5p^2+3)+p^4(3p^4-6p^2+4),$$

$$C_3=4m^4(p^2-3)+4m^2p^2(2-3p^2)+p^8$$

with V_1 being an arbitrary constant.

In Fig. 1, the periodic waves are plotted for small modular $(m=1/2)$ for the fields u, v, and w which are expressed by Eq. (45) with

$$U_0=c=1,\quad n=0.08,\quad A_0=-0.6996, \tag{47}$$

$$A_2=\mp 0.03961,\quad A_1=\pm 0.009238,$$

$$B_1=-0.528,\quad C_3=-0.3855,\quad p=-0.4829,$$

$$C_1=0.03546,\quad k=\pm 0.2919,$$

$$C_0=1.420,\quad C_2=0.03109,\quad V_1=\pm 0.002865,$$

$$a_1=\mp 92.13,\ \ B_2=2.264.$$

Figures 1(a) and 1(b). are related to the upper and lower signs of Eq. (47), respectively. The parameters of Eq. (47) are obtained by Eqs. (43) and (46) after fixing four arbitrary constants of them, say, m, n, c, and U_0.

As the modular increases, the periodic solution (45) becomes a soliton ''lattice'' solution. Figure 2 shows the soliton lattice structure of Eq. (45) with $m=0.9$ and

$$U_0=c=1,\quad n=0.08,\quad A_0=-0.5736,$$

$$C_2=0.7527,\quad C_0=0.04459,$$

$$C_3=-6.486,\quad B_2=2.211,\quad B_1=-0.4228,$$

$$k=\mp 0.00495,\quad p=-0.4373, \tag{48}$$

$$A_2=\mp 0.212,\quad V_1=0.0158,\quad A_1=\pm 0.04054,$$

$$a_1=\mp 13.38,\quad C_1=0.525.$$

As the modular increases further to 1, the periodic of the lattice becomes larger and larger. Finally, when $m=1$ the period becomes infinity and the function $\operatorname{sn} z$ becomes $\tanh z$, i.e.,

$$\operatorname{sn} z|_{m\to 1}\to \tanh z. \tag{49}$$

So when $m\to 1$, the periodic solution (45) will be reduced to the ring- or bell-type soliton solution.

In Fig. 3, the special type of bell and ring shape soliton solution [Eq. (45) with Eq. (49)] is plotted. The corresponding parameters related to Fig. 3 are

$$U_0=1,\quad c=1,\quad n=0.08,\quad k=\mp 0.01786,$$

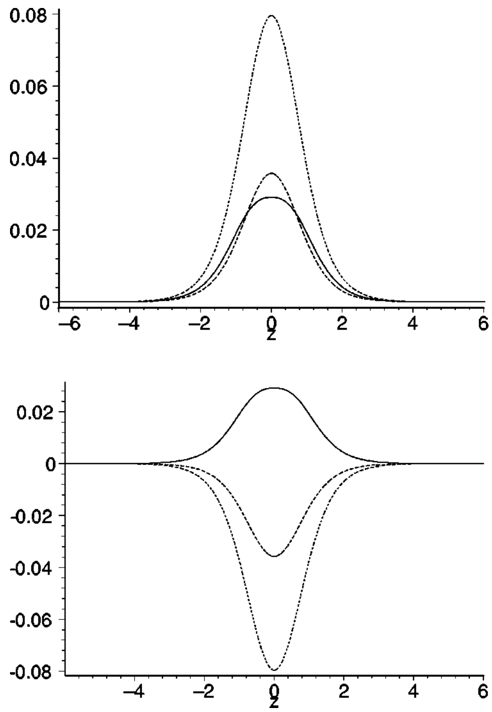

FIG. 3. Structures of a single soliton solution (45) with Eq. (50) for the WZ equation. (a) The bell-bell soliton solution described by Eqs. (45), (49), and (50) with the upper sign. (b) The bell-ring soliton solution (45), (49), and (50) with the lower sign. The solid lines, dotted lines, and dash-dotted lines denote the values of the fields w, v, and u, respectively.

$$B_2=2.205,\quad A_0=-0.5569,$$

$$a_1=-11.45,\quad A_2=\mp 0.2388,\quad C_1=0.8409,$$

$$C_0=0.03466,\quad p=-0.4309, \tag{50}$$

$$C_2=1.319,\quad B_1=-0.4093,\quad v_1=\pm 0.01786,$$

$$A_1=\pm 0.04433,\quad C_3=-10.18.$$

Figure 3(a) is related to the upper sign of Eq. (50) and the lower sign corresponds to Fig. 3(b). From Fig. 3, one can see that the soliton solution for the field w possesses always the bell shape, while the fields u and v may possess both the bell shape or the ring shape.

Similarly, for the second case (37), if we rewrite the parameters as

$$s_3=-\frac{3n^2}{p^6 2(a_2-a_1)^2}[p^2(a_1+a_2)(1+m^2) -2(a_2p^4+a_1m^2)],$$

$$s_4=\frac{2n^2}{p^2(a_1-a_2)^2}(m^2-p^2)(1-p^2),$$

$$s_1=\frac{n^2}{p^2(a_2-a_1)^2}[p^2a_2a_1(a_2+a_1)(1+m^2) -2p^4a_2^3-2m^2a_1^3], \tag{51}$$

$$s_2=\frac{n^2}{p^2(a_2-a_1)^2}[p^2(a_2^2-a_1^2)(m^2-1) +2p^2a_2(3p^2a_2-2a_1)+2m^2a_1(3a_1-2a_2p^2)],$$

$$U_0^2=\frac{4n^2(pa_2-a_1)}{3p^2(a_2-a_1)^2}(pa_2+a_1)(pa_2-a_1m)(pa_2+a_1m),$$

where m, n, a_1, a_2, and p are all arbitrary constants. Then the general solution of Eq. (32) with Eqs. (37) and (51) becomes

$$\xi=\frac{a_1+a_2p\ \mathrm{sn}\,z}{1+p\ \mathrm{sn}\,z},\quad z\equiv n(x+cy+kt). \tag{52}$$

In this case, the final periodic solution of Eq. (1) has the simple form

$$u=\frac{A_6-p\ \ A_7\,\mathrm{sn}\,z}{A_5(a_1+a_2p\ \ \mathrm{sn}\,z)},$$

$$v=\frac{(cU_0+V_1a_2)p\ \ \mathrm{sn}\,z+cU_0+V_1a_1}{a_1+a_2p\ \ \mathrm{sn}\,z}, \tag{53}$$

$$w=\frac{C_5(C_6+C_7\,\mathrm{sn}\,z+C_8\,\mathrm{sn}^2 z)}{(a_1+a_2p\ \ \mathrm{sn}\,z)^2},$$

with

$$A_5=3p^2U_0(a_2-a_1)^2,$$

$$A_6=p^2a_1^2a_2(1+m^2)(a_1+c^2a_1+c^2a_2-a_2) -2p^4a_2^3(a_1+c^2a_1+a_2)-2c^2a_1^4m^2 +3U_0p^2a_1(k+cV_1)(a_2-a_1)^2, \tag{54}$$

$$A_7=2p^2n^2a_1a_2^2(1+m^2)(a_2+c^2a_2-a_1+c^2a_1) -4n^2a_1^3m^2(a_2+c^2a_2-a_1)-4a_2^4c^2n^2p^4 -3U_0a^2p^2(k+cV_1)(a_2-a_1)^2,$$

$$C_5=\frac{4n^4(c^2+1)}{9p^2U_0^2(a_2-a_1)^2},\quad C_7=2a_2^3p^3a_1^3(m^2-1)^2,$$

$$C_6=a_1^2(a_1^4m^4+3a_2^4p^4)(1+m^2)-2p^2a_2^2(a_2^4p^4+3a_1^4m^2),$$

$$C_8=a_2^2p^2(a_2^4p^4+3a_1^4m^2)(1+m^2) -2a_1^2m^2(a_1^4m^2+3a_2^2p^4).$$

Similar to the first case, when the modular m tends to 1, the periodic solution (53) tend to the kink solution for the fields u and v and the bell soliton solution for the field w because of Eq. (49).

IV. SUMMARY AND DISCUSSIONS

In summary, for the WZ equation (1), there are three types of soliton solutions though it has no Painlevé property. The first type of multisoliton solutions [Eq. (9) with Eqs. (15) and (18)] with a special dispersion relation (19) can be obtained from the standard truncation of the WTC approach.

Usually, the solitons and the solitary wave solutions satisfy some dispersion relations. However, the results of this paper show us that for some types of nonlinear models there may be some types of solitons or solitary wave solutions without any dispersion relation. The second and third types of soliton solutions of the WZ equation (1) are not necessary to satisfy some special types of dispersion relations. However, if we put some special boundary conditions to the solitary wave solutions, then we have to put some special dispersive relations to the solitary wave solutions. These two types of solitary wave solutions are obtained from the nonstandard and standard truncations of an extended Conte's Painlevé expansion, respectively. For the second type of soliton solution [Eq. (25) with Eqs. (26) and (27)], all the fields u, v, and w are bell or ring type soliton solutions. The third type of soliton solution [Eq. (28) with Eqs. (27) and (29)] is a generalization of the first type of soliton solutions. In the third case, the fields u and v are kink or antikink soliton solutions while the field w is a bell-type soliton solution.

Though the general traveling wave solutions (and then the periodic solutions expressed by elliptic function solutions) of the WZ system (1) can be obtained by direct integration. However, any one of the standard and nonstandard truncations of the WTC and Conte expansions cannot be used to find the exact elliptic function solutions. In this paper, we use standard and nonstandard truncations of a special limited case of the extended Painlevé expansion to obtain some types of periodic solutions which are expressed by the Jacobi elliptic functions. When the modular of the elliptic function tends to 1, two types of periodic solutions tends to the equivalent two types of solitary wave solutions, respectively. The method proposed here to find the exact periodic solutions expressed by the Jacobi elliptic functions can be used for other types of models, which cannot be integrated directly.

Because the model (1) is derived for modeling nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters. More details on the results of this paper, especially on the soliton solutions without dispersion relations and the method to obtain these solutions, are worthy of further study.

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