Multichannel intermittencies induced by symmetries

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Type-I intermittencies are common phenomena that are often observed in the neighborhood of periodic windows when a control parameter is varied. These intermittencies usually have a single reinjection channel, that is, a single type of laminar phase was observed. Recently, type-I intermittencies with two reinjection channels were reported in several systems. In this paper, it will be shown that type-I intermittencies with n channels of reinjection are associated with the coexistence of n stable periodic orbits that are mapped into each other under a symmetry. A procedure to build type-I intermittency with n reinjection channels using the n-fold cover of an image system is presented. Cases up to $n=3$ are explicitly given with the covers of the centered Rössler system.

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I. INTRODUCTION

Intermittencies are often observed on the route to chaos. They were first described by Pomeau and Manneville [1]. The original intermittencies were all characterized by a single "reinjection channel." A reinjection channel is a virtual periodic orbit. This is an orbit that will be created in a saddle-node bifurcation under a small change in control parameter values. Before its creation, the virtual orbit traps nearby initial conditions and entrains the evolution in the neighbornood of the virtual orbit for many cycles. The average residence time in the neighborhood of the virtual periodic orbit increases with a canonical $-\frac{1}{2}$ power law dependence as the edge of the window is approached.

Recently intermittency with two reinjection channels has been reported in a number of systems, both driven and autonomous: a five-dimensional model of an externally forced laser with saturable absorber [2,3]; a driven one-dimensional model for the transition from periodic to chaotic motion in granular shear flows [4]; and a five-dimensional autonomous model of a laser with saturable absorber [4,6]. Also, type-I intermittency with two reinjection channels has been reported in two models of the Rayleigh-Bénard convection [5]: the celebrated Lorenz system [7]; and a nine-dimensional model of the Rayleigh-Benard convection recently introduced by Reiterer and coworkers [8] to study high-dimensional chaos.

All the systems in which two-channel intermittency has been observed are equivariant. This means they are unchanged under the action of some symmetry group. The symmetry raises the possibility that (virtual) periodic orbits can occur in disconnected pairs (more generally, multiplets). Each member of the pair provides a distinct reinjection channel under suitable circumstances.

In this paper the relation between symmetry and the existence of multichannel intermittency will be investigated. We will use the n-fold covers of the Rössler system [9] as benchmark models. These are covering dynamical systems invariant under the rotation group C_n, generated by the rotations $\mathcal{R}_z(2\pi/n)$ through $2\pi/n$ about the z axis. We show that intermittency with n channels (or $n/2$, or $n/3,\ldots$) can be observed. The number of channels is equal to the number of disconnected stable limit cycles that cover a stable limit cycle in a periodic window in the image dynamical system.

In Sec. II we describe two-channel intermittency in the Lorenz system. The main part of this paper is presented in Sec. III. Here we describe the n-fold covers of the centered Rössler system $(n=2,3)$. We describe how the number of reinjection channels is changed as the cover of this system is deformed by displacement. Our results are summarized in Sec. IV.

II. TWO-CHANNEL INTERMITTENCY IN THE LORENZ SYSTEM

The Lorenz system [7]

$$\begin{aligned}\dot{x}&=-\sigma x+\sigma y,\\ \dot{y}&=Rx-y-xz,\\ \dot{z}&=-bz+xy\end{aligned}\tag{1}$$

is one of the first systems in which a type-I intermittency has been investigated [10]. The Lorenz system is equivariant, i.e., it obeys the relation

$$\boldsymbol{\gamma}\cdot\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{\gamma}\cdot\boldsymbol{x}),\tag{2}$$

where $\boldsymbol{\gamma}$ is the 3×3 matrix

$$\boldsymbol{\gamma}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&+1\end{bmatrix}\tag{3}$$

defining a rotation symmetry $\mathcal{R}_z(\pi)$ by π around the z axis. Depending on the control parameter values, the attractor is either fully symmetric, i.e., globally unchanged under the action of the symmetry, or asymmetric. In the latter case, two attractors coexist in the phase space, one being mapped to

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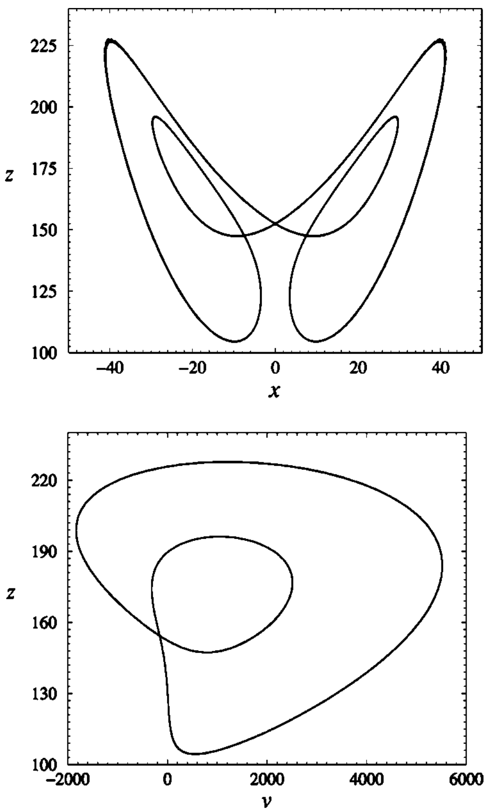

FIG. 1. (a) The period-4 limit cycle *LLRR* observed in the original phase space $\mathbb{R}^3(x,y,z)$ and (b) its image 01 in the image phase space $\mathbb{R}^3(u,v,w)$. Parameter values: $(R,\sigma,b)=(166.1,10,8/3)$.

the other under the action of the $\boldsymbol{\gamma}$ matrix. This feature results from the fact that the rotation symmetry $\mathcal{R}_z(\pi)$ is an order-2 symmetry, i.e., $\boldsymbol{\gamma}^2=\mathbb{I}$ where $\mathbb{I}$ is the identity matrix.

In the work that follows, we explore ranges of control parameter values in which low-period windows exist. These windows are bounded on both sides by chaotic regions in which the strange attractor exhibits the full symmetry of the equivariance group.

The first observation of an intermittent behavior was done for $(R,\sigma,b)=(166.1,10,8/3)$. For these control parameter values, a single stable limit cycle *LLRR* exists in the periodic window. It is symmetric, i.e., left globally unchanged under the action of the symmetry [Fig. 1(a)]. Thus, a unique reinjection channel is observed for this virtual symmetric periodic orbit.

Such an intermittency may be conveniently investigated in the image of the Lorenz system, i.e., in a representation of the Lorenz dynamics obtained by modding out the symmetry. Such a representation, the so-called image system, is locally equivalent to the original dynamics but without any residual symmetry [9]. The image system of the Lorenz system may be obtained using a $2\to 1$ mapping $\Psi_2:\mathbb{R}^3(x,y,z)\to\mathbb{R}^3(u,v,w)$. The coordinates (u,v,w) are linear combinations of elementary polynomials in (x,y,z) that are invariant under the symmetry. The coordinate transformation

$$\Psi_2 \equiv \left| \begin{array}{l} u=\mathrm{Re}(x+iy)^2=x^2-y^2, \\ v=\mathrm{Im}(x+iy)^2=2xy, \\ w=z \end{array} \right. \tag{4}$$

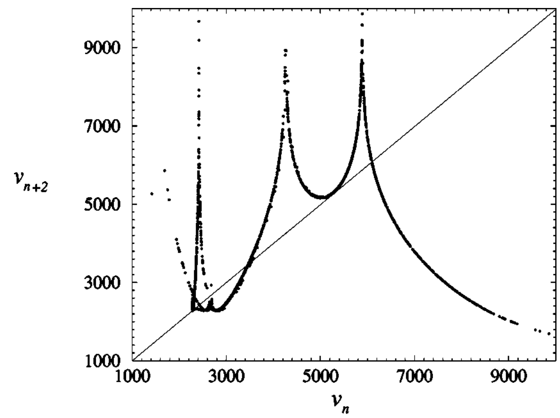

FIG. 2. The second-return map to a Poincaré section of the image of the Lorenz system. Parameter values: $(R,\sigma,b)=166.5,10,8/3)$.

introduced by Miranda and Stone [11] is used to mod out the rotation symmetry by π around the z axis. The image system is

$$\begin{aligned} \dot{u} &= -(\sigma+1)u+(\sigma-R)v+vw+(1-\sigma)\rho, \\ \dot{v} &= -(R-\sigma)u-(\sigma+1)v-uw+(R+\sigma)\rho-\rho w, \\ \dot{w} &= -bw+\frac{v}{2}, \end{aligned} \tag{5}$$

where $\rho=\sqrt{u^2+v^2}$. Under this map the image of the period-4 orbit *LLRR* [Fig. 1(a)] is the period-2 orbit 01 [Fig. 1(b)].

Computing the first-return map to a Poincaré section for the image system is equivalent to computing a first-return map to the maximum as Lorenz did in his original paper [7]. Since the period-2 stable limit cycle 01 is associated with the periodic window, a second-return map has to be computed for the image system. The tangent bifurcation (Fig. 2) is easily exhibited although spurious intersections are difficult to avoid. Two tangencies are observed in the return map at $v\simeq 2200$ and $v\simeq 5200$. These correspond to the local v maxima in the period-2 orbit 01 [Fig. 1(b)]. As reported in Ref. [10], the intermittency in the Lorenz system occurs with a single laminar phase associated with the virtual orbit *LLRR* [Fig. 1(a)].

In the periodic window at $R=100.795$ there are two symmetry-related period-3 orbits *LLR* and *RRL* (Fig. 3) created by simultaneous saddle-node bifurcations from the virtual orbits. The key point is that the disconnected virtual orbits occur in a strange attractor (Fig. 4) that is globally invariant under the action of $\boldsymbol{\gamma}$. The trajectory is "arbitrarily" reinjected into one of the two channels. Consequently, the laminar phases are associated with a trajectory visiting the neighborhood of either of the two symmetry-related virtual orbits *LLR* or *RRL*. Two different kinds of laminar phases can thus be distinguished (Fig. 5). This multichannel type-I intermittency is therefore closely related to the symmetry property of the Lorenz system.

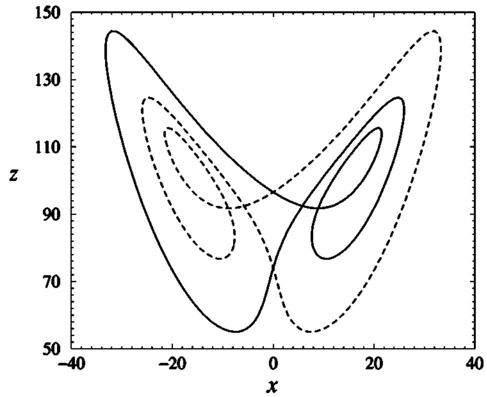

FIG. 3. The coexisting asymmetric period-3 limit cycles *LLR* and *RRL* generated by the Lorenz system. The initial conditions are related by the $\boldsymbol{\gamma}$ matrix. Parameter values: $(R,\sigma,b)=(100.795,10,8/3)$.

When the intermittent behavior is investigated in the image system, the symmetry properties are modded out and the two-channel type-I intermittency does not present any difference with the common type-I intermittency as observed in any system without any symmetry properties. Indeed, a single type of laminar phase is observed as suggested by the chaotic attractor of the image, which does not have any residual symmetry (Fig. 6). Both asymmetric orbits of the original phase space $\mathbb{R}^3(x,y,z)$ (Fig. 3) are mapped to the same period-3 orbit in the image phase space $\mathbb{R}^3(u,v,w)$. This is the effect of the $2\rightarrow 1$ mapping Ψ_2 between the equivariant Lorenz system and its image. The laminar phase is described by a single virtual orbit, 011, which is the image of both *LLR* and *RRL*.

Since, in the image system, the stable limit cycle has a period equal to 3, a third-return map to a Poincaré section has to be computed in the image system (Fig. 7). In such a third-return map, three tangencies to the bisector are clearly identified at $\nu \simeq 1000,1540$, and 3200. They correspond to the three periodic points of the limit cycle to appear.

Note that when the original Lorenz system is investigated, the third-return map is much more difficult to obtain. A third-return map should present 2×3 tangencies with the bisector, three points being associated with each asymmetric limit cycle observed for $R=100.795$ (Fig. 3).

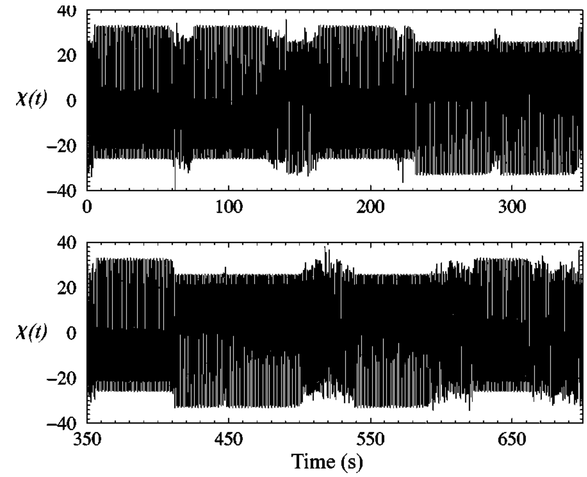

FIG. 5. Time series of the intermittent behavior with two reinjection channels in the Lorenz system. Parameter values: $(R,\sigma,b)=(100.799,10,8/3)$.

III. THE *n*-FOLD COVER OF THE CENTERED RÖSSLER SYSTEM

We would like to be able to build type-I intermittencies with n reinjection channels. To do that, the Rössler system [12] will be used as an image system. This choice results from the bifurcation diagram of the Rössler system which is quite well described [13]. To this end, we will construct the n-fold cover of the Rössler system.

In general, the n-fold cover $\dot{\boldsymbol{x}}=\boldsymbol{F}(\boldsymbol{x})$, of the image system $\dot{\boldsymbol{u}}=\boldsymbol{G}(\boldsymbol{u})$, with a rotation symmetry around the z axis is constructed by using

$$\frac{dx_i}{dt}=\frac{\partial x_i}{\partial u_j}\frac{du_j}{dt}=\left[\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^{-1}\right]_{ij} G_j(\boldsymbol{u})=F_i(\boldsymbol{x}), \qquad (6)$$

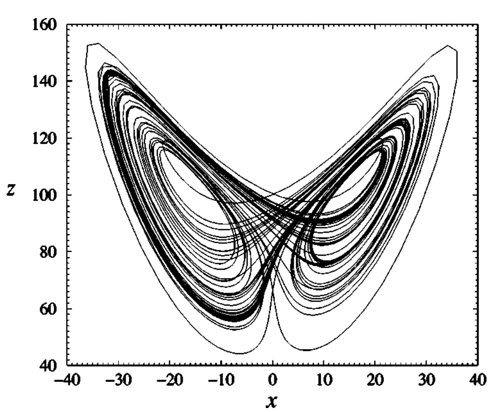

FIG. 4. Chaotic attractor, globally invariant under the action of the $\boldsymbol{\gamma}$ matrix, generated by the Lorenz system. Chaotic bursts from the two asymmetric laminar phases *LLR* and *RRL* follow this symmetric attractor. Parameter values: $(R,\sigma,b)=(100.799,10,8/3)$.

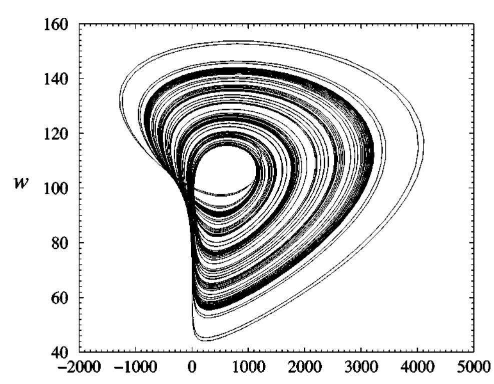

FIG. 6. Chaotic attractor generated by the Lorenz system projected in the image system without any residual symmetry. The image attractor is locally equivalent to the original one. Parameter values: $(R,\sigma,b)=(100.799,10,8/3)$.

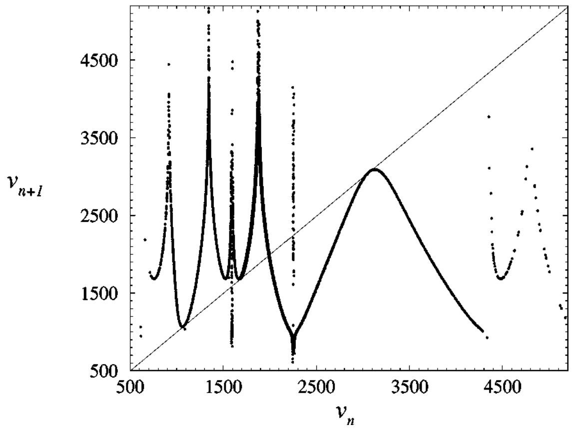

FIG. 7. Third-return map to a Poincaré section of the image attractor. Three tangencies with the bisecting lines are observed. They correspond to the image of both limit cycles observed in Fig. 3. Parameter values: $(R,\sigma,b)=(100.796,10,8/3)$.

with the obvious notations $\boldsymbol{x}=(x,y,z)=(x_1,x_2,x_3)$, $\boldsymbol{u}=(u,v,w)=(u_1,u_2,u_3)$ for the variables, the notations $\boldsymbol{F}=(F_1,F_2,F_3)$ and $\boldsymbol{G}=(G_1,G_2,G_3)$ for the vector fields, and $\partial\boldsymbol{u}/\partial\boldsymbol{x}$ for the Jacobian of the $n\rightarrow 1$ mapping $\Psi:\mathbb{R}^3(x,y,z)\rightarrow\mathbb{R}^3(u,v,w)$.

Following the procedure introduced in Ref. [9], the first step is to move the inner fixed point of the Rössler system to the origin of the phase space $\mathbb{R}^3(u,v,w)$. This is done using the rigid displacement $(u,v,w)\rightarrow(u+u_0,v+v_0,w+w_0)$. In the translated coordinate system, the equations for this image system (the Rössler system has no residual symmetry) are

$$\begin{aligned}\dot{u}&=-v-w-v_0-w_0,\\ \dot{v}&=u+av+u_0+av_0,\\ \dot{w}&=b+w(u+u_0-c)+w_0u+w_0(u_0-c),\end{aligned}\tag{7}$$

where $u_0=-av_0=aw_0=(c-\sqrt{c^2-4ab})/2$ are the coordinates of the inner fixed point of the original Rössler system. This system may thus be rewritten as

$$\begin{aligned}\dot{u}&=-v-w,\\ \dot{v}&=u+av,\\ \dot{w}&=\tilde{b}u+w(u-\tilde{c}),\end{aligned}\tag{8}$$

where $\tilde{b}=w_0$ and $\tilde{c}=c-u_0$.

In what follows, it will be necessary to move the rotation axis in order to change the number of coexisting stable limit cycles. Indeed, the *peeling bifurcation* (introduced in Ref. [9] and later described in terms of periodic orbits) allows to change the topology of the covers and, consequently, the connectivity properties of the orbit(s), both stable and virtual, that cover periodic orbits in the image. When the origin of coordinates is displaced along the u axis by a quantity equal to μ, the equations for the image Rössler system are

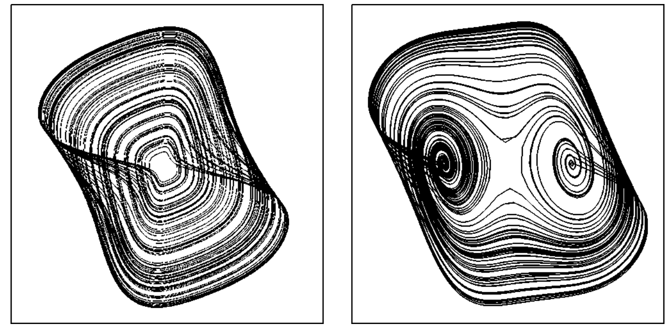

FIG. 8. Chaotic attractor generated by the twofold cover of the Rössler system. Both attractors are globally invariant under the rotation $\mathcal{R}_z(\pi)$. (a) $\mu=0.0$, (b) $\mu=-1.5$. Parameter values: $(a,b,c)=(0.432,2.0,4.0)$.

$$\begin{aligned}\dot{u}&=-v-w,\\ \dot{v}&=u+av+\mu,\\ \dot{w}&=\tilde{b}(u+\mu)+w(u-\tilde{c}+\mu).\end{aligned}\tag{9}$$

A. Twofold cover

Inverting Ψ_2 of Eq. (4) and injecting it in Eq. (6), the covering equations of the Rössler system (9) are [9]

$$\begin{aligned}\dot{x}&=\frac{1}{2\rho^2}[-\rho^2y+x(2ay^2-z)+\mu y],\\ \dot{y}&=\frac{1}{2\rho^2}[\rho^2x+y(2ax^2+z)+\mu x],\\ \dot{z}&=\tilde{b}(x^2-y^2+\mu)+z(x^2-y^2-\tilde{c}+\mu),\end{aligned}\tag{10}$$

where $\rho^2=x^2+y^2$. The chaotic attractor generated by these covering equations is globally invariant under a rotation symmetry $\mathcal{R}_z(\pi)$ (Fig. 8). For $(a,b,c)\simeq(0.4,2.0,4.0)$ it remains connected for $\mu=0.0$ down to $\mu=-4.0$.

We would like to investigate the type-I intermittency observed for $a\approx 0.4091$, just before the period-3 window. When $\mu=0$, the rotation axis is located at the origin of the phase space $\mathbb{R}^3(u,v,w)$. The period-3 limit cycle observed in the image Rössler system is thus "lifted" to a single symmetric period-6 orbit in the two-fold cover [Fig. 9(a)]. Since a single stable limit cycle exists, a type-I intermittency with a single channel is observed [Fig. 10(a)].

In order to change the number of stable limit cycles that coexist in the covering phase space for a control parameter value corresponding to a periodic window, it is sufficient to displace the rotation axis along the u axis [9]. Indeed, when the rotation axis intersects the chaotic attractor, the flow of the cover is deformed like the deformation of an apple skin when the apple is peeled. Hence the name *peeling bifurcation* for the global bifurcation [9]. The chaotic attractor of the cover is thus peeled around the rotation axis. But let us describe what happens to the period-3 orbit when the μ param-

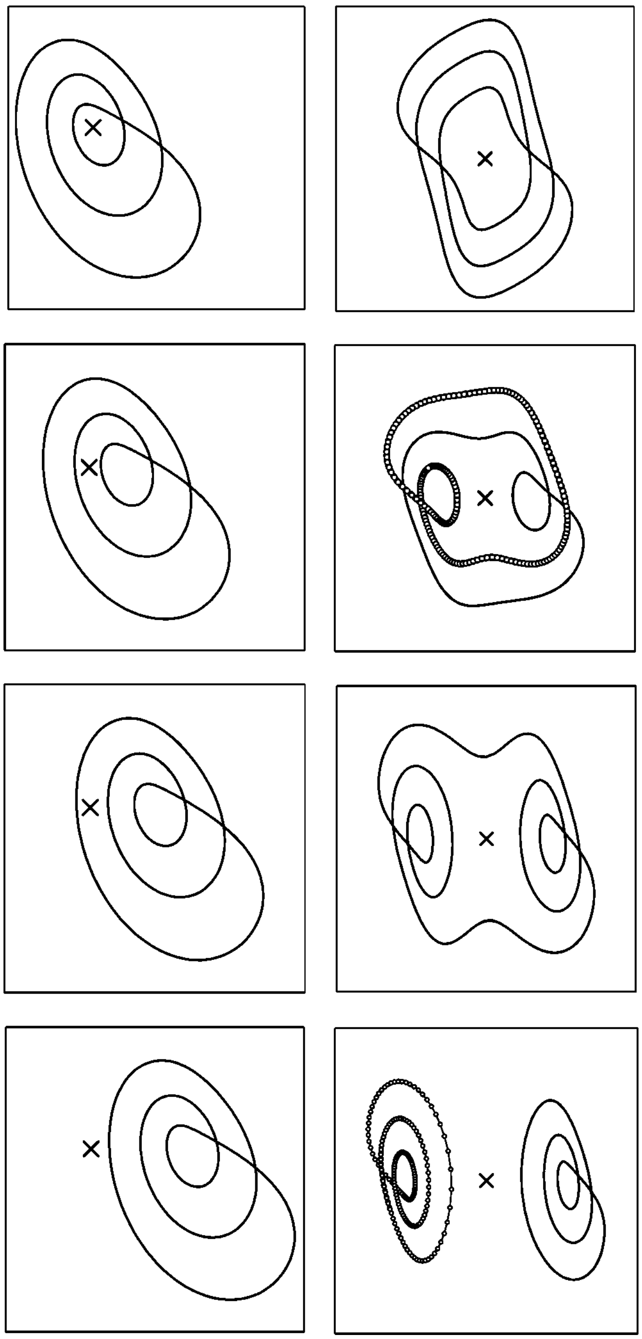

FIG. 9. Transformation of the period-3 orbits generated by the twofold cover of the Rössler under the peeling bifurcation when μ is varied. The location of the rotation axis is designated by the symbol $\times$. Parameter values: $(a,b,c)=(0.4096,2.0,4.0)$.

eter is progressively decreased. When the rotation axis is displaced toward the left side of the u axis ($\mu=-1.5$), it intersects a segment of the orbit in the image space $\mathbb{R}^3(u,v,w)$. One revolution of the image periodic orbit does not encircle the rotation axis anymore [Fig. 9(b)]. We introduce a topological index, N_α, that defines the number of times the orbit encircles the rotation axis. This index is decreased by 1 each time the rotation axis passes through the period-3 image orbit. After the first intersection, the period-3 image orbit is lifted into a symmetric pair of period-3 orbits [Fig. 9(b)]. The topological index $\tilde{N}_\alpha$ of this pair of periodic orbits is equal to 2, as in the image space.

As μ continues to decrease, the rotation axis cuts the image period-3 orbit a second time. At $\mu=-3.0$ [Fig. 9(c)] $N_\alpha=1$ and the period-3 image orbit is lifted to a single symmetric period-6 covering orbit. Finally, after the third crossing, at $\mu=-4.5$ [Fig. 9(d)] $N_\alpha=0$ and the period-3 image orbit is lifted to a pair of symmetry-related cover orbits. However, these orbits are embedded in two asymmetric symmetry-related strange attractors.

In this sequence, there is a single reinjection channel for $\mu=0.0$ and $\mu=-3.0$ [Figs. 9(a) and 9(c)] since the covering orbit is connected in a symmetric strange attractor. There are two reinjection channels at $\mu=-1.5$ since the cover consists of two disconnected symmetry-related orbits in a symmetric connected attractor. Finally, there is one reinjection channel (in each attractor) at $\mu=-4.5$, since the cover of the period-3 orbit consists of two symmetry-related orbits, but the covering attractor is itself not connected. It consists of two asymmetric symmetry-related attractors, each containing one of the two symmetry-related orbits. The $x(t)$ time series

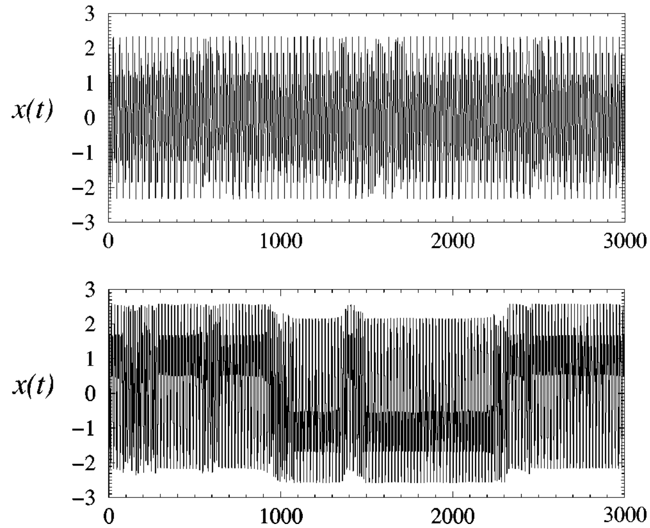

FIG. 10. Time series of the x variable corresponding to the case where one orbit (a) and two orbits (b) are associated with the period-3 window. In the first case, an intermittency with a single channel is observed. In the second case, two channels are identified and, consequently, two different laminar phases are described. Parameter values: $(a,b,c)=(0.409109,2.0,4.0)$.

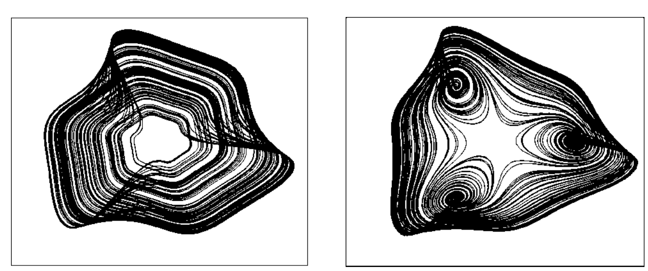

FIG. 11. Chaotic attractor generated by the threefold cover of the Rössler system for (a) $\mu=0.0$ and (b) $\mu=-1.15$. Parameter values: $(a,b,c)=(0.432,2.0,4.0)$.

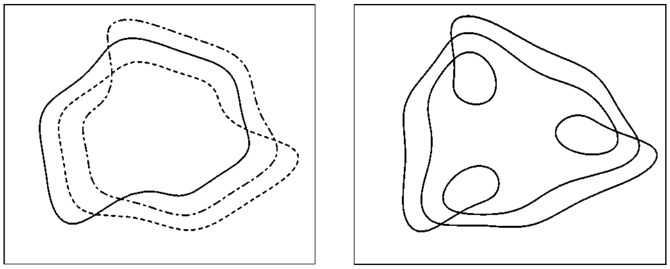

FIG. 12. At $(a,b,c)=(0.4096,2.0,4.0)$ the Rössler system has a stable period-3 orbit. (a) This is lifted to three coexisting disconnected symmetry-related period-3 orbits for $\mu=0.0$ and (b) one single symmetric period-9 orbit for $\mu=-1.15$.

for one ($\mu=0.0$) and two ($\mu=-1.5$) reinjection channels are shown in Figs. 10(a) and 10(b).

B. Threefold cover

A similar process can be observed in the threefold cover of the centered Rössler system. The rotation is now by an angle $2\pi/3$. Following the same procedure as for the twofold cover, the dynamical equations can be obtained using the coordinate transformation

$$\Psi_3 \equiv \left| \begin{array}{l} u=\mathrm{Re}(x+iy)^3=x^3-3xy^2 \\ v=\mathrm{Im}(x+iy)^3=3x^2y-y^3 \\ w=z. \end{array} \right. \qquad (11)$$

The chaotic attractor generated by the threefold cover of the centered Rössler system is shown in Figs. 11(a) and 11(b) for two different values of μ.

When the rotation axis is displaced, the attractor generated by the threefold cover is deformed as shown in Fig. 11(b). Since this is a threefold cover, upto three limit cycles may coexist in the cover space $\mathbb{R}^3(x,y,z)$ [Fig. 12(a)]. Depending on the location of the rotation axis, three period-3 [Fig. 12(a)] or one period-9 [Fig. 12(b)] limit cycles are observed. In both cases, the limit cycles are embedded in an attractor globally invariant under the rotation symmetry $\mathcal{R}_z(2\pi/3)$. Consequently, depending on μ, one or three types of laminar phases may be observed. An example with the three types of laminar phases is shown in Fig. 13.

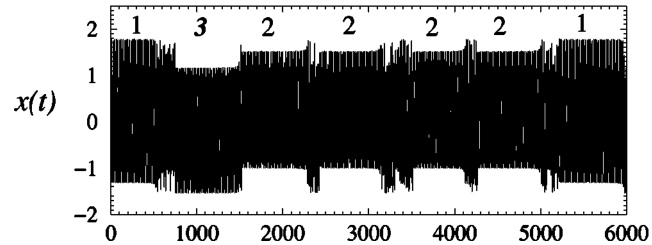

FIG. 13. Time series of the x variable corresponding to the case where three orbits are associated with the period-3 window. Three channels are identified and, consequently, three different laminar phases are described. Parameter values: $(a,b,c)=(0.409109,2.0,4.0)$.

IV. CONCLUSION

Type-I intermittency occurs when a trajectory in a strange attractor enters a neighborhood in phase space that generates near recurrent behavior. Such neighborhoods are typically associated with virtual orbits: periodic orbits about to be created by a saddle-node bifurcation. Intermittency typically occurs on the other side of the edge of a periodic window defined by the saddle-node bifurcation, and can only be observed easily near relatively low-period windows.

Multiple reinjection channels can exist when two or more virtual low-period orbits exist for the same control parameter values. The easiest way to enforce this condition is through a symmetry. We have studied the relation between the number of reinjection channels and symmetry in this work. In particular, we have looked at n-fold covers of the Rössler system that are invariant under the rotation group C_n generated by $\mathcal{R}_z(2\pi/n)$. Period-p orbits in the image system can lift to n period-p orbits in the cover, one period-np orbit, or other intermediate cases, depending on the symbolic dynamic name of the image orbit and some topological index (N_α) of that orbit. When m disconnected orbits cover the original orbit and when the covering attractor is connected, m reinjection channels are observed in type-I intermittency in the covering dynamical system. Since multistability phenomenon naturally arises when considering networks of identical oscillators ([14–17] among others) that have inherent symmetry properties, such systems typically possess a variety of coexisting attractors that can be good candidates for multichannel intermittencies induced by symmetries.

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